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Character Amenability of $C_0(X, A)$

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Abstract. Let A be a Banach algebra and X be a locally compact space. We derive various character amenability and construct the corresponding approximate character means of the A-valued continuous function algebra $C_0(X, A)$ directly from those of A.

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1. Introduction

The concept of amenable Banach algebras was introduced by Johnson [7]. Let A be a (complex) Banach algebra and let E be a Banach A-bimodule. A bounded linear map $D : A \to E$ is called a *derivation* if $D(ab) = a.D(b) + D(a).b$ for all a, b in A. Note that if E is a Banach A-bimodule then E^* is also a Banach A-bimodule with the module actions defined by

$$\langle a.x^*, x\rangle = \langle x^*, x.a\rangle \quad \text{and} \quad \langle x^*.a, x\rangle = \langle x^*, a.x\rangle, \quad \text{for } x^* \in E^*, x \in E, a \in A.$$

We call A *amenable* if for any Banach A-bimodule E, every continuous derivation $D : A \to E^*$ is inner, i.e., there is an $x^* \in E^*$ such that

$$D(a) = d_{x^*}(a) := a.x^* - x^*.a, \quad \forall a \in A.$$

Let ϕ be a (nonzero) *character* on A, that is, a nonzero homomorphism from A into $\mathbb{C}$. For any right Banach A-module E, we define a left action

$$a.x = \phi(a)x \quad \text{for } a \in A, x \in E,$$

to make E become a Banach A-bimodule, which we denote by E_ϕ. Following [6], we say that A is *ϕ-amenable* if for any right Banach A-module E, every continuous derivation $D : A \to E^*_\phi$ is inner. We call A *character amenable* if it is ϕ–amenable for every character ϕ on A. It is plain that every amenable Banach algebra is character amenable. However, the converse is not true in general. For example, the Fourier algebra $A(G)$ of a locally compact group G is character amenable but it is not always amenable (see [6, Example 2.6]).

As generalizations, we call a Banach algebra A *locally approximately ϕ–amenable* if for any right A-module E and any continuous derivation $D : A \to E^*_\phi$, there is a net $\{\xi_\alpha\}_\alpha$ in E^* such that the inner derivations $d_{\xi_\alpha}(a) = a.\xi_\alpha - \xi_\alpha.a = a.\xi_\alpha - \phi(a)\xi_\alpha$ converge to $D(a)$ in norm topology in E^* for each $a \in A$. If the convergence is always uniformly in $a \in K$ for any normed bounded (resp. compact) subset K of A, we call A *uniformly* (resp. *compactly*) *approximately ϕ–amenable*.

A net $\{u_\alpha\}_\alpha$ in A is called a *locally approximate ϕ–mean* if $\phi(u_\alpha) = 1$ for all α, and $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for all $a \in A$. If the convergence is uniformly on any norm bounded (resp. compact) subset K of A, then $\{u_\alpha\}_\alpha$ is called a *uniformly* (resp. *compactly*) *approximate ϕ–mean*. It turns out that a uniformly approximate ϕ–mean $\{u_\alpha\}_\alpha$ must be uniformly bounded, i.e., $\sup_\alpha \|u_\alpha\| < +\infty$ [1, Proposition 5.4]. Moreover, if a locally approximate ϕ–mean $\{u_\alpha\}_\alpha$ is uniformly bounded then it is a compactly approximate ϕ-mean; in this case we call it a *bounded approximate ϕ-mean*.

Lemma 1.1.

(i) [6, Theorem 1.4] *A Banach algebra A is ϕ–amenable if and only if it has a bounded approximate ϕ–mean.*

(ii) [1, Propositions 2.2 and 2.3] *A Banach algebra A is locally (resp. compactly) approximately ϕ–amenable if and only if it has a possibly unbounded locally (resp. compactly) approximate ϕ–mean.*

More precisely, if $\{u_\alpha\}_\alpha$ is a locally (resp. compactly) approximate ϕ–mean of A, then for any continuous derivation $D : A \to E^*_\phi$, by letting $g_\alpha = -D(u_\alpha) \in E^*$ we have $d_{g_\alpha}(a)$ converges to $D(a)$ in norm for each $a \in A$ (resp. uniformly for a in any norm compact subset of A) (see the proof of [1, Proposition 2.2]). If the net $\{u_\alpha\}_\alpha$ is uniformly bounded, then so is $\{g_\alpha\}_\alpha$. In this case, for any weak* cluster point g of $\{g_\alpha\}_\alpha$, we have $D = d_g$ is an inner derivation.

Examples of derivations which are approximately inner but not inner can be found in [2, Section 8]. In summary, we have the following lemma.

Lemma 1.2. [6, 8, 1] *Let ϕ be a character of a Banach algebra A. Then we have the following statements:*

(i) *A is ϕ–amenable $\Longleftrightarrow$ A is uniformly approximately ϕ-amenable $\Longleftrightarrow$ $\ker \phi$ has a bounded right approximate identity;*

(ii) *A is locally approximately ϕ-amenable $\Longleftrightarrow$ $\ker \phi$ has a right approximate identity.*

Example 1.3. For each $n = 1, 2, \ldots$, let $A_n = \ell^1_n(\mathbb{C})$ be the n-dimensional l_1 space with entrywise multiplication. Let $A = \oplus_{c_0} A_n$ be the c_0 direct sum of A_n. Let $B = A \oplus \mathbb{C}I$ be the unitilzation of A.

Let ϕ be the character of B such that $\phi(I) = 1$ and $\phi(A) = 0$. Since the kernel of ϕ, which is A, does not have a bounded approximate identity, B is not ϕ-amenable. However, the unbounded approximate identity of A provides an unbounded approximate ϕ-mean of B. Thus B is locally approximately ϕ-amenable.

We note that the amenability defined by Johnson in term of the existence of a *compactly approximate diagonal*, see [4], might not ensure the amenability of a Banach algebra; for example, when we consider the Segal algebra on a non-amenable locally compact group [3]. However, we do not know whether the compactly approximate ϕ–amenability is equivalent to either the local approximate ϕ–amenability or the (uniform) ϕ–amenability for a general Banach algebra.

Let X be a locally compact Hausdorff space, and A a Banach algebra. Let

$$C_0(X, A) := \{f : X \to A \text{ is continuous and vanishes at infinity}\}.$$

The uniform norm on $C_0(X, A)$ defined by

$$\|f\|_\infty = \sup\{\|f(x)\|_A : x \in X\}$$

makes $C_0(X, A)$ to be a Banach algebra.

Suggested by [6, Theorem 3.3], we shall see that if A is character amenable then $C_0(X, A)$ is character amenable. In fact, every character Φ of $C_0(X, A)$ assumes the form $\Phi(f) = \phi(f(t))$ for some character ϕ of A and for some point t in X. In this paper, we will construct explicitly an approximate Φ–mean for $C_0(X, A)$ from an approximate ϕ–mean of A. We also establish similar results involving the locally and the compactly approximate character amenability of A.

2. Results

In part (iii) below, if the positive constant $M < +\infty$ then $\{u_\alpha\}_\alpha$ will be a bounded approximate ϕ-mean. If $M = +\infty$ then it will be a (possibly unbounded) locally approximate ϕ-mean.

Lemma 2.1. *Let ϕ be a nonzero character of a Banach algebra A, and $0 < M \leq +\infty$. The following statements are equivalent:*

- (i) *For each finite subset F of A, there is a net $\{u_\alpha\}_\alpha$ in A with all $\|u_\alpha\| \leq M$ such that $\phi(u_\alpha) = 1$ and $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for all $a \in F$.*
- (ii) *For each finite subset F of A, there is a net $\{u_\alpha\}_\alpha$ in A with all $\|u_\alpha\| \leq M$ such that $\phi(u_\alpha) = 1$ and $au_\alpha - \phi(a)u_\alpha \to 0$ weakly for all $a \in F$.*
- (iii) *There is a net $\{u_\alpha\}_\alpha$ in A with all $\|u_\alpha\| \leq M$ such that $\phi(u_\alpha) = 1$ for all α, and $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for all $a \in A$.*

Proof. Obviously, (iii) $\implies$ (i) $\implies$ (ii).

Following the reasoning in [6, Theorem 1.4], we will show that (ii) implies (i). Consider a finite subset F of A. Let $\{w_\beta\}_\beta$ be a net satisfying the condition in (ii). Let A^F be the locally convex product space of n copies of the Banach space A, where n is the finite cardinality of the set F, equipped with the product topology of the norm topology of A. It is known that its dual space $(A^F)^* = \bigoplus_{a\in F} A^*$ is the locally convex direct sum of n copies of the dual space A^* of A. In particular, the $\sigma(A^F, (A^F)^*)$-topology, that is, the weak topology of A^F, agrees with the product topology of the weak topology of A. Define a linear map

$$T : A \to A^F \quad \text{by} \quad T(u) = (au - \phi(a)u)_{a\in F}.$$

Let $K = \{u \in A : \|u\| \leq M, \phi(u) = 1\}$. Since K is a convex subset of A, we see that $T(K)$ is convex in the product space A^F. Since $aw_\beta - \phi(a)w_\beta \to 0$ weakly for all $a \in F$, the point (0) with all coordinates 0 belongs to the closure of $T(K)$ in the $\sigma(A^F, (A^F)^*)$-topology. By the separation theorem, the closure of $T(K)$ in A^F in the product topology of the norm topology agrees with that taken in the $\sigma(A^F, (A^F)^*)$-topology. Hence there is a net $\{u_\alpha\}_\alpha$ in K such that $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ for each $a \in F$, as asserted.

Finally, suppose that (i) holds. Then for each given $\varepsilon > 0$ and each finite subset F of A, there is a $u_{F,\varepsilon}$ in A with $\|u_{F,\varepsilon}\| \leq M$ such that $\|au_{F,\varepsilon} - \phi(a)u_{F,\varepsilon}\| \leq \varepsilon$. Order $\mathcal{J} = \{(F, \varepsilon) : F \subset A \text{ is finite}, \varepsilon > 0\}$ by letting $(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2)$ if $F_1 \subseteq F_2$ and $\varepsilon_1 \geq \varepsilon_2$. Then the net $\{u_{F,\varepsilon} : (F, \varepsilon) \in \mathcal{J}\}$ satisfies the condition (iii). ∎

Let X be a locally compact Hausdorff space and A a complex Banach algebra. Recall that for any character Φ on $C_0(X, A)$ there exist a character ϕ on A and a point t in X such that $\Phi(f) = \phi(f(t))$ for all $f \in C_0(X, A)$ (see [5]). We write $\Phi = \delta_t \otimes \phi$ for this connection. Let $\mathcal{V}_t$ consist of all compact neighborhoods V of t in X, and ordered by reverse set inclusion. By the Urysohn lemma, for each $V \in \mathcal{V}_t$, there exists a nonnegative function $g_V \in C(X)$ such that

$$\operatorname{supp}(g_V) \subset V, \quad \|g_V\| = g_V(t) = 1 \quad \text{and} \quad g_V \to \mathbf{1}_t \text{ pointwise}, \tag{1}$$

where $\mathbf{1}_t$ is the indicator function of the singleton $\{t\}$.

Theorem 2.2. *Let ϕ be a nonzero character of a Banach algebra A. Let t be a point in a locally compact Hausdorff space X and $\Phi = \delta_t \otimes \phi$ be the associated character of $C_0(X, A)$. Assume that A is (resp. locally, compactly) ϕ–amenable. Then $C_0(X, A)$ is (resp. locally, compactly) Φ–amenable.*

Proof. (i) Suppose A is locally approximately ϕ–amenable with a locally approximate ϕ–mean $\{u_\alpha : \alpha \in \mathcal{J}\}$. Let $\varepsilon > 0$, and $\mathcal{F} = \{f_1, \ldots, f_n\}$ be any finite subset of $C_0(X, A)$. Let $a_j = f_j(t) \in A$ for $j = 1, 2, \ldots, n$. For large enough index α, we have

$$\|a_j.u_\alpha - \phi(a_j)u_\alpha\| < \varepsilon/2, \quad \forall j = 1, 2, \ldots, n.$$

Let $V \in \mathcal{V}_t$ be a compact neighborhood of t and g_V a nonnegative continuous function in $C_0(X)$ supported in V with $g_V(t) = \|g_V\| = 1$. For small enough V we can assume that

$$\|g_V a_j - g_V f_j\| < \frac{\varepsilon}{2\|u_\alpha\|}, \quad \forall j = 1, 2, \ldots, n.$$

Consider $g_V u_\alpha \in C_0(X, A)$. We have $\Phi(g_V u_\alpha) = \phi(g_V(t)u_\alpha) = \phi(u_\alpha) = 1$. Observe that

$$\|f_j g_V u_\alpha - \Phi(f_j) g_V u_\alpha\| \leq \|(g_V f_j - g_V a_j)u_\alpha\| + \|g_V(a_j u_\alpha - \phi(a_j)u_\alpha)\| < \varepsilon.$$

In view of Lemma 2.1, we see that $C_0(X,A)$ is locally approximately Φ–amenable.

(ii) Suppose A is ϕ–amenable with a uniformly bounded approximate ϕ–mean $\{u_\alpha\}_\alpha$. The construction in (a) gives us a locally approximate Φ–mean $\{g_V u_\alpha\}$ of $C_0(X, A)$. It is clear that $\sup_{\alpha,V} \|g_V u_\alpha\| = \sup_\alpha \|u_\alpha\| < +\infty$, and thus $C_0(X, A)$ is Φ–amenable.

(iii) Finally, suppose A is compactly ϕ–amenable with a compactly bounded approximate ϕ–mean $\{u_\alpha\}_\alpha$. In this case, $\{u_\alpha : \alpha \in \mathcal{J}\}$ is a locally approximate

ϕ–mean of A such that the convergence $\|au_\alpha - \phi(a)u_\alpha\| \to 0$ is uniform on any compact set in A.

We want to show that $C_0(X, A)$ has a compactly approximate Φ–mean. It suffices to show that for any $\epsilon > 0$ and any compact set $\mathcal{K}$ in $C_0(X, A)$, there is a v in $C_0(X, A)$ such that

$$\|fv - \Phi(f)v\| < \epsilon, \quad \forall f \in \mathcal{K}.$$

We adapt some arguments from [4, Proposition 3.1]. Let $X_0 = X \cup \{\infty\}$ be the one-point compactification of X. In this way, every f in $C_0(X, A)$ can be considered as an element in $C(X_0, A)$ by assigning $f(\infty) = 0$. Let

$$K = \{f(x) : f \in \mathcal{K}, x \in X_0\}.$$

Since $\mathcal{K}$ is compact and X_0 is compact, K is a compact subset of A.

Let $\varepsilon > 0$. By assumption, there is an index α_0 such that for all $\alpha \geq \alpha_0$ we have $\phi(u_\alpha) = 1$ and

$$\|bu_\alpha - \phi(b)u_\alpha\| \leq \frac{\varepsilon}{6}, \quad \forall b \in K. \tag{2}$$

Let $u = u_\alpha$ for any $\alpha \geq \alpha_0$. Since $\mathcal{K}$ is compact, we can choose $f_1, \ldots, f_m$ from $\mathcal{K}$ such that every f in $\mathcal{K}$ is within norm distance $\frac{\epsilon}{9\|u\|}$ from some f_i. Then the uniform continuity of $f_1, \ldots, f_m$ ensures that there are nonempty open subsets $X_1, \ldots, X_n$ of X_0 such that $X_0 = \bigcup_{k=1}^n X_k$, and

$$\|f_i(x) - f_i(y)\| \leq \frac{\varepsilon}{9\|u\|}, \quad \forall x, y \in X_k, \forall i = 1, 2, \ldots, m, \forall k = 1, \ldots, n.$$

Consequently,

$$\|f(x) - f(y)\| \leq \frac{\varepsilon}{3\|u\|}, \quad \forall x, y \in X_k, \forall f \in \mathcal{K}, \forall k = 1, 2, \ldots, n.$$

Choose a continuous partition of unity, $h_1, \ldots, h_n$ in $C(X_0)$, such that $0 \leq h_k \leq 1$, $\mathrm{supp}(h_k) \subset X_k$ and $\sum_{k=1}^n h_k = 1$. For each $k = 1, \ldots, n$, choose $x_k \in X_k$. For each $f \in \mathcal{K}$, let $f_\varepsilon = \sum_{k=1}^n h_k f(x_k)$. Then

$$\|f - f_\varepsilon\| = \left\| \sum_{k=1}^n (h_k f - h_k f(x_k)) \right\| \leq \frac{\varepsilon}{3\|u\|}, \quad \forall f \in \mathcal{K}. \tag{3}$$

Let the finite positive number

$$L = \sup\{|\phi(f(x_k))| : f \in \mathcal{K}, k = 1, \ldots, n\} + 1.$$

Then there exists $V_0 \in \mathcal{V}_t$ such that for any $V \in \mathcal{V}_t$ with $V \subseteq V_0$ we have

$$\|(h_k - h_k(t))g_V\| \leq \frac{\varepsilon}{6\|u\|nL}, \quad \forall k \in 1, \ldots, n. \tag{4}$$

Since $f(x_k)\in K$ for any $f\in\mathcal{K}$, it follows from (2) and (4) that

$$\begin{aligned}
&\|f_\varepsilon u g_V-\Phi(f_\varepsilon)ug_V\|\\
&=\|ug_V\sum_{k=1}^{n}h_kf(x_k)-ug_V\sum_{k=1}^{n}h_k(t)\phi(f(x_k))\|\\
&\le \sum_{k=1}^{n}\|\,[uf(x_k)-\phi(f(x_k))u]\,h_kg_V\|+\sum_{k=1}^{n}\|\phi(f(x_k))u\|\|\,[h_k-h_k(t)]\,g_V\|\\
&\le \frac{\varepsilon}{6}+\sum_{k=1}^{n}L\|u\|\frac{\varepsilon}{6\|u\|nL}=\frac{\varepsilon}{3}.
\end{aligned}\tag{5}$$

Therefore, for all $f\in\mathcal{K}$ it follows from (3) and (5) that

$$\begin{aligned}
&\|fug_V-\Phi(f)ug_V\|\\
&\le\|(f-f_\varepsilon)ug_V\|+\|(f_\varepsilon u-\Phi(f_\varepsilon)u)g_V\|+\|(\Phi(f_\varepsilon)-\Phi(f))ug_V\|\\
&\le 2\|u\|\frac{\varepsilon}{3\|u\|}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

Let $v=ug_V$. We have $\|fv-\Phi(f)v\|\le\varepsilon$ for all $f\in\mathcal{K}$. This completes the proof. ∎

3. Remarks and Open Problem

We end this paper with an open problem about possible extensions of our result. Recall that a Banach algebra is *weakly ϕ–amenable* (resp. *weakly amenable*) if every continuous derivation $D:A\to A^*_\phi$ (resp. $D:A\to A^*$) is inner. It is showed in [4] that if X is a compact Hausdorff space and A is a commutative Banach algebra then $\mathrm{C}(X,A)$ is weakly amenable if and only if A is weak amenable. We ask for a similar result as in Theorem 2.2 for the weakly character amenability.

Question 3.1. Does the weak ϕ–amenability of A ensure the weak Φ–amenability of $C_0(X,A)$?

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