Southeast Asian Bulletin of Mathematics (2022) 46: 591–600

Southeast Asian Bulletin of Mathematics

On Representation of Functionals on Ultraproduct Banach Spaces and its Applications to Super Weakly Compact Sets*

L.X. Cheng†, Q.J. Cheng, and W.Y. He
School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China
Email: lxcheng@xmu.edu.cn; qjcheng@xmu.edu.cn; wuyihe@stu.xmu.edu.cn

J.J. Wang
School of Applied Mathematics, Xiamen University of Technology, Xiamen, 361021, China
Email: 1057383151@qq.com

Received 17 September 2021
Accepted 23 December 2021

Communicated by Ngai-Ching Wong

Dedicated to the memory of Professor Ky Fan (1914–2010)

AMS Mathematics Subject Classification(2020): 46A20; 46B50; 46A55, 46B03

Abstract. In this paper, we first present a brief review for the study of super weakly compact sets of Banach spaces. Then we show a representation theorem of the dual of the ultraproduct space $(X_i)_{\mathfrak{U}}$ for a family $(X_i)_{i\in\Omega}$ of Banach spaces X_i and a free ultrafilter $\mathfrak{U}$ on Ω. As its applications, we show that for any family of subsets $A_i \subset X_i, i \in \Omega$, we have

$$\overline{\mathrm{co}}\Big((A_i)_{\mathfrak{U}}\Big) = \overline{(\mathrm{co}A_i)_{\mathfrak{U}}}.$$

Consequently, we give two basic questions related to topologies of subsets in Banach spaces affirmative answers: The closed convex hull of a relative super weakly compact subset of a Banach space is again super weakly compact; and every super weakly compact set is a uniformly Eberlein-compact set.

*supported in partial by the Natural Science Foundation of China, grant No. 11731010.
†Corresponding author.

Keywords: Super weakly compact set; Representation of functional; Ultraproduct; Uniformly Eberlein compact set; Banach space.

1. Introduction

It is well-known that the notion of compact set is a key concept in mathematics, especially, in topology and analysis, which guarantees that every continuous scalar-valued function attains a minimum and maximum on a Hausdorff compact space, and every closed bounded subset of a finite dimensional topological linear space is compact. Nevertheless, the latter is not true in an infinite dimensional topological linear space, even if in a Banach space. Therefore, the notion of weakly compact set is one of appropriate substitute for that of compact set. Now, the theory of weak compactness has been widely studied and formed a complete system in functional analysis and in linear topology. On the other hand, the notion of weakly compact set can be understood as a generalization and localization of the notion of "reflexive Banach space", because a Banach space is reflexive if and only if every bounded subset of the space is relatively weakly compact. We are also known that the class of super reflexive Banach spaces is one of most important classes of Banach spaces, which was introduced by C.R. James in 1972 [26]: A Banach space X is said to be super reflexive provided that every Banach space Y is reflexive if it is finitely representable in X, i.e. for every finite dimensional subspace Y_0 of Y, and for all $\varepsilon > 0$, there exists a finite dimensional subspace X_0 of X and a linear isomorphism T from Y_0 to X_0 such that $\|T\|\|T^{-1}\| < 1+\varepsilon$. This class is a natural extension of the classes of Hilbert spaces and L_p-spaces, and every such a space admits many nice properties. For example, James [26] showed that a Banach space is super reflexive if and only if its unit ball does not contain arbitrarily finite ε-trees for for all $\varepsilon > 0$. In the same year, Enflo [17], applying James' characterization, further showed the deep renorming theorem that every super reflexive Banach space can be renormed to be uniformly convex. Later in 1975, Pisier [37] applying deep martingale technique proved that every super reflexive Banach space admits an equivalent norm with uniform convexity of power type.

After more than half a century development, the research achievements related to super reflexive spaces in both theory and application have formed a rich system. However, comparing with reflexive Banach space theory and its localization, the theory of super reflexive spaces is far from complete, especially, there is not a localized notion of super reflexive spaces. Indeed, in many cases of application, we need only a subset of a Banach space admitting some nice properties similar to those of a subset in a super reflexive space, rather than that of the whole space. For example, in the past two decades, with the development of coarse geometry, non-commutative geometry and non-commutative group theory, it has been found that super-reflexive spaces are closely related to some important problems in the field of coarse geometry such as coarse Novikov's conjecture (see [48, 30, 32, 36, 49]), and many other new questions. In 2006, Kasparov and Yu [30] proved that the coarse Novikov conjecture holds true if the bounded geometry space in question can be coarsely embedded into a super-reflexive space. But then Lafforgue, Mendel and Naor discovered that there is a class of bounded geometry spaces (expanded graphs) that cannot be coarsely embedded into any super-reflexive space [32, 36]. Therefore, in many cases the assumption of super-reflexivity on the whole spaces in question is neither necessary nor possible. Unfortunately, this concept had not been proposed for more than 30 years after 1970s. By our understanding, the main difficulty is that there was not an appropriate localized notion of the "finite representability" which is a core concept to define super reflexive Banach spaces.

Motivated by the facts mentioned above, the first two named authors of this paper started to search for appropriate localization of super reflexive spaces in 2004. In 2007, they (coauthored with W. Zhang) [14, 12] substituted "simplexes" for "finite dimensional spaces" to define finite representability between two general sets: A subset A of a Banach space X is said to be finitely representable in a subset B of a Banach space Y provided for every possible n-simplex $S_1 \subset X$ with its vertices in A, and for all $\varepsilon > 0$, there exists an n-simplex $S_2 \subset Y$ with its vertices in B and an affine isomorphism T from S_1 onto S_2 so that

$$(1-\varepsilon)\|x-y\| \leq \|Tx-Ty\| \leq (1+\varepsilon)\|x-y\|, \ \forall x, y \in S_1.$$

Thus, the localized notion "super weakly compact sets" of super reflexive Banach spaces was introduced naturally: We say that a subset A of a Banach space X is relatively super weakly compact provided for every Banach space Y, every subset B of Y which is finitely representable in A is relatively weakly compact. They showed a series of nice properties of super weakly compact sets paralleling to bounded convex sets in super reflexive spaces, especially, proved a localized version of Enflo's deep renorming theorem: A closed bounded convex set of a Banach space is super weakly compact if and only if it can be renormed to be uniformly convex [9]. In 2018, they [8] further showed a series of characterizations and nice properties of super weakly compacts, such as, the super Banach-Saks property of a bounded subset is equivalent to its relatively super weak compactness, super weak compactness is invariant under ultrafilter powers and Grothendieck's theorem holds for super weakly compact sets.

After consulting the literature when the article [9] was published in 2010, we found that seeking for localization of super-reflexivity is painstaking process exploration in history. Indeed, this process can be traced back to the 1970s. In 1976, Beauzamy [3] extended the super reflexivity of Banach spaces to linear operators between two Banach spaces. He introduced the notion of "uniform convex operators" (which is different from the present notion of super weak compactness). He established an Enflo's theorem of operator version. For more generalized concepts in this direction, we refer the reader to [24, 43, 6, 7]. Motived by Lancien [33], Raja [40] introduced the notion of finite index property of a bounded closed convex set in Banach spaces, and showed that such a set can always be given a "uniformly convex norm" on it. In 2009, Fabian, Montesinos, Zizler [21] localized Lancien's dual index property, called the finite dual index property to the general bounded subset of a Banach space, and proved that such a set can be given some kind of "consistent" Gâteaux differentiability norm. In 2018, the first two named author of this paper coauthored with other ones [8] further studied basic properties of super weakly compact sets. What is particularly interesting is that they discovered that both Raja's finite index property and Fabian-Montesinos-Zizler's finite duality index property of bounded convex sets are all equivalent to the super weak compactness of convex sets. The concept of super weak compactness has been regarded as a reasonable, unified and appropriate localization of super reflexive Banach spaces (see, for example, [41, 15, 34])) and has been widely studied.

However, there is a basic question in the theory of super weak compactness:

Problem 1.1. Is the closed convex hull of a super weakly compact set again super weakly compact?

This question was first raised in [8]. In [34], the authors announced that the answer to this question are affirmative, and exhibited many interesting applications of this result. But unfortunately, they found soon that there is a gap in their proof. In the second part of this paper, with an entirely different approach, we will show that the closed convex hull of a relatively super weakly compact set is again super weakly compact. The result is actually contained in Wang's doctoral dissertation [44].

In 1968, Amir and Lindenstrauss introduced the concept of Eberlein compactness [1]: A compact Hausdorff space K is said to be Eberlein compact, provided that K is isomorphic to a weakly compact set of a Banach space endowed with its weak topology. They showed that a Hausdorff space K is Eberlein compact if and only if K is isomorphic to a weakly compact set of $c_0(\Gamma)$ for some index set Γ .

In 1976, Benyamini and Starbird [5] further introduced the notion of uniform Eberlein compactness: A compact Hausdorff space K is called uniformly Eberlein compact if it is Eberlein compact and it is isomorphic to a weakly compact set K_1 of $c_0(\Gamma)$ for some index set Γ such that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ satisfying

$$^\sharp\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} < N_\varepsilon,\ \forall k \in K_1,$$

where $^\sharp A$ denotes the cardinality of A. For more information related to uniformly Eberlein compact sets, we refer the reader to [2, 18, 19].

In 2008, Raja [40] showed that every super weakly compact convex set of a Banach space is uniformly Eberlein compact by applying the Davis-Figiel-Johnson-Pelzyński embedding theorem [16] and Fabian-Godefroy-Zizler's characterization [18]. However, we do not know whether a general super weakly compact set is necessarily uniformly Eberlein compact. Therefore, the following question raises naturally.

Problem 1.2. Is a super weakly compact set necessarily uniformly Eberlein compact?

As a consequence of the affirmative answer to Problem 1.1, we obtain that the answer to Problem 1.2 is again positive.

2. A Representation Theorem of Functionals on Ultraproducts and its Applications

In this section, we first give a representation theorem of functional in the dual space $(E_i)^*_{\mathfrak{U}}$ of the $\mathfrak{U}$-ultraproducts of a collection $\{X_i\}_{i\in\Omega}$ of Banach spaces X_i (Theorem 2.2), where $\mathfrak{U}$ is a free ultrafilter on some index set Ω. Then making use of it, we show the following somewhat surprising result (Theorem 2.3) that for any collection $\{A_i\}_{i\in\Omega}$ of subsets $A_i \subset X_i$, which are uniformly bounded in X_i, we have

$$\overline{\text{co}}\Big((A_i)_{\mathfrak{U}}\Big) = \overline{(\text{co}A_i)_{\mathfrak{U}}}.$$

As a consequence, we obtain that the closed convex hull of a super weakly compact set is again super weakly compact, and further, every super weakly compact set is uniformly Eberlein compact.

The following theorem was independently showed by K.D. Kürsten [31] and J. Stern [42], which shows that in the general case the duality between $(E_i)_{\mathfrak{U}}$ and $(E_i^*)_{\mathfrak{U}}$ still holds "locality". This is similar to the principle of local reflexivity (also called the local duality of ultraproducts by S. Heinrich [23, Theorem 7.3]), and it will play a central rule in the construction of the "embedding theorem" of this section.

Theorem 2.1. *(Local duality of ultraproducts) Let $(E_i)_{i\in\Omega}$ be a collection of Banach spaces and $\mathfrak{U}$ be an ultrafilter on an index set Ω. For each pair (M, N) of finite dimensional subspaces $M \subset (E_i)^*_{\mathfrak{U}}$ and $N \subset (E_i)_{\mathfrak{U}}$ and each $\varepsilon > 0$, there exists a linear operator $T : M \to (E_i^*)_{\mathfrak{U}}$ satisfying*

$$(1-\varepsilon)\|x^* - y^*\| \le \|Tx^* - Ty^*\| \le (1+\varepsilon)\|x^* - y^*\|,\ \forall\, x^*, y^* \in M, \quad (1)$$

$$T|_{M\cap(E_i^*)_{\mathcal{U}}} = id_{M\cap(E_i^*)_{\mathfrak{U}}}, \quad (2)$$

$$\langle Tx^*, x\rangle = \langle x^*, x\rangle,\ \forall\, x \in N. \quad (3)$$

Applying the local duality theorem above, S. Heinrich [23, Corollary 7.6]) showed that there is another ultrafilter $\mathcal{V}$ on some index set Γ so that the dual $(E_i)^*_{\mathcal{U}}$ of the ultraproduct $(E_i)_{\mathcal{U}}$ is (linearly) isometric to a 1-complemented subspace of the space $((E_i^*)_{\mathcal{U}})_{\mathcal{V}}$, and the restriction of the isometry to $(E_i^*)_{\mathcal{U}}$ is canonical embedding of the space to its ultrapower. Nevertheless, we require a stronger version, i.e. the following w^*-to-w^* continuously isometric embedding.

Before stating the theorem, we should give some explanation of several symbols which will be used in the sequel. For a Banach space E and an ultrafilter $\mathfrak{U}$ on some index set Ω, $I_{\mathfrak{U}}$ denotes the canonical isometric embedding of E into its ultrapower $(E)_{\mathfrak{U}}$, i.e. $I_{\mathfrak{U}}x = (x_i)_{\mathfrak{U}}$ with $x_i = x \in E$ for all $i \in \Omega$. For distinction, we use $J_{\mathcal{U}}$ to denote the canonical isometric embedding E^* into its ultrapower $(E^*)_{\mathcal{U}}$. For two ultrafilters $\mathcal{U}_j$ on Ω_j $(j = 1, 2)$, $\mathfrak{U}_1\mathfrak{U}_2$ stands for the product ultrafilter on $\Omega_1 \times \Omega_2$ generated by $\mathfrak{U}_1 \times \mathfrak{U}_2$. For a mapping T from a Banach space E to another Banach space F, and a subset $G \subset E$, $T|_G$ denotes the restriction of T to the subset G. If F is a function space consisting of functions defined on D and A is a subset of D, we use $T_{R|A}$ to denote the image of T restricted to A, i.e. $T_{R|A}(x) = T(x)|_A$ for $x \in E$.

Theorem 2.2. *Let $(E_i)_{i\in\Omega}$ be a collection of Banach spaces and $\mathfrak{U}$ be an ultrafilter on an index set Ω. Then we have the following statements:*

(i) *there exists a free ultrafilter $\mathfrak{V}$ on some index set and a linear isometry $T : (E_i)^*_{\mathfrak{U}} \to (E^*_i)_{\mathfrak{U}\mathfrak{V}}$;*
(ii) $T|_{(E^*_i)_{\mathfrak{U}}} = J_{\mathfrak{V}}$;
(iii) $T_{R|I_{\mathfrak{V}}\left((E_i)_{\mathfrak{U}}\right)} = [J_{\mathfrak{V}}]_{R|I_{\mathfrak{V}}\left((E_i)_{\mathfrak{U}}\right)}$ *is again an isometry;*
(iv) $T_{R|I_{\mathfrak{V}}\left((E_i)_{\mathfrak{U}}\right)}^{-1} = J_{\mathfrak{V}}^{-1} = I^*_{\mathfrak{V}}$, *hence*
(v) $T_{R|I_{\mathfrak{V}}\left((E_i)_{\mathfrak{U}}\right)}$ *is w^*-to-w^* continuous.*

Proof. Let $X = (E_i)_{\mathfrak{U}}$, and $Z = (E^*_i)_{\mathfrak{U}}$. Then $Z \subset X^*$ is an exact norming subspace of X, i.e. for each $x \in X$, there is $x^* \in Z$ with $\|x^*\| = 1$ so that $\langle x^*, x\rangle = \|x\|$. Indeed, given $x = (x_i)_{\mathcal{U}}$, there is $(x^*_i)_{\mathcal{U}}$ with $x^*_i \in E^*_i$ satisfying $\|x^*_i\| = 1$ and $\langle x^*_i, x_i\rangle = \|x_i\|$ for all $i \in \Omega$. Therefore,

$$\langle x^*, x\rangle = \lim_{\mathfrak{U}} \langle x^*_i, x_i\rangle = \lim_{\mathfrak{U}} \|x_i\| = \|x\|.$$

This implies that B_Z is $\tau(X^*, X)$-dense in B_{X^*}. Consequently, Z is a $\tau(X^*, X)$-dense subspace of X^*.

Fix any finite dimensional subspaces $M \subset X^*$ and $N \subset X$ with $\dim M = m \geq 1$. By Theorem 2.1, there exists a linear operator $T_{MN} : M \to Z$ satisfying

$$(1 - \frac{1}{m})\|x^* - y^*\| \leq \|T_{MN}x^* - T_{MN}y^*\| \leq (1 + \frac{1}{m})\|x^* - y^*\|, \tag{4}$$

$$T_{MN}|_{M\cap Z} = id_{M\cap Z}, \tag{5}$$

$$\langle T_{MN}x^*, x\rangle = \langle x^*, x\rangle, \tag{6}$$

for all $x^*, y^* \in M, x \in N$. Let Ω_X be the collection of all finite dimensional subspaces of X, and let $\mathfrak{N}$ be a free ultrafilter filter containing all cofinal subsets of Ω_X. Now, we define the following mapping $T_M : M \to (Z)_{\mathfrak{N}}$ by

$$T_M x^* = (T_{MN}x^*)_{\mathfrak{N}}, \ \ \forall\, x^* \in M. \tag{7}$$

It follows from (4)–(6) that T_M is an isometric embedding from M into $(Z)_{\mathfrak{N}}$ satisfying

$$\langle T_M x^*, I_{\mathfrak{N}}x\rangle = \langle x^*, x\rangle, \ \forall\, x^* \in M, x \in X, \tag{8}$$

$$T_M|_{M\cap Z} = I_{\mathfrak{N}}|_{M\cap Z}. \tag{9}$$

Next, let $\widetilde{T_M} : X^* \to (Z)_{\mathfrak{N}}$ be an extension of T_M (not necessarily linear) defined for $x^* \in X^*$ by

$$\widetilde{T_M}x^* = \begin{cases} T_M x^* & \text{if } x^* \in M, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Let Ω_{X^*} be the collection of all finite dimensional subspaces of X^*, $\mathfrak{M}$ be a free ultrafilter containing all cofinal subsets of Ω_{X^*}, and let $\mathfrak{V} = \mathfrak{N}\mathfrak{M}$. Then we define a mapping $T : X^* \to ((Z)_{\mathfrak{N}})_{\mathfrak{M}} = (Z)_{\mathfrak{V}}$ for $x^* \in X^*$ by

$$Tx^* = (\widetilde{T_M}x^*)_{\mathfrak{M}}. \tag{11}$$

Then we can easily get that $T : X^* \to (Z)_{\mathfrak{V}}$ is an isometry. Thus, (i) holds. (ii) and (iii) follow from (5)–(11), and which further imply (v). It remains to show (iv).

Note that $I_{\mathfrak{V}}$ is defined for $x \in X$ by $I_{\mathfrak{V}}x = (x)_{\mathfrak{V}}$. Then $I^*_{\mathfrak{V}} : [I_{\mathfrak{V}}(X)]^* = (X)^*_{\mathfrak{V}}/[I_{\mathfrak{V}}X]^{\perp} \to X^*$ is a w^*-to-w^* continuous isometry satisfying $I^*_{\mathcal{V}}(x^*)_{\mathcal{V}} = x^*$ for each $x^* \in X^*$. In particular, $I^*_{\mathfrak{V}}(x^*)_{\mathfrak{V}} = x^* = J^{-1}_{\mathfrak{V}}[(x^*)_{\mathfrak{V}}]$ for each $x^* \in Z$. This and (ii) entail that for each $x^* \in Z$ we have

$$J^{-1}_{\mathfrak{V}}[(x^*)_{\mathfrak{V}}] = x^* = (T|_Z)^{-1}[(x^*)_{\mathfrak{V}}].$$

Since Z is w^*-dense in X^*, and since both $T_{R|I_{\mathfrak{V}}(X)}$ and $[J_{\mathfrak{V}}]_{R|I_{\mathfrak{V}}(X)}$ are w^*-to-w^* continuous on Z, (iv) follows. ∎

Theorem 2.3. *Let $(E_i)_{i\in\Omega}$ be a collection of Banach spaces and $\mathfrak{U}$ be an ultrafilter on an index set Ω, $(A_i \subset E_i,\ i \in \Omega)$ be a collection of uniformly bounded subsets with $C_i = \overline{\mathrm{co}}(A_i)$, i.e. $\sup_{x_i\in E_i, i\in\Omega} \|x_i\| < \infty$ and let $A = (A_i)_{\mathfrak{U}}$ and $C = (C_i)_{\mathfrak{U}}$. Then for each $x^* \in (E_i)^*_{\mathfrak{U}}$*

$$\sup_{x\in A}\langle x^*, x\rangle = \sup_{x\in C}\langle x^*, x\rangle. \tag{12}$$

Proof. Let the Banach spaces X, Z, the linear isometry T and the free ultrafilter $\mathfrak{V}$ with respect to $(E_i)_{i\in\Omega}$ be defined as the same way in Theorem 2.2. By Theorem 2.2 (i), for each $x^* \in X^*$, there exist $x^*_j = (x^*_{ij})_{\mathfrak{U}} \in Z$ with $i \in \Omega$ so that $Tx^* = (x^*_j)_{\mathfrak{V}}$. For every $x = (x_i)_{\mathfrak{U}} \in X$, by Theorem 2.2 (iii), we have

$$\begin{aligned} \langle x^*, x\rangle &= \langle J_{\mathcal{V}}x^*, I_{\mathcal{V}}x\rangle = \langle T_{R|I_{\mathcal{V}}(X)}x^*, I_{\mathcal{V}}x\rangle = \lim_{\mathcal{V}}\langle x^*_j, x\rangle \\ &= \lim_{j,\mathfrak{V}}\lim_{i,\mathfrak{U}}\langle x^*_{ij}, x_i\rangle = \lim_{i,\mathfrak{U}}\lim_{j,\mathfrak{V}}\langle x^*_{ij}, x_i\rangle. \end{aligned}$$

For every $\varepsilon > 0$, and for each fixed i, let $z_i \in A_i$ so that

$$\lim_{j,\mathfrak{V}}\langle x^*_{ij}, z_i\rangle > \sup_{x_i\in C_i}\lim_{j,\mathfrak{V}}\langle x^*_{ij}, x_i\rangle - \varepsilon. \tag{13}$$

Let $z = (z_i)_{\mathfrak{U}}$. Then $z \in A$ satisfies

$$\begin{aligned}\langle x^*, z\rangle &= \lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{V}} \langle x^*_{ij}, z_i\rangle > \lim_{i,\mathfrak{U}} \{ \sup_{x_i \in C_i} \lim_{j,\mathfrak{V}} \langle x^*_{ij}, x_i\rangle \} - \varepsilon \\ &\geq \sup_{x_i \in C_i} \{ \lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{V}} \langle x^*_{ij}, x_i\rangle \} - \varepsilon = \sup_{x \in C} \langle x^*, x\rangle - \varepsilon.\end{aligned}$$

Therefore, we have shown

$$\sup_{x \in A} \langle x^*, x\rangle > \sup_{x \in C} \langle x^*, x\rangle - \varepsilon.$$

Since ε is arbitrary, our proof is complete. ∎

Now, we use Theorem 2.3 to show the following result.

Theorem 2.4. *The closed convex hull of a relatively super weakly compact set is again super weakly compact.*

Proof. Let A be a nonempty relatively super weakly compact set of a Banach space X, and let C be the closed convex hull of A. Suppose that C is not super weakly compact. Then there is a free ultrafilter $\mathfrak{U}$ on some index set Ω so that the $\mathfrak{U}$-ultrapower $(C)_{\mathfrak{U}}$ is not weakly compact [8]. Since the weak closure of A is again super weakly compact [8], we can assume that itself is super weakly compact. Thus, the $\mathfrak{U}$-ultrapower $(A)_{\mathfrak{U}}$ of A is weakly compact [8]. By James' characterization of weakly compact sets, there exists $x^* \in {(X)_{\mathfrak{U}}}^*$ which does not attain its maximum on $(C)_{\mathfrak{U}}$. On the other hand, there exists $x_0 \in (A)_{\mathfrak{U}}$ such that

$$\sup_{x \in (A)_{\mathfrak{U}}} \langle x^*, x\rangle = \langle x^*, x_0\rangle < \sup_{x \in (C)_{\mathfrak{U}}} \langle x^*, x\rangle.$$

This is clearly a contradiction to Theorem 2.3. ∎

Corollary 2.5. *Every super weakly compact set of a Banach space is uniformly Eberlein compact.*

Proof. Let A be a super weakly compact set of a Banach space X. Then by Theorem 2.4, its closed convex hull $C = \overline{\text{co}}(A)$ is again super weakly compact. By a theorem of Raja [41], C is uniformly Eberlein compact. Consequently, A is uniformly Eberlein compact. ∎

References

[1] D. Amir, J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.* **88** (1968) 35–46.

[2] A. Argyros, On the structure of weakly compact subsets of Hilbert spacs and applications to the geometry of Banach spaces, *Trans. Amer. Math. Soc.* **289** (1985) 409–427.

[3] B. Beauzamy, Opérateurs uniformément convexifiants, *Studia Math.* **57** (1976) 103–139.

[4] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, Amer. Math. Soc. Colloq. Publ. **48**, Amer. Math. Soc. Providence, RI, 2000.

[5] Y. Benyamini, T. Starbird, Embedding weakly compact sets into Hilbert space, *Israel J. Math.* **23** (1976) 137–141.

[6] R.M. Causey, S.J. Dilworth, Metric characterizations of super weakly compact operators, *Studia Math.* **239** (2017) 175–188.

[7] R.M. Causey, S.J. Dilworth, ξ-asymptotically uniformly smooth, ξ-asymptotically uniformly convex, and (β) operators, *J. Funct. Anal.* **274** (2018) 2906–2954.

[8] L. Cheng, Q. Cheng, S. Luo, K. Tu, W. Zhang, On super weak compactness and its equivalences, *J. Convex Anal.* **25** (2018) 899–926.

[9] L. Cheng, Q. Cheng, B. Wang, W. Zhang, On super-weakly compact sets and uniformly convexifiable sets, *Studia. Math.* **199** (2010) 145–169.

[10] L. Cheng, Q. Cheng, J. Wang, A note on absolute uniform retracts, uniform approximation property and super weakly compact sets of Banach spaces, *Acta Math. Sin. (Eng. Ser.)* **37** (2021) 731–739.

[11] L. Cheng, Q. Cheng, J. Zhang, On super fixed point property and super weak compactness of convex subsets in Banach spaces, *J. Math. Anal. Appl.* **428** (2015) 1209–1224.

[12] L. Cheng, Q. Cheng, W. Zhang, On super-weakly compact sets and generalized renormings, www.ims.cuhk.edu.hk/publications/reports/2008-01.pdf.

[13] L. Cheng, Z. Luo, Y. Zhou, On super weakly compact convex sets and representation of the dual of the normed semigroup they generate, *Canad. Math. Bull.* **56** (2013) 272–282.

[14] Q. Cheng, *On Local Embedding to Banach Spaces*, Ph.D. Thesis, Xiamen University, 2007. (in Chinese)

[15] Q. Cheng, On super weakly compact subsets of Banach spaces, *Sci. China Math.* **50** (2020) 1695–1720. (in Chinese)

[16] W.J. Davis, T. Figiel, W.B. Johnson, A. Pelczynski, Factoring weakly compact operators, *J. Funct. Anal.* **17** (1974) 311–327.

[17] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, *Israel J. Math.* **13** (1972) 281–288.

[18] M. Fabian, G. Godefroy, V. Zizler, The structure of uniformly Gateaux smooth Banach spaces, *Israel J. Math.* **124** (2001) 243–252.

[19] M. Fabian, G. Godefroy, P. Hájek, V. Zizler, Hilbert-generated spaces, *J. Funct. Anal.* **200** (2003) 301–323.

[20] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, *Functional Analysis and Infinite Dimensional Geometry*, Canadian Math. Soc. Books in Math., Vol. **8**, Springer-Verlag, New York, 2001.

[21] M. Fabian, V. Montesinos, V. Zizler, Sigma-finite dual dentability indices, *J. Math. Anal. Appl.* **350** (2009) 498–507.

[22] K. Floret, *Weakly Compact Sets*, Lecture Notes in Math., Vol. **801**, Springer, 1980.

[23] S. Heinrich, Ultraproducts in Banach space theory, *J. Reine Angew. Mat.* **313** (1980) 72–104.

[24] S. Heinrich, Finite representability and super-ideals of operators, Dissertationes Math, *Rozprawy Mat.* **172** (1980), 37 pages.

[25] R.C. James, Weak compactness and reflexivity, *Israel J. Math.* **2** (1964) 101–119.

[26] R.C. James, Super-reflexive Banach spaces, *Canad J. Math.* **24** (1972) 896–904.

[27] M. Junge, N.J. Nielsen, Z. Ruan, Q. Xu, $\mathcal{COL}_p$ spaces–the local structure of non-commutative L_p spaces, *Adv. Math.* **187** (2) (2004) 257–319.

[28] M. Junge, Q. Xu, Noncommutative maximal ergodic theorems, *J. Amer. Math.*

Soc. **20** (2) (2007) 385–439.

[29] M. Junge, Q. Xu, Representation of certain homogeneous Hilbertian operator spaces and applications, *Invent. Math.* **179** (1) (2010) 75–118.

[30] G. Kasparov, G. Yu, The coarse geometric Novikov conjecture and uniform convexity, *Adv. Math.* **26** (2006) 1–56.

[31] K.D. Kürsten, On some questions of A. Pietsch. II, *Teor. Funct. Funct. Anal. i Pril.* **29** (1978) 61–73.

[32] Lafforgue, Vincent Un renforcement de la propriété (T) (French) [A strengthening of property (T)], *Duke Math. J.* **143** (2008) 559–602.

[33] G. Lancien, On uniformly convex and uniformly Kadec-Klee renormings, *Serdica Math. J.* **21** (1995) 1–18.

[34] G. Lancien, M. Raja, Nonlinear aspects of super weakly compact sets, *Annales de l'Institut Fourier*, Online first (2022). doi: 10.5802/aif.3488.

[35] A. Lima, O. Nygaard, E. Oja, Isometric factorization of weakly compact operators and the approximation property, *Israel J. Math.* **119** (2000) 325–348.

[36] M. Mendel, A. Naor, Nonlinear spectral calculus and super-expanders, *Publ. Math. Inst. Hautes Études Sci.* **119** (2014) 1–95.

[37] G. Pisier, Martingales with values in uniformly convex spaces, *Israel J. Math.* **20** (1975) 326–350.

[38] Y. Raynaud, Q. Xu, On subspaces of non-commutative L_p-spaces, *J. Funct. Anal.* **203** (1) (2003) 149–196.

[39] G. Pisier, Q. Xu, Non-commutative martingale inequalities, *Comm. Math. Phys.* **189** (3) (1997) 667–698.

[40] M. Raja, Finitely dentable functions, operators and sets, *J. Convex Anal.* **15** (2008) 219–233.

[41] M. Raja, Super WCG Banach spaces, *J. Math. Anal. Appl.* **439** (2016) 183–196.

[42] J. Stern, Ultrapowers and local properties of Banach spaces, *Trans. Amer. Math. Soc.* **240** (1978) 231–252.

[43] D.G. Tacon, Nonstandard extensions of transformations between Banach spaces, *Trans. Amer. Math. Soc.* **260** (1980) 147–158.

[44] J. Wang, *Nonlinear Properties of Super Weakly Compact Convex Sets*, Ph.D. Thesis, Xiamen University, 2020. (in Chinese)

[45] Q. Xu, Littlewood-Paley theory for functions with values in uniformly convex spaces, *J. Reine Angew. Math.* **504** (1998) 195–226.

[46] Q. Xu, Operator-space Grothendieck inequalities for noncommutative L_p-spaces, *Duke Math. J.* **131** (3) (2006) 525–574.

[47] V. Zizler, Nonseparable Banach spaces, In: *Handbook of the Geometry of Banach Spaces*, Vol. **2**, North-Holland, Amsterdam, 2003.

[48] G. Yu, Higher index theory of elliptic operators and geometry of groups, *Proceedings of International Congress of Mathematicians. Marid.* **2** (2006) 1623–1639.

[49] G. Yu, The Novikov conjecture, *(Russian) Uspekhi Mat. Nauk.* **74** (2019) 167–184, translation in *Russian Math. Surveys* **74** (2019) 525–541.