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An Ideal Based Regular Digraph of Ideals of Commutative Rings

Masoud Karimi
Department of Mathematics, Bojnourd Branch, Islamic Azad University, Bojnourd, Iran
Email: karimimth@yahoo.com

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Abstract. Let R be a commutative ring. The regular digraph of ideals of R, denoted by $\Gamma(R)$, is a digraph whose vertex-set is the set of all non-trivial ideals of R and for every two distinct vertices I and J, there is an arc from I to J, whenever I contains a non-zero zero divisor on J. We generalize this notion with respect to an ideal I_0 of R and denote it by $\Gamma_{I_0}(R)$ in such a way that $I_0 = 0$ gives us $\Gamma(R)$. Also by verifying connectedness and diameter of $\Gamma_{I_0}(R)$, we will observe that there is a strong relation between $\Gamma_{I_0}(R)$ and $\Gamma(R/I_0)$. Finally, we consider the number of non-singular connected components of $\Gamma_{I_0}(R)$.

Keywords: Regular digraph; Diameter; Connectedness; Commutative ring; Zero divisor.

1. Introduction

Let R be a commutative ring. Then, the regular digraph of ideals of R, denoted by $\Gamma(R)$, is a digraph whose vertex-set is the set of all nontrivial ideals of R, and there is an arc from I to J, whenever I contains a J-regular element, that is, an element $x \in I$ such that $xy \neq 0$ for all $y \in J$ with $y \neq 0$.

Authors in [9] introduced the notion of regular digraph of ideals of a commutative ring; they denoted it by $\overrightarrow{\Gamma_{\mathrm{reg}}}(R)$. This work was mostly concerned with only the colouring of Artinain rings and some results on diameter and connectedness. Soon after, authors in [3, 2, 1] extensively discussed this graph and directed their study towards Noetherian rings.

A motivation of our attempt, is the work of Redmond in [10] (see also [6, 8]), in which the concept of an ideal based zero divisor graph was introduced. He showed that the zero divisor graph and its ideal based one, are strongly related to each other by using the notion of quotient rings.

In this paper, for a given ideal I_0 of R, by $\Gamma_{I_0}(R)$ we denote the regular digraph of ideals of R with respect to I_0. It will be observed that in the case of $I_0 = 0$ we turn to $\Gamma(R)$, indeed, $\Gamma(R/I_0)$ is isomorphic to an induced subgraph of $\Gamma_{I_0}(R)$.

We use standard terminology of ring theory following [4, 11], and [5, 7] to that of graph theory. But for the sake of completeness, we state some definitions and notations used throughout.

Throughout, all rings are assumed to be commutative and Noetherian, with non-zero identity. For given ring R, the *Jacobson radical* denoted by $\mathrm{J}(R)$, is the intersection of all maximal ideals of R, also by $\sqrt{I}$ we denote the set

$$\{x \in R \mid x^k \in I, \text{for some non-negative integer } k\},$$

and $\sqrt{0}$ is called *Nilradical* of R, but by way of exception we denote it by $\mathrm{Nil}(R)$. Also, the set of all zero-divisors of an R-module M, which is denoted by $Z_R(M)$, is the set

$$Z_R(M) = \{r \in R \mid rx = 0 \text{ for some non} - \text{zero element } x \text{ in } M\}.$$

An element $r \in R$ is called *M-regular* if $r \notin Z(M)$. We say that $\mathrm{depth}(R) = 0$ if every non-unit element of R is a zero-divisor. Let G be a simple graph. A vertex x is *isolated* if no vertex of G is adjacent to x. G is called *complete* when all vertices of G are adjacent to each other. Let A be a set of vertices of G. Then the subgraph of G induced by vertices in A denoted by $G[A]$.

The distance between two vertices x and y in G denoted by $\mathrm{d}_G(x, y)$, is the length of the shortest path between x and y. The *diameter* of G is the size of the longest distances between vertices of G, and we denote it by $\mathrm{diam}(G)$. If G is a directed graph (or digraph), an arc from a vertex x to a vertex y denoted by $x \longrightarrow_G y$, if no confusion, only with $x \longrightarrow y$.

2. Preliminaries

In this section we introduce the regular digraph of ideals of a commutative ring with respect an ideal, and state some preliminary results including some theorems and lemmas from [3, 2].

Definition 2.1. *Let I_0 be an ideal of a commutative ring R. The regular digraph of ideals of R with respect to I_0, denoted by $\Gamma_{I_0}(R)$. For distinct ideals I and J of R, there is an arc from I to J if and only if there exists $x \in I \setminus I_0$ such that $xy \notin I_0$ for all $y \in J \setminus I_0$. In this case we use $I \longrightarrow_{I_0} J$, to denote this arc.*

Clearly, $I \subseteq I_0$ implies that I is an isolated vertices of $\Gamma_{I_0}(R)$, thus the vertex-set of $\Gamma_{I_0}(R)$ is considered as the set of all ideals of R such that are not contained in I_0.

By definition, $\Gamma_{I_0}(R) = \Gamma(R)$ if and only if $I_0 = 0$.

Lemma 2.2. *$I \longrightarrow_{I_0} J$ if and only if $(I + I_0)/I_0 \nsubseteq Z_{R/I_0}((J + I_0)/I_0)$.*

Proof. Suppose that $I \longrightarrow_{I_0} J$. Then there exists $x \in I \setminus I_0$ such that $xy \notin I_0$ for all $y \in J \setminus I_0$. Since $x + I_0$ is a non-zero element in $(I + I_0)/I_0$ so $xy + I_0$ is a non-zero in $(J + I_0)/I_0$ for all $y \in J \setminus I_0$. This follows that $(I + I_0)/I_0 \nsubseteq Z_{R/I_0}((I + I_0)/I_0)$.

Conversely, let $(I+I_0)/I_0 \nsubseteq Z_{R/I_0}((I+I_0)/I_0)$. Hence there exists a non-zero element $x + I_0$ in $(I + I_0)/I_0$ such that $(x + I_0)(y + I_0) = xy + I_0$ is non-zero element in $(J + I_0)/I_0$ for all $y \in J \setminus I_0$. On the other hand, $x = x_1 + x_2$ such that $x_1 \in I \setminus I_0$. Thus, $x_1y \notin I_0$ for all $y \in J \setminus I_0$, and so $I \longrightarrow_{I_0} J$. ■

In the following lemma, we can establish some primary adjacencies in $\Gamma_{I_0}(R)$. The proof is routine and is omitted.

Lemma 2.3. *Suppose that I_0, I and J are distinct non-trivial ideals of R. Then*

(i) *If $I_0 + I = R$, then I is adjacent to all other vertices of $\Gamma_{I_0}(R)$.*
(ii) *If $IJ \subseteq I_0$, then I and J are not adjacent.*
(iii) *Let $I + I_0 \neq R$ and $I + I_0 \neq J$ and $I \neq J$. Then, $I \longrightarrow_{I_0} J$ if and only if $I + I_0 \longrightarrow_{I_0} J$.*
(iv) *Let $J + I_0 \neq R$ and $J + I_0 \neq I$ and $I \neq J$. Then, $I \longrightarrow_{I_0} J$ if and only if $I \longrightarrow_{I_0} J + I_0$.*

Theorem 2.4. *$\Gamma(R/I_0)$ is isomorphic to $\Gamma_{I_0}(R)[A]$.*

Proof. Set $A := \{I \mid I_0 \subset I\}$. Then the mapping $I/I_0 \longmapsto I$ provides a one to one correspondence between vertex-set of $\Gamma(R/I_0)$ and A. Furthermore, by Lemma 2.2, $I/I_0 \longrightarrow J/I_0$ is an arc in $\Gamma(R/I_0)$ if and only if $I \longrightarrow_{I_0} J$. Thus, the mapping gives an isomorphism between $\Gamma(R/I_0)$ and $\Gamma_{I_0}(R)[A]$. ■

Corollary 2.5. *$\Gamma_{I_0}(R)$ is a complete graph if and only if I_0 is a prime ideal of R.*

Proof. Suppose that I_0 is a prime ideal. Since R/I_0 is integral domain, we have $Z_{R/I_0}(R/I_0) = \{I_0\}$. This, together with Lemma 2.2, implies that all vertices of $\Gamma_{I_0}(R)$ (if any) are adjacent to each other. Thus, $\Gamma_{I_0}(R)$ is complete graph.

To the converse, suppose that $\Gamma_{I_0}(R)$ is complete. Then in view of Theorem 2.4, $\Gamma(R/I_0)$ is also complete. Now [9, Theorem 3.1] implies that R/I_0 is an integral domain, and so I_0 is prime ideal. ■

Theorem 2.6. *Let R be a Noetherian ring. The graph $\Gamma(R)$ is connected if and only if one of the following statements holds:*

(i) $\operatorname{depth}(R) \neq 0$.
(ii) $\operatorname{depth}(R) = 0$ *and* $R = F \times R'$, *where F is a field and R' is not an Artinian local ring.*

Corollary 2.7. [3, Corollary 3.2] *If R is a non-reduced ring such that $\Gamma(R)$ contains an isolated vertex, then $\operatorname{Nil}(R)$ is an isolated vertex in $\Gamma(R)$.*

3. Connectedness

We are interested to finding conditions under which $\Gamma_{I_0}(R)$ to be connected. First we need to discover isolated vertices. It is worth to recall from [3] that for a given ring R we say $\operatorname{depth}(R) \neq 0$ when there exists regular element.

Remark 3.1. Suppose that $\operatorname{depth}(R/I_0) \neq 0$. Let $x + I_0 \in R/I_0$ be a regular element. Because $(Rx + I_0)/I_0 \nsubseteq Z_{R/I_0}(R/I_0)$, by Lemma 2.2, Rx is adjacent to all other vertices of $\Gamma_{I_0}(R)$ and so $\Gamma_{I_0}(R)$ is connected. Moreover, in view of Lemma 2.3 (i), by condition $I_0 \nsubseteq \mathrm{J}(R)$ we are guaranteed that $\Gamma_{I_0}(R)$ is connected.

Next lemma is a routine statement in introductory ring-theory, and we will apply it frequently.

Lemma 3.2. *Suppose that R is a ring and I and J are distinct ideals. Then the mapping*

$$\phi\colon \frac{R}{\ker(\phi)} \longrightarrow \frac{R}{I} \times \frac{R}{J}$$

by $\phi(x) = (x + I, x + J)$, is a ring isomorphism if and only if $I + J = R$ and $ker(\phi) = I \cap J$.

Theorem 3.3. *Let* $\operatorname{depth}(R/I_0) = 0$ *and* $I_0 \subseteq \mathrm{J}(R)$. *Then $\Gamma_{I_0}(R)$ is empty if and only if one of the following statements holds:*

(i) *$R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*
(ii) *R/I_0 is an Artinian local ring.*

Proof. Suppose that (i) holds. Then R has only two distinct non-trivial ideals, and so easily $\Gamma_{I_0}(R)$ is empty.

Now suppose that (ii) holds. In view of [3, Theorem 2.5], $\Gamma(R/I_0)$ is empty, so Lemma 2.2 shows that $\Gamma_{I_0}(R)$ is empty.

Conversely, suppose that $\Gamma_{I_0}(R)$ is empty. Then, Theorem 2.4 together with [3, Theorem 2.5] show that either R/I_0 is an Artinian local ring, which implies (ii), or $R/I_0 \cong F_1 \times F_2$ where F_1 and F_2 are fields. In the preceding case, there exist maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R in such a way that $I_0 = \mathfrak{m} \cap \mathfrak{n}$, hence by Lemma 3.2 we can assume $F_1 = R/\mathfrak{m}$ and $F_2 = R/\mathfrak{n}$.

We deduce from the assumption and Lemma 2.3(i) that $I_0 = \mathrm{J}(R)$. To the rest of proof, we claim that $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathfrak{n}^2 = \mathfrak{n}$. To do this, assume, without loss of generality and contrary to the claim, that $\mathfrak{m}^2 \neq \mathfrak{m}$. Because

$$\frac{\mathfrak{m}^2 + I_0}{I_0} \cong \frac{\mathfrak{m}^2 + \mathfrak{m}}{\mathfrak{m}} \times \frac{\mathfrak{m}^2 + \mathfrak{n}}{\mathfrak{n}} = 0 \times F_2,$$

and $Z_{F_1 \times F_2}(0 \times F_2) = F_1 \times 0$, thus

$$Z_{R/I_0}(\frac{\mathfrak{m}^2 + I_0}{I_0}) = \frac{\mathfrak{n}}{I_0}.$$

On the other hand, it is clear that $\mathfrak{m}/I_0 \nsubseteq \mathfrak{n}/I_0$. This means that in contrast to the assumption we obtain $\mathfrak{m} \longrightarrow_{I_0} \mathfrak{m}^2$. So we must have $\mathfrak{m}^2 = \mathfrak{m}$ and similarly $\mathfrak{n}^2 = \mathfrak{n}$. Accordingly, $I_0^2 = I_0$ and Nakayama's Lemma imply that $I_0 = 0$. So we get (i). ■

Theorem 3.4. *Let* $\operatorname{depth}(R/I_0) = 0$ *and* $I_0 \subseteq \mathrm{J}(R)$. *Then* $\Gamma_{I_0}(R)$ *has an isolated vertex if and only if one of the following statements holds:*

(i) $R \cong F_1 \times F_2$ *where* F_1 *and* F_2 *are fields.*

(ii) $\Gamma(R/I_0)$ *has an isolated vertex and* R/I_0 *is not reduced.*

Proof. First suppose that (i) holds. Since $\mathrm{J}(R) = 0$ and $I_0 \subseteq \mathrm{J}(R)$, thus $I_0 = 0$. This means $\Gamma_{I_0}(R) = \Gamma(R)$, which comprises two isolated vertices.

Now suppose that (ii) holds. We have the following facts concerning $\sqrt{I_0}$.

(1) In view of [3, Corollary 3.2], $\sqrt{I_0}/I_0 = \mathrm{Nil}(R/I_0)$ is an isolated vertex of $\Gamma(R/I_0)$.

(2) By [3, Lemma 2.6], it follows from Lemma 2.2 that $\sqrt{I_0}$ is not the initial of any arc in $\Gamma_{I_0}(R)$

(3) By using [3, Proposition 3.1], $Z_{R/I_0}(R/I_0) = Z_{R/I_0}(\sqrt{I_0}/I_0)$.

(4) Since $\operatorname{depth}(R/I_0) = 0$, for given ideal I of R we have

$$\frac{I + I_0}{I_0} \subseteq Z_{R/I_0}\left(\frac{R}{I_0}\right) = Z_{R/I_0}\left(\frac{\sqrt{I_0}}{I_0}\right)$$

(5) Since $I_0 \subseteq \mathrm{J}(R)$, we have $I + I_0 \neq R$.

In view of Lemma 2.2 and (4), there is not any arc to $\sqrt{I_0}$, that is, $\sqrt{I_0}$ is not the terminal point of an arc; so (3) implies that $\sqrt{I_0}$ is an isolated vertex of $\Gamma_{I_0}(R)$.

Conversely, suppose that $\Gamma_{I_0}(R)$ meets an isolated vertex, say I. Then, in view of Lemma 2.3, it is easy to see that $(I + I_0)/I_0$ is an isolated vertex of $\Gamma(R/I_0)$. In the case that R/I_0 is not reduced, we have (ii). So assume that R/I_0

is reduced. Since $\operatorname{depth}(R/I_0) = 0$, by [3, Lemma 3.4(ii)], R/I_0 is a finite direct product of fields. If there appears more than two fields, then by Theorem 2.6(ii), $\Gamma(R/I_0)$ is connected, which is not possible. Thus $R/I_0 \cong F_1 \times F_2$. By using Lemma 3.2, $F_1 = R/\mathfrak{m}$ and $F_2 = R/\mathfrak{n}$ where $\mathfrak{m}$ and $\mathfrak{n}$ are maximal ideals of R with $I_0 = \mathfrak{m} \cap \mathfrak{n}$.

Since I assumed to be a vertex, $I \nsubseteq I_0 = \mathfrak{m} \cap \mathfrak{n}$. From this, without loss of generality, let $I \subseteq \mathfrak{m}$ and $I \nsubseteq \mathfrak{n}$. Then we have

$$Z_{R/I_0}\left(\frac{I + I_0}{I_0}\right) = \frac{\mathfrak{n}}{I_0}.$$

If $I \neq \mathfrak{m}$, then Lemma 2.2 implies $\mathfrak{m} \longrightarrow_{I_0} I$ which is impossible. Hence we must have $I = \mathfrak{m}$. By a similar argument one can conclude that $\mathfrak{m}$ and $\mathfrak{n}$ are the only possible isolated vertices of $\Gamma_{I_0}(R)$. Therefore, $\Gamma_{I_0}(R)$ is empty, and Theorem 3.3(i) gives that $I_0 = 0$ as desired. ■

Corollary 3.5. *Under the assumptions of Theorem 3.4, if moreover, $I_0 \neq \sqrt{I_0}$, then the ideal I is an isolated vertex if and only if $I \subseteq \sqrt{I_0}$ and $Z_{R/I_0}((I + I_0)/I_0) = Z_{R/I_0}(R/I_0)$.*

Remark 3.6. Let R be a non-reduced ring with $\operatorname{depth}(R) = 0$. Then $\Gamma(R)$ has no isolated vertex if and only if $R \cong F \times R'$, where F is a field and R' is not a field.

Proof. Suppose that $\Gamma(R)$ has no isolated vertex. Because $\operatorname{Nil}(R)$ is not an isolated vertex of $\Gamma(R)$, so by using [3, Lemma 2.4(i)] and [3, Lemma 2.6(ii)] there exists a maximal ideal $\mathfrak{m}$ such that $\operatorname{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Hence, Lemma 3.2 implies that $R \cong R/\operatorname{Ann}(\mathfrak{m}_0) \times R/\mathfrak{m}_0$. Since R is a non-reduced ring, $R/\operatorname{Ann}(\mathfrak{m}_0)$ may not be a field. This shows that $R \cong F \times R'$, where F is a field and R' is not a field.

Conversely, let $R \cong F \times R'$, where F is a field. Then it is easy to verify that vertices of $\Gamma(R)$ are adjacent to $F \times 0$ or $0 \times R'$. In other words, $\Gamma(R)$ has no isolated vertices. ■

Lemma 3.7. *Let $\operatorname{depth}(R/I_0) = 0$ and $I_0 \subseteq \mathrm{J}(R)$. If $\Gamma_{I_0}(R)$ has no isolated vertex, then $I_0 = \mathfrak{m} \cap J_0$ at which $\mathfrak{m}$ is a maximal ideal of R, and $\mathfrak{m} + J_0 = R$ for some ideal J_0 of R.*

Proof. The proof falls into two cases.

Case 1. R/I_0 is reduced. Since R/I_0 is a finite direct product of fields, we have $I_0 = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$, where $\mathfrak{m}_i$ are maximal ideals of R for $i = 1, \ldots, n$. Letting $\mathfrak{m} = \mathfrak{m}_1$ and $J_0 = \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$ gives the result.

Case 2. R/I_0 is not reduced. Since $\Gamma_{I_0}(R)$ does not meet any isolated vertices, so Theorem 3.4 and Remark 3.6 imply that $R/I_0 \cong F \times R'$, where F is a field and R' is a non-reduced ring. Consequently, $I_0 = \mathfrak{m} \cap J_0$ at which $\mathfrak{m}$ is a maximal ideal of R with $\mathfrak{m} + J_0 = R$ for some ideal J_0 of R. ■

Remark 3.8. Suppose that $\operatorname{depth}(R/I_0) = 0$ and $I_0 \subseteq \mathrm{J}(R)$. Furthermore, assume that $\Gamma_{I_0}(R)$ has no isolated vertices. In view of Lemma 3.7, $R/I_0 \cong F \times R'$, where $\mathfrak{m}$ is a maximal ideal of R with $\mathfrak{m} + J_0 = R$ for some ideal J_0 of R, $F = R/\mathfrak{m}$, $R' = R/J_0$ also we have $I_0 = \mathfrak{m} \cap J_0$.

Put

$$\Theta := \{I \in \mathrm{V}(\Gamma_{I_0}(R)) \mid I \subseteq \mathfrak{m}, I \nsubseteq J_0\},$$

$$\Sigma := \{I \in \mathrm{V}(\Gamma_{I_0}(R)) \mid I \nsubseteq \mathfrak{m}, I \subseteq J_0\},$$

$$\Omega := \{I \in \mathrm{V}(\Gamma_{I_0}(R)) \mid I \nsubseteq \mathfrak{m}, I \nsubseteq J_0\}.$$

Clearly $\mathrm{V}(\Gamma_{I_0}(R)) = \Theta \cup \Sigma \cup \Omega$. Since $\mathfrak{m} \in \Theta$ and $J_0 \in \Sigma$, thus Θ and Σ are non-empty sets. Moreover, if $\Omega = \emptyset$, then J_0 and $\mathfrak{m}$ are the only maximal ideals of R.

Lemma 3.9. *Under notations and assumptions of Remark 3.8, for given ideals I and J of R the following statements hold:*

(i) *If $J \in \Theta$ and $J \neq \mathfrak{m}$, then $\mathfrak{m} \longrightarrow_{I_0} J$.*
(ii) *$I \longrightarrow_{I_0} \mathfrak{m}$ if and only if $I + J_0 = R$, $I \neq J_0$ and $I \neq \mathfrak{m}$.*
(iii) *If $J \in \Sigma$ and $J \neq J_0$, then $J_0 \longrightarrow_{I_0} J$.*
(iv) *$I \longrightarrow_{I_0} J_0$ if and only if $I \nsubseteq \mathfrak{m}$ and $I \neq J_0$.*
(v) *Let $J \in \Theta$ and $I \in \Omega$. Then $I \longrightarrow_{I_0} J$ if and only if $I \longrightarrow_{J_0} J$. Furthermore, there is no arc from J to I in $\Gamma_{I_0}(R)$.*
(vi) *Vertices in Θ are not adjacent to vertices in Σ.*

Proof. First note that because

$$Z_{R_1 \times R_2}(I \times 0) = Z_{R_1}(I) \times R_2$$

and

$$Z_{R_1 \times R_2}(0 \times J) = R_1 \times Z_{R_2}(J),$$

and that $(I \times 0) \cap (0 \times J) = 0$, we can verify that

$$Z_{R_1 \times R_2}(I \times J) = Z_{R_1 \times R_2}((I \times 0) + (0 \times J)) = (Z_{R_1}(I) \times R_2) \cup (R_1 \times Z_{R_2}(J)).$$

Moreover, to simplicity of notation, we let " $=$ " stand for " $\cong$ " in Lemma 3.2 which we frequently use it.

(i) Suppose that $J \in \Theta$ and $J \neq \mathfrak{m}$. Since

$$\mathrm{Z}_{R/I_0}\left(\frac{J}{I_0}\right) = \mathrm{Z}_{F \times R'}\left(0 \times \frac{J + J_0}{J_0}\right) = F \times \mathrm{Z}_{R/J_0}\left(\frac{J + J_0}{J_0}\right)$$

and

$$\frac{\mathfrak{m}}{I_0} = 0 \times R' \nsubseteq F \times \mathrm{Z}_{R/J_0}\left(\frac{J + J_0}{J_0}\right)$$

thus, by Lemma 2.2, $\mathfrak{m} \longrightarrow_{I_0} J$.

(ii) Suppose that $I \longrightarrow_{I_0} \mathfrak{m}$. Then $\dfrac{I+I_0}{I_0} \nsubseteq Z_{R/I_0}(\dfrac{\mathfrak{m}}{I_0})$. In other words

$$\frac{I+\mathfrak{m}}{\mathfrak{m}} \times \frac{I+J_0}{J_0} \nsubseteq Z_{F\times R'}(0\times R') = F \times \mathrm{Z}_{R'}(R').$$

This implies that $(I+J_0)/J_0 \nsubseteq \mathrm{Z}_{R'}(R')$. On the other hand, $\operatorname{depth}(R/I_0) = 0$ gives $\operatorname{depth}(R') = 0$. Thus, we must have $(I+J_0)/J_0 = R'$, and so $I + J_0 = R$.

The converse implication is clear.

(iii) Suppose that $J \in \Sigma$ and $I \neq J_0$. Since

$$\frac{J_0}{I_0} = F \times 0 \nsubseteq 0 \times R',$$

and

$$\mathrm{Z}_{R/I_0}(\frac{J}{I_0}) = \mathrm{Z}_{F\times R'}(F\times 0) = 0 \times R'$$

thus, $J_0 \longrightarrow_{I_0} J$.

(iv) Suppose that $I \nsubseteq J_0$. Then, $(I+I_0)/I_0 \nsubseteq Z_{R/I_0}(J_0/I_0)$ if and only if

$$\frac{I+\mathfrak{m}}{\mathfrak{m}} \times \frac{I+J_0}{J_0} \nsubseteq Z_{F\times R'}(F\times 0) = 0 \times R'.$$

This is equivalent to say that $I \nsubseteq \mathfrak{m}$. So we obtain (iv).

(v) Suppose that $I \in \Omega$ and $J \in \Theta$. Then, $I \longrightarrow_{I_0} J$ if and only if

$$\frac{I+\mathfrak{m}}{\mathfrak{m}} \times \frac{I+J_0}{J_0} \nsubseteq Z_{F\times R'}(\frac{J+\mathfrak{m}}{\mathfrak{m}} \times \frac{J+J_0}{J_0}).$$

Equivalently,

$$F \times \frac{I+J_0}{J_0} \nsubseteq Z_{F\times R'}(0 \times \frac{J+J_0}{J_0}).$$

Since

$$Z_{F\times R'}(0\times \frac{J+J_0}{J_0}) = F \times \mathrm{Z}_{R/J_0}(\frac{J+J_0}{J_0}),$$

we have

$$\frac{I+J_0}{J_0} \nsubseteq \mathrm{Z}_{R/J_0}(\frac{J+J_0}{J_0}).$$

Therefore, $I \longrightarrow_{J_0} J$ if and only if $I \longrightarrow_{I_0} J$.

To the furthermore, note that by

$$Z_{F\times R'}(\frac{I+\mathfrak{m}}{\mathfrak{m}} \times \frac{I+J_0}{J_0}) = Z_{F\times R'}(F\times \frac{I+J_0}{J_0}) = (F\times \mathrm{Z}_{R/J_0}(\frac{I+J_0}{J_0}))\cup(0\times R'),$$

we obtain

$$\frac{J+\mathfrak{m}}{\mathfrak{m}} \times \frac{J+J_0}{J_0} = 0 \times \frac{J+J_0}{J_0} \subseteq 0 \times R' \subseteq Z_{F\times R'}(F\times \frac{I+J_0}{J_0})$$

which shows that there is no arc from J to I in $\Gamma_{I_0}(R)$.

(vi) Suppose that $I \in \Sigma$ and $J \in \Theta$. Since $IJ \subseteq I_0$, by Lemma 2.3 (ii), they are not adjacent. ■

Now we are ready to prove the main result of this section.

Theorem 3.10. $\Gamma_{I_0}(R)$ *is connected if and only if one of the following statements holds:*

(i) $I_0 \nsubseteq \mathrm{J}(R)$.

(ii) $\Gamma(R/I_0)$ *is connected.*

Proof. Suppose that $\Gamma_{I_0}(R)$ is connected. In the case that $\mathrm{depth}(R/I_0) \neq 0$, clearly $\Gamma(R/I_0)$ is connected, so (ii) holds. Assume that $\mathrm{depth}(R/I_0) = 0$.

If (i) holds, we have nothings to prove. So suppose that $I_0 \subseteq \mathrm{J}(R)$. In view of Lemma 3.7, $R/I_0 \cong F \times R'$, where $F = R/\mathfrak{m}$ and $R' = R/J_0$. In view of Theorem 2.6, we need only to show that R/J_0 is not Artinian local ring.

The subgraph of $\Gamma_{I_0}(R)$ induced by $\Sigma \cup \Theta$ is disconnected (see Lemma 3.9 (vi)), however, the subgraph of $\Gamma_{I_0}(R)$ induced by $\Omega \cup \Sigma$ is connected (see Lemma 3.9 (v)). Consequently, there must exist an arc $I \longrightarrow_{I_0} J$ where $I \in \Omega$ and $J \in \Theta$ because $\Gamma_{I_0}(R)$ is connected. So, as a result of Lemma 3.9 (v), $I \longrightarrow_{J_0} J$. This, indeed, shows that $\Gamma_{J_0}(R)$ is not empty, hence by Theorem 3.3, R/J_0 is not Artinian local ring. This is enough to deduce that $\Gamma(R/I_0)$ is connected.

Conversely, if (i) holds, then by applying Lemma 2.3(i), $\Gamma_{I_0}(R)$ is connected. So, suppose that $I_0 \subseteq \mathrm{J}(R)$ and $\Gamma(R/I_0)$ is connected. Then by Theorem 3.4 $\Gamma_{I_0}(R)$ has no isolated vertices. In view of Lemma 3.9 all vertices of $\Gamma_{I_0}(R)$ are adjacent to $\mathfrak{m}$ or J_0, so we show that $\mathfrak{m}$ and J_0 can be connected via a path.

Since $\Gamma(R/I_0)$ is connected, there is a path in $\Gamma(R/I_0)$ (in fact in $\Gamma(F \times R')$) as below.

$$F \times 0 \longleftarrow F \times I \longrightarrow 0 \times J \longleftarrow 0 \times R'.$$

This is a path in $\Gamma_{I_0}(R)$, such that connects $\mathfrak{m}$ and J_0 because by Theorem 2.4, $\Gamma(R/I_0)$ is a subgraph of $\Gamma_{I_0}(R)$. ■

4. Diameter and Connected Components

This section is devoted to computing two numerical invariants, the diameter and the number of connected components. Suppose that $I_0 \nsubseteq \mathrm{J}(R)$. Then, in view of Lemma 2.3 (i), $\mathrm{diam}(\Gamma_{I_0}(R)) \leq 2$. By Corollary 2.5, $\mathrm{diam}(\Gamma_{I_0}(R)) = 1$ if and only if $\mathrm{diam}(\Gamma(R/I_0) = 1$. Therefore $\mathrm{diam}(\Gamma_{I_0}(R)) = 2$ if and only if $\mathrm{diam}(\Gamma(R/I_0) = 2$.

Hence by the above discussion, in the reminder of this section we assume that $I_0 \subseteq \mathrm{J}(R)$ and R is.

Lemma 4.1. *Let R be a ring. Then* $\operatorname{diam}(\Gamma(R/I_0)) \leq \operatorname{diam}(\Gamma_{I_0}(R))$

Proof. Let us A be the set that we used in the proof of Theorem 2.4. First we show that for any two vertices I and J in A

$$\mathrm{d}_{\Gamma_{I_0}(R)[A]}(I, J) = \mathrm{d}_{\Gamma_{I_0}(R)}(I, J). \tag{1}$$

From the fact $\Gamma_{I_0}(R)[A]$ is a subgraph of $\Gamma_{I_0}(R)$ it is clear that

$$\mathrm{d}_{\Gamma_{I_0}(R)[A]}(I, J) \geq \mathrm{d}_{\Gamma_{I_0}(R)}(I, J).$$

Now suppose that

$$I = I_1 - I_2 - \cdots - I_{n-1} - I_n = J$$

is a path in $\Gamma_{I_0}(R)$ connecting I and J. In view of Lemma 2.3, if $I_i + I_0 \neq I_{i+1} + I_0$, then they are adjacent for $i = 1, \ldots, n$. Hence

$$\mathrm{d}_{\Gamma_{I_0}(R)[A]}(I, J) \leq \mathrm{d}_{\Gamma_{I_0}(R)}(I, J).$$

because $I_i + I_0 \in A$. So we have the identity (1).

Finally, the fact that $\mathrm{d}_{\Gamma_{I_0}(R)}(I, J) \leq \operatorname{diam}(\Gamma_{I_0}(R))$ gives us $\operatorname{diam}(\Gamma(R/I_0)) \leq \operatorname{diam}(\Gamma_{I_0}(R))$. ∎

Theorem 4.2. *Let R be a ring. Then* $\operatorname{diam}(\Gamma(R/I_0)) = \operatorname{diam}(\Gamma_{I_0}(R))$. *In particular* $\operatorname{diam}(\Gamma_{I_0}(R)) \in \{1, 2, 3, 4, 5\}$.

Proof. We have three cases:

Case 1. If $\operatorname{diam}(\Gamma_{I_0}(R)) \leq 2$, then by Corollary 2.5 I_0 is not a prime ideal, and so $2 \leq \operatorname{diam}(\Gamma(R/I_0))$. Now, by Lemma 4.1 we deduce that $\operatorname{diam}(\Gamma(R/I_0)) = 2$. A similar discussion implies that if $\operatorname{diam}(\Gamma_{I_0}(R)) = 1$, then $\operatorname{diam}(\Gamma(R/I_0)) = 1$. Therefore, in the case that the diameter is at most two, we have the identity $\operatorname{diam}(\Gamma(R/I_0)) = \operatorname{diam}(\Gamma_{I_0}(R))$.

Case 2. $\operatorname{diam}(\Gamma_{I_0}(R)) \geq 3$. We claim that there are ideals I and J such that $I + I_0 \neq J + I_0$. To see this, we note that if $I + I_0 = J + I_0$, then I and J belong to same sets Θ, Σ or Ω were introduced in Remark 3.8. Thus, in view of Lemma 3.9, $\mathrm{d}_{\Gamma_{I_0}(R)}(I, J) \leq 2$. Therefore, $\operatorname{diam}(\Gamma_{I_0}(R)) \geq 3$ implies that there exist ideals I and J such that $I + I_0 \neq J + I_0$. Let

$$I + I_0 - I_1 - \cdots - I_n - J + I_0$$

be a path in $\Gamma_{I_0}(R)$ connecting $I + I_0$ and $J + I_0$. Then, by using Lemma 2.3(iii), (iv), we can see that

$$\mathrm{d}_{\Gamma_{I_0}(R)}(I, J) \leq \mathrm{d}_{\Gamma_{I_0}(R)}(I + I_0, J + I_0). \tag{2}$$

Since $I + I_0, J + I_0 \in A$, by (1), (2) and Theorem 2.4, we have

$$\begin{aligned} \mathrm{d}_{\Gamma_{I_0}(R)}(I, J) \leq \mathrm{d}_{\Gamma_{I_0}(R)[A]}(I + I_0, J + I_0) &\leq \operatorname{diam}(\Gamma_{I_0}(R))[A] \\ &= \operatorname{diam}(\Gamma(R/I_0)). \end{aligned}$$

Hence $\mathrm{diam}(\Gamma_{I_0}(R)) \leq \mathrm{diam}(\Gamma(R/I_0))$. Now the result follows from Lemma 4.1. To see the "*In particular*" statement, we note that [3, Theorem 2.10] asserts $\mathrm{diam}(\Gamma(R/I_0)) \in \{1, 2, 3, 4, 5\}$, so proof is complete. ■

For given ring R, let us $\pi_{I_0}(R)$ and $\pi(R)$ denote the number of non-singular connected components of $\Gamma(R)$ and $\Gamma_{I_0}(R)$, respectively. In the light of [3, Theorem 2.9] we will show that $\pi_{I_0}(R) = \pi(R/I_0)$. Before proceeding further, we need to define our notations and some relevant properties.

Remark 4.3. For a vertex I of $\Gamma_{I_0}(R)$ put $T(I) := \{J \mid I + I_0 = J + I_0\}$ and for a set of vertices C of $\Gamma_{I_0}(R)$ put $T(C) := \bigcup_{I \in C} T(I)$. Then it can easily be seen that we have the following facts:

(i) $T(I) \cap T(I') \neq \emptyset$ if and only if $T(I) = T(I')$.
(ii) $T(I) = T(I + I_0)$.
(iii) Every in $T(I)$ is adjacent to every vertex in $T(J)$, or no edge within $T(I)$ and $T(J)$.
(iv) The subgraph of $\Gamma_{I_0}(R)$ induced by $T(I)$ is either empty or complete.
(v) $T(C \cup C') = T(C) \cup T(C')$.
(vi) C induces a connected subgraph if and only if $T(C)$ induces a connected subgraph.

Theorem 4.4. *Suppose that $\Gamma_{I_0}(R)$ is disconnected with $I_0 \neq 0$. If R/I_0 is a direct product of two fields, then $\pi_{I_0}(R) \in \{1, 2\}$, otherwise $\pi_{I_0}(R) = \pi(R/I_0)$.*

Proof. Let $R/I_0 \cong F_1 \times F_2$, where F_1 and F_2 are fields. Then, there exist maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R (in fact the only maximal ideals) such that $I_0 = \mathfrak{m} \cap \mathfrak{n}$. Clearly, every connected component meets a maximal ideal and, moreover, in view of Theorem 3.3, $\Gamma_{I_0}(R)$ is not empty, so $1 \leq \pi_{I_0}(R) \leq 2$.

As for the "otherwise" statement, due to $\Gamma_{I_0}(R)[A] \cong \Gamma(R/I_0)$ where $A = \{J | I_0 \subset J\}$, we prefer to handle with $\Gamma_{I_0}(R)[A]$.

Here, we are going to stablish a one to one correspondence between connected components of $\Gamma_{I_0}(R)[A]$ and those of $\Gamma_{I_0}(R)$. To attain this, let C be a non-singular connected component of $\Gamma_{I_0}(R)[A]$. Then the correspondence will be derived from the following three steps:

Step 1. We claim that $T(C)$ is a connected component of $\Gamma_{I_0}(R)$. That $T(C)$ induces a connected subgraph of $\Gamma_{I_0}(R)$ is easily obtained from Remark 4.3(vi). But as to the maximality of $T(C)$, assume that for given vertex J, $\{J\} \cup T(C)$ induces a connected subgraph of $\Gamma_{I_0}(R)$; hence by Remark 4.3(iv),(v),(vi), $T(J) \cup T(C) = T(\{J\} \cup C)$ and then $\{J\} \cup T(C)$ induce connected subgraphs of $\Gamma_{I_0}(R)$ and then $\Gamma_{I_0}(R)[A]$, respectively. This implies that $J + I_0 \in C$ and consequently $T(J + I_0) = T(J) \subseteq T(C)$, so we have $J \in T(C)$.

Step 2. We claim that any connected component of $\Gamma_{I_0}(R)$ is in the form of $T(C)$ at which C is a connected component of $\Gamma_{I_0}(R)[A]$. To get this, assume that H is a connected component of $\Gamma_{I_0}(R)$. Easily we can see that

$$T(H) = \bigcup_{I\in H} T(I) = \bigcup_{I\in H} T(I+I_0) \subseteq T(H\cap A). \tag{3}$$

Because $T(H\cap A)\subseteq T(H)$, (3) give us $H = T(H\cap A)$ at which by Remark 4.3, $H\cap A$ induces a connected subgraph of $\Gamma_{I_0}(R)[A]$.

Now, assume that C_1 is connected components of $\Gamma_{I_0}(R)[A]$ such that $H\cap A\subseteq C_1$. Then we can see that $H = T(H) = T(H\cap A)\subseteq T(C_1)$. By Step 1, $T(C_1)$ is a connected component of $\Gamma_{I_0}(R)$, so $H = T(C_1)$, as desired.

Step 3. Let C and C' be connected components of $\Gamma_{I_0}(R)[A]$ in such a way that $T(C) = T(C')$. Then, $T(C\cup C') = T(C) = T(C')$ is a connected component, and so $C\cup C'$ is a connected component of $\Gamma_{I_0}(R)[A]$. This shows that $C = C'$. ∎

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