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Hybrid Pairs of Nonself Multi-Valued Mappings in Metrically Convex Spaces

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Abstract. Using the concept of weakly compatible mappings between set-valued mappings and single valued mappings due to Ćirić and Ume [5], we prove some results on common fixed points for hybrid pairs of family of mappings satisfying a generalized contraction type condition on a complete metrically convex metric spaces which generalize relevant results in [2, 6, 5, 23, 17]. As an application of our main result, we also prove a common fixed point theorem in Banach spaces besides furnishing an illustrative example.

Keywords: Metrically convex metric space; Quasi-coincidentally commuting mappings; Coincidentally idempotent; Occasionally coincidentally idempotent; Weakly compatible mappings.

1. Introduction

In 1972, Assad and Kirk [2] initiated the study of nonself multi-valued mappings and also proved fixed point theorems for such mappings. After this result, several fixed point theorems for such mappings were proved which include relevant results in [1, 6, 5, 14, 19].

On the other hand there exists fixed point theorems for hybrid pairs of self mappings which are presented in [9, 16]. Combining these two ideas, Ahmad and Imdad [4, 3], Imdad and Khan [13, 12, 11, 10], Khan and Imdad [18] and Khan

[17] have recently proved results on fixed and coincidence points of generalized hybrid contractions for compatible mappings, pointwise R-weakly commuting mappings as well as weakly compatible mappings.

The purpose of this paper is to extend and generalize the fixed point theorem due to Ćirić and Ume [5] proved for nonself multi-valued mappings to family of hybrid pairs of weakly compatible mappings which is either partially or completely generalizes the results in [2, 6, 5, 23, 17]. Here for the sake of completeness, we state the main theorem of Ćirić and Ume [5] which runs as follows:

Theorem 1.1. *Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let S,T be mappings of K into $CB(X)$ such that*

$$\begin{aligned} H(Sx,Ty) \leq & \alpha\ d(x,y)+\beta \max\ \{D(x,Sx),D(y,Ty)\} \\ & +\gamma \max\ \{D(x,Sx)+D(y,Ty),D(x,Ty)+D(y,Sx)\} \end{aligned} \tag{1}$$

for all $x,y \in X$ where $\alpha,\beta,\gamma \geq 0$ are such that $\alpha+2\beta+3\gamma+\alpha\gamma<1$.

If $Sx \subseteq K$ and $Tx \subseteq K$ for each $x \in \delta K$(The boundary of K), then there exists a point $u \in K$ such that $u \in Su, u \in Tu$ and $Su = Tu$.

Notice that $D(x,Sx) = \inf\{\ d(x,y) : y \in Sx\}$.

2. Preliminaries

Let (X,d) be a metric space. We denote

(i) $CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$,

(ii) for nonempty subsets A,B of $X, x \in X$, $d(x,A) = \inf\{d(x,a) : a \in A\}$ and $H(A,B) = \max[\{\sup\ d(a,B) : a \in A\},\{\sup\ d(A,b) : b \in B\}]$.

It is well known (cf. Kuratowski [20]) that $CB(X)$ is a metric space with the distance function H which is known as Hausdorff-Pompeiu metric on X.

Definition 2.1. [22, Definition] *Two self mappings F and T of a metric space (X,d) are T-weak compatible iff the following limits exist and satisfy:*

(i) $\lim_{n\to\infty} d(TFx_n,FTx_n) \leq \lim_{n\to\infty} d(FTx_n,Fx_n)$, *and*

(ii) $\lim_{n\to\infty} d(TFx_n,Tx_n) \leq \lim_{n\to\infty} d(FTx_n,Fx_n)$.

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to t, Fx_n \to t$ for some $t \in X$.

Definition 2.2. [22, Definition] *Mappings $F : X \to CB(X)$ and $T : X \to X$ are T-weak compatible if $TFx \in CB(X)$ for all $x \in X$ and the following limits exist and satisfy:*

(i) $\lim_{n\to\infty} H(TFx_n, FTx_n) \leq \lim_{n\to\infty} H(FTx_n, Fx_n)$, *and*
(ii) $\lim_{n\to\infty} H(TFx_n, Tx_n) \leq \lim_{n\to\infty} H(FTx_n, Fx_n)$,

whenever $\{x_n\}$ *is a sequence in* X *such that* $Tx_n \to t \in A$ *and* $Fx_n \to A \in CB(X)$.

Motivated by the definition of weak compatability in [22, 7] adopted the same to nonself setting.

Definition 2.3. [7, Definition] *Let* K *be a nonempty subset of a metric space* (X, d). *Mappings* $F : K \to CB(X)$ *and* $T : K \to X$ *are said to be weakly compatible on* K *if for any sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* K *such that* $Tx_n \in K, F(x_n) \cap K \neq \phi$, *the following limits exist and satisfy:*

(i) $\limsup_{n\to\infty} d(Ty_n, FTx_n) \leq \limsup_{n\to\infty} H(FTx_n, Fx_n)$, *and*
(ii) $\limsup_{n\to\infty} d(Ty_n, Tx_n) \leq \limsup_{n\to\infty} H(FTx_n, Fx_n)$,

whenever $y_n \in F(x_n) \cap K$ *and* $\lim_{n\to\infty} d(y_n, Tx_n) = 0$.

Notice that if $F : X \to X$ and $T : X \to X$ are self mappings of X and lim sup is replaced by lim, then this definition reduces to [22, Definition 2.2].

Definition 2.4. [13, Definition] *Let* K *be a nonempty subset of a metric space* (X, d), $T : K \to X$ *and* $F : K \to CB(X)$. *The pair* (F, T) *is said to be quasi-coincidentally commuting if for all coincidence points '*x*' of* (F, T), $TFx \subset FTx$ *whenever* $Fx \subset K$ *and* $Tx \in K$ *for all* $x \in K$.

Definition 2.5. [13, Definition] *A mapping* $T : K \to X$ *is said to be coincidentally idempotent w.r.t mapping* $F : K \to CB(X)$, *if* T *is idempotent at the coincidence points of the pair* (F, T).

Definition 2.6. [18, Definition] *A mapping* $T : K \to X$ *is said to be occasionally coincidentally idempotent w.r.t mapping* $F : K \to CB(X)$, *if there exists a point* $z \in K$ *such that* T *is idempotent at the coincidence points of the pair* (F, T).

Definition 2.7. [2, Definition] *A metric space* (X, d) *is said to be metrically convex if for any* $x, y \in X$ *with* $x \neq y$ *there exists a point* $z \in X, x \neq z \neq y$ *such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.8. [2, Lemma 2.8] *Let* K *be a nonempty closed subset of a metrically convex metric space* (X, d). *If* $x \in K$ *and* $y \notin K$ *then there exists a point* $z \in \delta K$ *(the boundary of* K*) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.9. [6, Lemma 2.9] *Let $A, B \in CB(X)$ and $a \in A$. Then for any positive number $q < 1$ there exists $b = b(a)$ in B such that $q\ d(a, b) \leq H(A, B)$.*

3. Main Results

Theorem 3.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying the conditions:*

(i) $\delta K \subseteq SK \cap TK, F_i(K) \cap K \subseteq SK, F_j(K) \cap K \subseteq TK,$

(ii) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K,$ *and*

$$\begin{aligned} &H(F_i(x), F_j(y)) \\ \leq &\alpha\ d(Tx, Sy) + \beta \max\{d(Tx, F_i(x)), d(Sy, F_j(y))\} \\ &+ \gamma \max\{d(Tx, F_i(x)) + d(Sy, F_j(y)), d(Tx, F_j(y)) \\ &+ d(Sy, F_i(x))\} \end{aligned} \tag{2}$$

where $i = 2n-1, j = 2n,\ (n \in N), i \neq j$ for all $x, y \in X$ with $x \neq y$, where $\alpha, \beta, \gamma \geq 0,\ q < 1$ such that $\alpha + 2\beta + 3\gamma + \alpha\gamma < q < 1$,

(iii) $(F_i(x), T)$ *and* $(F_j(y), S)$ *are weakly compatible pairs,*

(iv) T *and* S *are continuous on* K.

Then there exists a point $z \in K$ such that $z = Tz = Sz \in F_i(z) \cap F_j(z)$.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x \in \delta K$. Then due to $\delta K \subseteq TK$ there exists a point $x_0 \in K$ such that $x = Tx_0$. From $Tx \in \delta K \Rightarrow F_i(x) \subseteq K$, one concludes that $F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$ such that

$$q\ d(y_1, y_2) \leq H(F_1(x_0), F_2(x_1)).$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K) \cap K \subseteq TK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \delta K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ and so

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 \in F_3(x_2)$ be such that $q\ d(y_2, y_3) \leq H(F_2(x_1), F_3(x_2))$.

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{2n} \in F_{2n}(x_{2n-1})$, for all $n \in N$, $y_{2n+1} \in F_{2n+1}(x_{2n})$ for all $n \in N_0 = N \cup \{0\}$,

(b) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$, and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$$

(c) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$, and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

(d) $q\ d(y_{2n-1}, y_{2n}) \leq H(F_{2n-1}(x_{2n-2}), F_{2n}(x_{2n-1}))$ and $q\ d(y_{2n}, y_{2n+1}) \leq H(F_{2n}(x_{2n-1}), F_{2n+1}(x_{2n}))$.

We denote

$$\begin{aligned} P_\circ &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, \\ P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}, \\ Q_\circ &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}, \\ Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}. \end{aligned}$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$. Now we distinguish the following three cases:
Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned} & q\ d(Tx_{2n}, Sx_{2n+1}) \\ = {} & q\ d(y_{2n}, y_{2n+1}) \leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) \\ \leq {} & \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} \\ & + \gamma\ \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})) + d(Sx_{2n-1}, F_{2n}(x_{2n-1})), \\ & d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\} \\ \leq {} & \alpha\ d(y_{2n-1}, y_{2n}) + \beta\ \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ & +\gamma\ \max\{d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n+1})\} \\ \leq {} & \alpha\ d(y_{2n-1}, y_{2n}) + \beta\ \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ & +\gamma\ \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, \end{aligned}$$

$$\begin{aligned} & q\ d(Tx_{2n}, Sx_{2n+1}) \\ \leq {} & (\alpha + \gamma)\ d(y_{2n-1}, y_{2n}) + \beta\ \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ & +\gamma\ d(y_{2n}, y_{2n+1}). \end{aligned} \tag{3}$$

If we suppose that $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$ then we obtain

$$q\ d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + 2\gamma)\ d(y_{2n}, y_{2n+1})$$

which is a contradiction. Therefore from (3) we obtain

$$q\ d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + \gamma)\ d(y_{2n}, y_{2n-1}) + \gamma\ d(y_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)\; d(Sx_{2n-1}, Tx_{2n}). \tag{4}$$

Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_\circ \times P_\circ$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)\; d(Sx_{2n-1}, Tx_{2n-2}). \tag{5}$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_1$ then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}),$$

which in turn yields $d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$, and hence

$$q\; d(Tx_{2n}, Sx_{2n+1}) \leq q\; d(y_{2n}, y_{2n+1}) \leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})).$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)\; d(Sx_{2n-1}, Tx_{2n}).$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_\circ$ then as earlier, we also obtain

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)\; d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_\circ$ then $Sx_{2n-1} = y_{2n-1}$. As in Case 1, we get

$$\begin{aligned}
&q\; d(Tx_{2n}, Sx_{2n+1})\\
&= q\; d(Tx_{2n}, y_{2n+1}) \leq q\; \{d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1})\}\\
&\leq q\; d(Tx_{2n}, y_{2n}) + q\; d(y_{2n}, y_{2n+1})\\
&\leq q\; d(Tx_{2n}, y_{2n}) + H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1}))\\
&\leq q d(Tx_{2n}, y_{2n}) + \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}\\
&\quad + \gamma \max\{d(Tx_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1})\}.
\end{aligned}$$

Since $\alpha < q$ and $d(Tx_{2n}, y_{2n}) + d(Tx_{2n}, Sx_{2n-1}) = d(Sx_{2n-1}, y_{2n})$ we obtain

$$q\; d(Tx_{2n}, y_{2n}) + \alpha\; d(Tx_{2n}, Sx_{2n-1}) \leq q\; d(Sx_{2n-1}, y_{2n}).$$

Also, by the triangle inequality, we obtain

$$\begin{aligned}
&d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1})\\
&\leq d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1})\\
&\leq d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, Sx_{2n+1}).
\end{aligned}$$

Therefore

$$\begin{aligned} & q\ d(Tx_{2n}, Sx_{2n+1}) \\ \le\ & q\ d(Sx_{2n-1}, y_{2n}) + \beta\ \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ & +\gamma\ \{d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, y_{2n+1})\} \end{aligned}$$

If $d(Tx_{2n}, y_{2n+1}) \ge d(y_{2n-1}, y_{2n})$ then we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \le \left(\frac{q+\gamma}{q-\beta-\gamma}\right)\ d(Sx_{2n-1}, y_{2n}).$$

Otherwise, if $d(Tx_{2n}, y_{2n+1}) \le d(y_{2n-1}, y_{2n})$ then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\le \left(\frac{q+\beta+\gamma}{q-\gamma}\right)\ d(Sx_{2n-1}, y_{2n}) \\ &\le \left(\frac{q+\gamma}{q-\beta-\gamma}\right)\ d(Sx_{2n-1}, y_{2n}). \end{aligned}$$

Now, proceeding as earlier, we also obtain

$$d(Sx_{2n-1}, y_{2n}) \le \left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)\ d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le kd(Sx_{2n-1}, Tx_{2n-2}),$$

where $k = \left(\frac{q+\gamma}{q-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{q-\gamma}\right)$.

Thus in all the cases, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le k \max\ \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}, \tag{6}$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+2}) \le k \max\ \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}. \tag{7}$$

Now on the lines of Assad and Kirk [3], it can be shown by induction that for $n \ge 1$, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le k^n\delta \quad \text{and} \quad d(Sx_{2n+1}, Tx_{2n+2}) \le k^{n+\frac{1}{2}}\delta,$$

whereas

$$\delta = k^{\frac{-1}{2}} \max\ \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, \ldots\}$ is Cauchy and converges to some point z. It follows that $\{Tx_{2n}\} \to z$ as $n \to \infty$ that is $y_{2n} \to z$ as $n \to \infty$. For each y_{2n} denoted by Y_{2n} one of the subsets $\{F_{2n}(x_{2n-1})\}$ which contains y_{2n} and also y_{2n+1} denoted by Y_{2n+1} one of the subsets $\{F_{2n+1}(x_{2n})\}$ which contains y_{2n+1}. Then $H(Y_{2n}, Y_{2n+1}) \leq (\frac{\alpha+\beta+\gamma}{q-\gamma})d(Sx_{2n-1}, Tx_{2n})$ (See (4)), or

$$H(Y_{2n+1}, Y_{2n+2}) \leq k \ \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\} \quad \text{(by (7))} \quad (8)$$

Since the sequence is Cauchy, it follows that $\{Y_n\}$ is Cauchy in the complete metric space $(CB(X), H)$. Thus $\lim_{n\to\infty} Y_n = Y$ for some $Y \in CB(X)$. Now we have

$$D(z, Y) \leq d(z, y_n) + H(Y_n, Y) \to 0 \quad \text{as} \quad n \to \infty$$

and hence $z \in Y$. By the construction of the sequence there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively. Consequently the subsequence $\{Sx_{2n_k+1}\}$ which is contained in $Q_\circ$ for each $k \in N$, converges to z. Since $Sx_{2n+1} = y_{2n+1}$, $\{SSx_{2n_k+1}\}$ and $\{F_{2n}(Sx_{2n_k-1})\}$ are well defined. Set

$$L_k = d(Sx_{2n_k+1}, F_{2n}(Sx_{2n_k-1})) \text{ and } R_k = H(F_{2n}(x_{2n_k-1}), F_{2n}(Sx_{2n_k-1})).$$

Therefore

$$\begin{aligned}
R_k \leq\ & H(Y_{2n_k}, Y_{2n_k+1}) + H(F_{2n+1}(x_{2n_k}), F_{2n}(Sx_{2n_k-1})) \\
\leq\ & H(Y_{2n_k}, Y_{2n_k+1}) + \alpha d(Tx_{2n_k}, SSx_{2n_k-1}) + \beta \max\{d(Tx_{2n_k}, F_{2n+1}(x_{2n_k})), \\
& d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1}))\} + \gamma \ \max\{d(Tx_{2n_k}, F_{2n+1}(x_{2n_k})) \\
& + d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1})), \\
& d(Tx_{2n_k}, F_{2n}(Sx_{2n_k-1})) + d(F_{2n+1}(x_{2n_k}), SSx_{2n_k-1})\} \\
\leq\ & H(Y_{2n_k}, Y_{2n_k+1}) + \alpha \ d(y_{2n_k}, SSx_{2n_k-1}) + \beta \ \max\{d(y_{2n_k}, y_{2n_k+1}), \\
& d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1}))\} + \gamma \ \max\{d(y_{2n_k}, y_{2n_k+1}) \\
& + d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1})), \\
& d(y_{2n_k}, F_{2n}(Sx_{2n_k-1})) + d(y_{2n_k+1}, SSx_{2n_k-1})\} \\
\leq\ & H(Y_{2n_k}, Y_{2n_k+1}) + \alpha \ [d(F_{2n}(Sx_{2n_k-1}), SSx_{2n_k-1}) + R_k] \\
& + \beta \ [d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1})) + R_k] \\
& + \gamma \ [d(Sx_{2n_k+1}, SSx_{2n_k-1}) + R_k].
\end{aligned}$$

Hence

$$\begin{aligned}
R_k \leq & H(Y_{2n_k}, Y_{2n_k+1}) + (\alpha + \beta) \ [d(SSx_{2n_k-1}, F_{2n}(Sx_{2n_k-1})) + R_k] \\
& + \gamma \ [d(Sx_{2n_k+1}, SSx_{2n_k-1}) + R_k]
\end{aligned} \quad (9)$$

Since $Y_n \to Y$, $Sx_{2n_k+1} \to z$ and as S is continuous, it follows that the real sequence $\{R_k\}$ is bounded. Thus $\limsup_{n\to\infty} R_k$ exists. Since F_{2n} and S are weakly compatible and $Sx_{2n_k+1} = y_{2n_k+1} \in K, y_{2n_k+1} \in F_{2n}(K) \cap K$ and $\lim_{n\to\infty} d(Sx_{2n_k+1}, y_{2n_k}) = 0$. Using Definition 2.3, one gets

$$\limsup_{n\to\infty} L_k \leq \limsup_{n\to\infty} R_k, \quad (10)$$

$$\limsup_{n\to\infty} d(SSx_{2n_k+1}, Sx_{2n_k-1}) \leq \limsup_{n\to\infty} R_k. \quad (11)$$

Denoting $\limsup_{n\to\infty} R_k$ as R and taking the upper limit in (8), by (9) and (10), we get $R \leq (\alpha+\beta+\gamma)R$. Hence $R = 0$. Then from (11), $d(Sz, z) = 0$. Thus $Sz = z$. Similarly, using the foregoing arguments, we obtain $Tz = z$. In order to show that $z \in F_{2n}(z)$, consider

$$\begin{aligned}
&q\ d(y_{2n_k+1}, F_{2n}(z)) \\
\leq\ & H(F_{2n+1}(x_{2n_k}), F_{2n}(z)) \\
\leq\ & \alpha\ d(Tx_{2n_k}, Sz) + \beta\ \max\{d(Tx_{2n_k}, F_{2n+1}(x_{2n_k})), d(Sz, F_{2n}(z))\} \\
&+\gamma\ \max\{d(Tx_{2n_k}, F_{2n+1}(x_{2n_k})) + d(Sz, F_{2n}(z)), d(Tx_{2n_k}, F_{2n}(z)) \\
&+d(Sz, F_{2n+1}(x_{2n_k}))\} \\
\leq\ & \beta\ d(z, F_{2n}(z)) + \gamma\ d(z, F_{2n}(z)),
\end{aligned}$$

implying thereby $z \in F_{2n}(z)$.

Next consider

$$\begin{aligned}
&q\ d(F_{2n+1}(z), z) \\
\leq\ & H(F_{2n+1}(z), F_{2n}(z)) \\
\leq\ & \alpha\ d(Tz, Sz) + \beta\ \max\{d(Tz, F_{2n+1}(z)), d(Sz, F_{2n}(z))\} \\
&+\gamma \max\{d(Tz, F_{2n+1}(z)) + d(Sz, F_{2n}(z)), d(Tz, F_{2n}(z)) + d(Sz, F_{2n+1}(z))\} \\
\leq\ & \beta\ d(z, F_{2n+1}(z)) + \gamma\ d(z, F_{2n+1}(z)),
\end{aligned}$$

implying thereby $z \in F_{2n+1}(z)$. Thus we obtained $z = Tz = Sz \in F_i(z) \cap F_j(z)$, which shows that z is a common coincidence point of $\{F_n\}, S$ and T. This completes the proof. ■

Remark 3.2. Setting $F_i = F_j = F$ for all (i and j), $S = T = I_K$ and $\beta = 0 = \gamma$ in Theorem 3.1, we deduce a theorem due to Assad and Kirk [3].

Remark 3.3. By setting $F_i = F$ for all i, $F_j = G$ for all j and $S = T = I_K$ in Theorem 3.1, we deduce a theorem due to Ćirić and Ume [7].

Remark 3.4. By setting $F_i = F$ for all i, $F_j = G$ for all j and $S = T = I_K$ in Theorem 3.1, we deduce a partially generalized form of a result due to Ćirić and Ume [6].

Remark 3.5. By setting $F_i = F_j = F$ for all (i and j) and $S = T = I_K$ in Theorem 3.1, we deduce a theorem due to Rhoades [23].

Theorem 3.6. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $F : K \to CB(X)$ satisfying the contraction condition:*

$$H(Fx, Fy) \leq \alpha\ d(x, y) + \beta \max\{d(x, Fx), d(y, Fy)\}\ +\ \gamma\ \{d(x, Fy) + d(y, Fx)\}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ such that $(\frac{1+\alpha+\gamma}{1-\beta-\gamma})(\frac{\alpha+\beta+\gamma}{1-\gamma}) < 1$.

If $Fx \subseteq K$ for each $x \in \delta K$, then there exists $z \in K$ such that $z \in Fz$.

Proof. From the condition of Rhoades [23], one can write

$$\begin{aligned} &\left(\frac{1+\alpha+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) \\ =\ &\frac{\alpha+\beta+\gamma+\alpha^2+\alpha\beta+\alpha\gamma+\alpha\gamma+\beta\gamma+\gamma^2}{1-\gamma-\beta+\beta\gamma-\gamma+\gamma^2} \\ =\ &\frac{\alpha+2\beta+3\gamma+\alpha\gamma+\alpha^2+\alpha\beta+\alpha\gamma+\beta\gamma+\gamma^2-\beta-2\gamma}{1+\gamma(\beta+\gamma)-\beta-2\gamma} \\ =\ &\frac{\alpha+2\beta+3\gamma+\alpha\gamma+\alpha(\alpha+\beta+\gamma)+(\beta+\gamma)\gamma-\beta-2\gamma}{1+\gamma(\beta+\gamma)-\beta-2\gamma} \end{aligned}$$

Rhoades condition implies that $\alpha+2\beta+3\gamma+\alpha\gamma+\alpha(\alpha+\beta+\gamma) < 1$. Thus, we can say that Theorem 3.1 is a generalization of the theorem of Rhoades. Notice that condition (2) for $\beta = 0$ and $\gamma = 0$ reduces to $\alpha < 1$. ■

Remark 3.7. Theorem 3.1 is a generalization and extension of Theorem 2.1 due to Ćirić and Ume [7] and generalization of theorem due to Assad and Kirk [3], Assad [4], Khan [21], Itoh [14] and Rhoades [23]. The present technique of proof gives a simplication of the corresponding proofs given by Ćirić and Ume [7], Itoh [14], Khan [21] and Rhoades [23].

In the next theorem we utilize the closedness of TK and SK (or $F_i(K)$ and $F_j(K)$) to relax the continuity requirements besides minimizing the commutativity requirements to merely coincidence points.

Theorem 3.8. *Let (X, d) be a metrically convex metric space and K be a nonempty closed subset of X. Let $F_n : K \to CB(X)$ and $S, T : K \to X$ satisfying (2) and the conditions (i) and (ii) of the Theorem 3.1. Suppose that TK and SK (or $F_i(K)$ and $F_j(K)$) are closed subspaces of X. Then the pair (F_j, S) as well as (F_i, T) has a point of coincidence.*

Moreover, (F_i, T) has a common fixed point if T is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t F_i, whereas (F_j, S) has a common fixed point provided S is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t F_j.

Proof. Proceeding as in Theorem 3.1, we assume that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$. Then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (2) we can write

$$\begin{aligned} &q\ d(F_i(v), Tx_{2n_k}) \\ \leq\ &H(F_i(v), F_j(x_{2n_k-1})) \end{aligned}$$

$$\leq \alpha\ d(Tv, Sx_{2n_k-1}) + \beta\ \max\{d(Sx_{2n_k-1}, F_j(x_{2n_k-1})), d(Tv, F_i(v))\}$$
$$+ \gamma\ \max\{d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) + d(Tv, F_i(v)),$$
$$d(Tv, F_j(x_{2n_k-1})) + d(Sx_{2n_k-1}, F_i(v))\},$$

which on letting $k \to \infty$, reduces to

$$q\ d(F_i(v), u) \leq \beta\ \max\{d(u, F_i(v)), 0\} + \gamma\ \max\{d(F_i(v), u), d(F_i(v), u)\}$$
$$\leq (\beta + \gamma)\ d(u, F_i(v)),$$

yielding thereby $u \in F_i(v)$, which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n_k}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (2) we get

$$q\ d(Sw, F_j(w))$$
$$= q\ d(Tv, F_j(w)) \leq H(F_i(v), F_j(w))$$
$$\leq \alpha\ d(Tv, Sw) + \beta\ \max\{d(Tv, F_i(v)), d(Sw, F_j(w))\}$$
$$+ \gamma\ \max\{d(Tv, F_i(v)) + d(Sw, F_j(w)), d(Tv, F_j(w)) + d(Sw, F_i(v))\}$$
$$\leq (\alpha + \beta + \gamma)\ d(Sw, F_j(w)),$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

In case $F_i(K)$ and $F_j(K)$ are closed subspaces, then $u \in F_i(K) \cap K \subseteq SK$ or $F_j(K) \cap K \subseteq TK$. The analogous arguments establish the desired conclusions. If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK are closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tz \in F_i(z)$ and $Sw \in F_j(w)$.

Since v is a coincidence point of (F_i, T) therefore using quasi-coincidentally commuting property of (F_i, T) and occasionally coincidentally idempotent property of T w.r.t F_i we have

$$Tv \in F_i(v) \text{ and } u = Tv \Rightarrow Tu = TTv = Tv = u.$$

Therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$, which shows that u is the common fixed point of (F_i, T). Similarly using the quasi-coincidentally commuting property of (F_j, S) and occasionally coincidentally idempotent property of S w.r.t F_j we can show that (F_j, S) has a common fixed point as well. This completes the proof. ■

Finally, we prove a theorem when closedness of K is replaced by compactness of K.

Theorem 3.9. *Let (X, d) be a complete metrically convex metric space and K be a nonempty compact subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $T : K \to X$ satisfying:*

(i) $\delta K \subseteq TK$, $(F_i(K) \cup F_j(K)) \cap K \subseteq TK$,

(ii) $Tx \in \delta K \Rightarrow F_i(x) \cup F_j(x) \subseteq K$, *with*

$$H(F_i(x), F_j(y)) < M(x,y), \text{ when } M(x,y) > 0, \text{ for all } x, y \in K,$$

where

$$\begin{aligned} M(x,y) =& \alpha\ d(Tx,Ty) + \beta\ \max\{d(Tx,F_i(x)), d(Ty,F_j(y))\} \\ &+\ \gamma\ \max\{d(Tx,F_i(x)) + d(Ty,F_j(y)), d(Tx,F_j(y)) \\ &+ d(Ty,F_i(x))\} \end{aligned} \tag{12}$$

for all $x, y \in X$ with $x \neq y$, where α, β and γ are non-negative reals with $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq q \leq 1$. If T is weakly compatible with F_n then F_n and T have a common coincidence. Moreover F_n and T have a common fixed point provided Tz remains fixed under T.

Proof. We assert that $M(x,y) = 0$ for some $x, y \in K$. Otherwise $M(x,y) \neq 0$, for any $x, y \in K$ implies that

$$f(x,y) = \frac{H(F_i(x), F_j(y))}{M(x,y)}$$

is continuous and satisfies $f(x,y) < 1$ for all $(x,y) \in K \times K$. Since $K \times K$ is compact, there exists $(u,v) \in K \times K$ such that $f(x,y) \leq f(u,v) = c < 1$ for $x, y \in K$, which in turn yields $H(F_i(x), F_j(y)) \leq cM(x,y)$ for $x, y \in K$ and $0 < c < 1$. Therefore using (12) we obtain

$$\max\left\{\frac{\alpha+\beta+\gamma}{q-\gamma}, \frac{\alpha+\gamma}{q-\beta-\gamma}\right\} < 1.$$

Now by Theorem 3.1 (with restrictionS = T,) we get $Tz \in F_i(z) \cap F_j(z)$ for some $z \in K$ and one concludes $M(z,z) = 0$, contradicting the facts that $M(x,y) > 0$. Therefore $M(x,y) = 0$ for some $x, y \in K$ which implies $Tx \in F_i(x)$ and $Tx = Ty \in F_j(y)$. If $M(x,x) = 0$ then $Tx \in F_j(x)$ and if $M(x,x) \neq 0$ then using (12) we infer that $d(Tx, F_j(x)) \leq 0$ yielding thereby $Tx \in F_j(x)$. Similarly in either of the cases $M(y,y) = 0$ or $M(y,y) > 0$ we conclude that $Ty \in F_i(y)$. Thus we have shown that F_n and T have a common point of coincidence. ∎

By setting $T = I_K$ in Theorem 3.9, we deduce the following corollary for a family of maps.

Corollary 3.10. *Let (X,d) be a complete metrically convex metric space and K be a nonempty compact subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ satisfying:*

$$\begin{aligned} x \in \delta K \ &\Rightarrow F_i(x), F_j(x) \subset K, \\ H(F_i(x), F_j(y)) \ &< \ \alpha\ d(x,y) + \beta\ \max\{d(x,F_i(x)), d(y,F_j(y))\} \\ &\quad +\gamma \max\{d(x,F_i(x)) + d(y,F_j(y)), d(x,F_j(y)) + d(y,F_i(x))\} \end{aligned}$$

for all $x, y \in X$ *with* $x \neq y$, *where* α, β *and* γ *are non-negative reals with* $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq q \leq 1$. *Then* $z \in F_i(z)$ *and* $z \in F_j(z)$.

Remark 3.11. Similar corollaries can also be derived for a family of maps by setting $F_i = F_j$ and $T = I_K$ in Theorem 3.9.

4. An Illustrative Example

Now, we furnish an example which demonstrates the validity of the hypotheses of Theorem 3.1 besides establishing the genuineness of our extension over other relevant results of the existing literature.

Example 4.1. Let $X = \Re$ be the set of reals equipped with natural distance and $K = \{\frac{-1}{3}\} \cup [0, 1]$. Define $F_n : K \to CB(X)$ and $S, T : K \to X$ by

$$F_i(x) = \begin{cases} [\frac{-x}{2}, 0] & \text{if } 0 \leq x < 1, \\ \{0\} & \text{if } x \in \{\frac{-1}{3}, 1\}, \end{cases} \qquad Tx = \begin{cases} \frac{-x}{3} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = \{\frac{-1}{3}, 1\}, \end{cases}$$

$$F_j(x) = \begin{cases} [\frac{-x}{4}, 0] & \text{if } 0 \leq x < 1, \\ \{0\} & \text{if } x \in \{\frac{-1}{3}, 1\} \end{cases} \qquad Sx = \begin{cases} \frac{-x}{4} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = \{\frac{-1}{3}, 1\}, \end{cases}$$

where $i = 2n - 1$ and $j = 2n$. Since δK(the boundary of K) $= \{\frac{-1}{3}, 0, 1\}$. Clearly $\delta K \subset TK \cap SK$.

Further, $F_i(K) \cap K = (\frac{-1}{2}, 0] \cap K \subset TK$ and $F_j(K) \cap K = (\frac{-1}{4}, 0] \cap K \subset SK$. Also

$$\begin{aligned} T\left(\frac{-1}{3}\right) &= 1 \in \delta K \Rightarrow F_i\left(\frac{-1}{3}\right) = \{0\} \subseteq K, \\ T(0) &= 0 \in \delta K \Rightarrow F_i(0) = \{0\} \subseteq K, \\ T(1) &= 1 \in \delta K \Rightarrow F_i(1) = \{0\} \subseteq K, \\ S\left(\frac{-1}{3}\right) &= 1 \in \delta K \Rightarrow F_j\left(\frac{-1}{3}\right) = \{0\} \subseteq K, \\ S(0) &= 0 \in \delta K \Rightarrow F_j(0) = \{0\} \subseteq K, \\ S(1) &= 1 \in \delta K \Rightarrow F_j(1) = \{0\} \subseteq K. \end{aligned}$$

Similarly, by a routine calculation we can show that (F_i, F_j) is a generalized (S, T) contraction pair of K into $CB(X)$ with $\alpha = \frac{1}{3}$ and $\beta = \gamma = \frac{1}{7}$. And also '0' is a point of coincidence as $T0 = F_i(0)$ and $S0 = F_j(0)$. Thus F_n, S and T have a point of common coincidence, whereas both the pairs (F_i, T) and (F_j, S) are weakly compatible pairs and the pair (T, S) is continuous. Thus all the conditions of Theorem 3.1 are satisfied and note that '0' is the unique common fixed point of F_n, S and T.

5. Application

Now we state and prove the following theorem in Banach spaces as an application of our main Theorem 3.1 for two pairs of single valued nonself mappings. While proving this result the notion of demiclosed and starshaped subset are utilized.

Definition 5.1. [8, Definition] *Let K be a non empty subset of a normed linear space X. A mapping $T : K \to X$ is said to be demiclosed if $\{x_n\} \subset K, x_n \to x$ and $Tx_n \to y \in X$ then $Tx = y$.*

Definition 5.2. [10, Definition] *Let K be a non empty subset of a normed linear space X. K is said to be starshaped if there exists at least one point $p \in K$ such that for each $x \in K$ and $t \in [0,1], (1-t)p + tx \in K$.*

Definition 5.3. [15, Definition] *A pair (F,T) of nonself mappings defined on a nonempty subset K of a set X is said to be weakly compatible if $Fx = Tx$ for some $x \in K$ with $Fx, Tx \in K \Rightarrow FTx = TFx$.*

Theorem 5.4. *Let K be a nonempty weakly compact starshaped subset of a Banach space X. Let (F,G) a generalized (S,T) nonexpansive mappings of K into X satisfying:*

(i) $\delta K \subseteq SK \cap TK, FK \cap K \subseteq SK, GK \cap K \subseteq TK$,

(ii) $Tx \in \delta K \Rightarrow Fx \in K, Sx \in \delta K \Rightarrow Gx \in K$,

(iii) *(F,T) and (G,S) are weakly compatibles pairs, with*

$$\begin{aligned} &d(Fx, Gy) \\ <&\alpha\ d(Tx, Ty) + \beta\ \max\{d(Tx, Fx), d(Ty, Gy)\} \\ &+ \gamma\ \max\{d(Tx, Fx) + d(Ty, Gy), d(Tx, Gy) + d(Ty, Fx)\} \end{aligned} \tag{13}$$

for all $x, y \in X$ with $x \neq y$, where α, β and γ are non-negative reals with $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq 1$. Moreover, if $(I - F)$ and $(I - G)$ are demiclosed, then the mappings F, G, S and T have a common fixed point $z \in K$ provided S and T are continuous.

Proof. Choose $p \in K$ such that $(1-t)p + tx \in K$ for all $x \in K$ and all $t \in (0,1)$. Take $k_n = 1 - \frac{1}{n}, n = 2, 3, 4, \ldots$ and define $F_n, G_n : K \to X$ by $F_n(x) = (1 - k_n)p + k_n(Fx)$ and $G_n(x) = (1 - k_n)p + k_n(Gx)$ for all $x \in K$. It is easy to verify that (F_n, G_n) is a generalized (S,T) contractive mapping of K into X and F_n, G_n satisfy conditions (i), (ii) and (iii). Since weak topology is Hausdorff and K is weakly compact, we can conclude that K is weakly closed and hence strongly closed. Now by Theorem 3.1 (for single valued setting) for each $n \geq 2, F_n, G_n, S$ and T have a unique common fixed point, say $z_n \in K$. Now it follows that z_n has a weakly convergent subsequence and we can assume that z_n itself converges to $z \in K$ weakly. Since weakly convergent sequences are norm bounded, therefore we can find a constant $M > 0$ such that $||z_n|| < M$ for all $n \geq 2$. Now for every $n \geq 2$, we can have

$$\begin{aligned} ||(I-F)z_n|| &= ||z_n - [k_n^{-1}\{Fz_n - (1-k_n)p\}]|| \\ &= ||z_n - [k_n^{-1}Fz_n - k_n^{-1}(1-k_n)p]|| \\ &= ||(1-k_n^{-1})z_n - (1-k_n^{-1})p|| \\ &= ||(1-k_n^{-1})(z_n - p)|| \\ &\leq |(1-k_n^{-1})|(||(z_n|| + ||p||) \\ &\leq |(1-k_n^{-1})|(M + ||p||) \end{aligned}$$

since $||z_n|| \leq M$. Since $k_n^{-1} \to 1$ as $n \to \infty$, we have $(I-F)z_n \to 0 \in K$. Also $z_n \to z \in K$ and $(I-F)$ is demiclosed, it follows that $(I-F)z = 0$ giving thereby $Fz = z$. Similarly using the demiclosedness of $(I-G)$ we can show that $Gz = z$. Since for each $n \geq 2, Tz_n = z_n$ and $Sz_n = z_n$, therefore taking the limit as $n \to \infty$, we obtain $Tz = Sz = z$ as T and S are continuous. This completes the proof. ■

By setting $S = I_K$ in Theorem 5.1 we deduce the following corollary for three maps.

Corollary 5.5. *Let K be a nonempty weakly compact starshaped subset of Banach space X. Let (F, G) be a generalized T non-expansive mappings of K into X satisfying:*

(i) $Tx \in \delta K \Rightarrow Fx, Gx \in K$,

(ii) $\delta K \subseteq TK,\ (FK \cup GK) \cap K \subseteq TK$

(iii) *F and T are weakly compatible pairs.*

Moreover, if $(I-F)$ is demiclosed, then the mappings F, G and T have a common fixed point $z \in K$ provided T is continuous.

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