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Some Open Problems Related to Fixed Point Properties

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Abstract. There are intrinsic connections between the common fixed point properties of a semigroup and the amenability properties of the semigroup. In this note, we survey open problems concerning these relations.

Keywords: Nonexpansive; Weakly compact; Weak* compact; Common fixed point; Invariant mean.

1. Preliminaries

Let S be a semigroup. We call S a *semitopological semigroup* if S is equipped with a Hausdorff topology such that, for each $a \in S$, the mappings $s \mapsto sa$ and

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$s \mapsto as$ from S into S are continuous. Denote by $\ell^\infty(S)$ the C*-algebra of all bounded complex-valued functions on S with the uniform norm topology and the pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, denote by $\ell_a f$ and $r_a f$ the left and, respectively, right translates of f by a, i.e. $\ell_a f(s) = f(as)$ and $r_a f(s) = f(sa)$ $(s \in S)$. Let X be a closed subspace of $\ell^\infty(S)$ containing constants and be invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$; m is called a *left (resp. right) invariant mean*, abbreviated LIM (resp. RIM), if $m(\ell_a f) = m(f)$ (resp. $m(r_a f) = m(f)$) for all $a \in S$, $f \in X$. Let X be a C*-subalgebra of $\ell^\infty(S)$. Then a *multiplicative linear functional* is an element $\phi \in X^*$ that satisfies $\langle \phi, f \cdot g\rangle = \langle \phi, f\rangle\langle \phi, g\rangle$ for all $f, g \in X$. Every $s \in S$ is a multiplicative linear functional on X if we regard it as the evaluation functional: $\langle s, f\rangle = f(s)$ for $f \in X$.

Denote by $C_b(S)$ the space of all bounded continuous complex-valued functions on S. A function $f \in C_b(S)$ is called *left uniformly continuous functions* if the mapping $s \mapsto \ell_s f$: $S \to C_b(S)$ is continuous. We denote by $LUC(S)$ the space of all left uniformly continuous functions on S. We have $LUC(S) \subset C_b(S) \subset \ell^\infty(S)$. The semitopological semigroup S is called left amenable if $LUC(S)$ has a LIM. We note that if S is discrete, then $LUC(S) = C_b(S) = \ell^\infty(S)$. Denote by $AP(S)$ the space of all $f \in C_b(S)$ such that $\mathcal{LO}(f) = \{\ell_s f :\ s \in S\}$ is relatively compact in the norm topology of $C_b(S)$, and denote by $WAP(S)$ the space of all $f \in C_b(S)$ such that $\mathcal{LO}(f)$ is relatively compact in the weak topology of $C_b(S)$. Functions in $AP(S)$ (resp. $WAP(S)$) are called almost periodic (resp. weakly almost periodic) functions on S. In general we have $AP(S) \subset LUC(S) \cap WAP(S) \subset C_b(S)$.

The semitopological semigroup S is called *left reversible* if any two closed right ideals of S have non-void intersection, i.e. $\overline{aS} \cap \overline{bS} \neq \emptyset$ for all $a, b \in S$. When S is a discrete semigroup the following implication relation chain is known.

$$\begin{array}{c} S \text{ is left amenable} \\ \Downarrow \quad \not\Uparrow \\ S \text{ is left reversible} \\ \Downarrow \quad \not\Uparrow \\ WAP(S) \text{ has LIM} \\ \Downarrow \quad \not\Uparrow \\ AP(S) \text{ has LIM} \end{array}$$

An action of a semigroup S on a topological space K is a mapping ψ from $S \times K$ into K such that $T_{s_1 s_2} x = T_{s_1}(T_{s_2} x)$ for all $s_1, s_2 \in S$ and $x \in K$, where $T_s x = \psi(s, x)$. The action is *separately continuous* or *jointly continuous* if the mapping ψ is, respectively, separately or jointly continuous. We call $\mathcal{S} = \{T_s : s \in S\}$ a representation of S on K. We say that $x \in K$ is a *common fixed point* for (the representation of) S if $T_s(x) = x$ for all $s \in S$.

A locally convex topological space E with the topology generated by a family Q of seminorms will be denoted by (E, Q). A representation $\mathcal{S} = \{T_s : s \in S\}$ of S on a subset K of a seprated locally convex space (E, Q) is Q-*nonexpansive* if $p(T_s x - T_s y) \leq p(x - y)$ for all $s \in S$, all $p \in Q$ and all $x, y \in K$. If E is a normed space with the norm$\|\cdot\|$, then the representation $\mathcal{S} = \{T_s : s \in S\}$ of S on $K \subset E$ is norm nonexpansive if $\|T_s x - T_s y\| \le \|x - y\|$ for all $s \in S$ and all $x, y \in K$.

2. Open Problems Concerning Common Fixed Points

For a semitopological semigroup S, simply examining the representation of S on the weak* compact convex subset of all means on $LUC(S)$ defined by the dual of left translations on $LUC(S)$, we have that if the following fixed point property holds then $LUC(S)$ has a left invariant mean.

(fpp$_*$) Whenever $\mathcal{S} = \{T_s : \ s \in S\}$ is a representation of S as norm nonexpansive mappings on a nonempty weak* compact convex set C of the dual space of a Banach space E and the mapping $(s, x) \mapsto T_s(x)$ from $S \times C$ to C is jointly continuous, where C is equipped with the weak* topology of E^*, then there is a common fixed point for S in C.

Whether the converse is true is an open problem.

Problem 2.1. Does a semitopological semigroup S have the fixed point property (fpp$_*$) if $LUC(S)$ has a LIM?

The problem is open even for discrete case [8]. It was shown in [14, Proposition 6.1] that a weak version of property (fpp$_*$) holds if $LUC(S)$ has a LIM.

Problem 2.2. Suppose that $LUC(S)$ has a left invariant mean. Does the linear span of the set of left invariant means on $LUC(S)$ (i.e. the fixed point set of the adjoint operators of left translations on the set of means) form a finite dimensional space?

For discrete S this question was answered affirmatively by E. E. Graniner [1].

An F-algebra is a Banach algebra $\mathfrak{A}$ which is a predual of a von Neumann algebra $\mathfrak{M}$ such that the identity 1 of $\mathfrak{M}$ is a multiplicative linear functional on $\mathfrak{A}$ [10]. The F-algebra $\mathfrak{A}$ is left amenable if there is a *topological left invariant mean* (abbreviated TLIM) m on $\mathfrak{A}^* = \mathfrak{M}$, i.e. if there is $m \in \mathfrak{M}^*$ such that $\|m\| = 1$ and $\langle m, \varphi \cdot f\rangle = \langle m, f\rangle$ for all $f \in \mathfrak{M}$ and all $\varphi \in \mathfrak{A}$ with $\|\varphi\| = \langle 1, \varphi\rangle = 1$, where $\langle \varphi \cdot f, \psi\rangle = \langle f, \psi\varphi\rangle$ for $\psi \in \mathfrak{A}$. In [15] the authors showed that $\mathfrak{A}$ is left amenable if and only if the metric semigroup $S = P_1(\mathfrak{A}) = \{\varphi \in \mathfrak{A} : \varphi \ge 0, \|\varphi\| = 1\}$ with the product and topology inherited from $\mathfrak{A}$ has the following fixed point property:

(fpp$_U$): Whenever S acts on a compact subset K of a locally convex space such that the mapping $(s, y) \mapsto T_s y : S \times K \to K$ is separately continuous and uniformly continuous in s for each $y \in K$, then K has a common fixed point for S.

Related to Problem 2.2 we pose the following problem.

Problem 2.3. Suppose that the F-algebra $\mathfrak{A}$ is left amenable. When is the space spanned by the set of topological left invariant means on $\mathfrak{A}$ finite dimensional?

Related to Problem 2.1, it is proved in [14] that if S is a left reversible or a left amenable semitopological semigroup, then the following fixed point property holds:

(fpp$_{*s}$) Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a norm nonexpansive representation of S on a nonempty norm separable weak* compact convex set C of the dual space of a Banach space E and the mapping $(s, x) \mapsto T_s(x)$ from $S \times C$ to C is jointly continuous when C is endowed with the weak* topology of E^*, then there is a common fixed point for S in C.

Problem 2.4. Let S be a (discrete) semigroup. If the fixed point property (fpp$_{*s}$) holds, does $WAP(S)$ have a LIM? We also do not know whether the existence of a LIM on $WAP(S)$ is sufficient to ensure the fixed point property (fpp$_{*s}$).

A partial affirmative answer to Problem 2.4 was given in [14, Proposition 6.5], which we quote as follows.

Proposition 2.5. *Suppose that S has the fixed point property (fpp$_{*s}$). Then*

(i) *$AP(S)$ has a LIM;*
(ii) *$WAP(S)$ has a LIM if S has a countable left ideal.*

Consider partially bicyclic semigroups $S_2 = \langle e, a, b, c \mid ab = e, ac = e\rangle$ and $S_{1,1} = \langle e, a, b, c, d \mid ab = e, cd = e\rangle$. We know that they are not left amenable (see [13, Proposition 4.3]). So they do not have the fixed point property (fpp$_*$). It is worth mentioning that $WAP(S_2)$ and $AP(S_{1,1})$ both have a LIM as shown in [13].

Problem 2.6. Does the partially bicyclic semigroup S_2 have the fixed point property (fpp$_{*s}$)?

If the answer to the above question is yes, then S having (fpp$_{*s}$) is not equivalent to S being left reversible (or left amenable); if the answer is no, then the converse of Proposition 2.5 (ii) does not hold even for a countable semigroup S.

It was shown in [7] that $AP(S)$ has a LIM if and only if S has the following fixed point property:

(fpp$_Q$) Whenever S acts on a compact convex subset K of a separated locally convex space (E, Q) and the action is separately continuous and Q-nonexpansive, then K contains a common fixed point for S.

If S is separable, then the existence of a LIM on $AP(S)$ is also equivalent to the following fixed point property [13, Theorem 3.6]:

(fpp$_{we}$) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) and the action is weakly separately continuous, weakly equicontinuous and Q-nonexpansive, then K contains a common fixed point for S.

An action of a semitopological semigroup S on a Hausdorff space X is called *quasi-equicontinuous* if $\overline{S}^p$, the closure of S in the product space X^X with the product topology, consists of only continuous mappings. Obviously, an equicontinuous action on a closed subset of a topological vector space is always quasi-equicontinuous (simply because if a net of equicontinuous functions converges pointwise to a function, then the limit function is also continuous). But a quasi-equicontinuous action on a convex compact subset of a topological vector space may not be equicontinuous [13, Example 4.14]. It is well-known that $WAP(S)$ has a LIM if S has the fixed point property stated as follows.

(fpp$_{wq}$) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) and the action is weakly separately continuous, weakly quasi-equicontinuous and Q-nonexpansive, then K contains a common fixed point for S.

The converse is an open problem.

Problem 2.7. Does fpp (fpp$_{wq}$) hold for a semitopological semigroup S if $WAP(S)$ has a LIM?

For the case that S is separable, the problem is settled affirmatively in [13, Theorem 3.4].

It is well-known that a discrete left reversible semigroup S has the follwoing fixed point property [6].

(fpp$_w$) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) and the action is weakly separately continuous and Q-nonexpansive, then K contains a common fixed point for S.

Clearly, we have the implication relations

$$\text{fpp}_w \Rightarrow \text{fpp}_{wq} \Rightarrow \text{fpp}_{we} \Rightarrow \text{fpp}_Q.$$

Problem 2.8. Can any of the above implications be reversed?

Problem 2.9. Let S be a left reversible semitopological semigroup acting on a weakly closed subset $C \neq \emptyset$ of a Hilbert space as norm nonexpansive and weakly jointly continuous self mappings. Suppose that there is $c \in C$ such that $\{T_s c : s \in S\}$ is bounded. Does C contain a common fixed point for S?

The question has been answered affirmatively in [16, Theorem 4.8] for the case that S is separable. We wonder whether the separability condition is removable.

It was also shown in [16, Theorem 4.3] that a nonexpansive representation of a semitopological semigroup S on a nonempty closed convex subset C of a Hilbert space H has a common fixed point in C if there is $c \in C$ such that $\{T_s c : s \in S\}$ is bounded and any of the following conditions holds:

(i) $C_b(S)$ has a left invariant mean and the mapping $s \mapsto T_s c$ is continuous from S into (C, wk);
(ii) S is left amenable and the action of S on C is weakly jointly continuous;
(iii) $AP(S)$ has a left invariant mean and the action of S on C is weakly separately continuous and weakly equicontinuous continuous;
(iv) $WAP(S)$ has a left invariant mean and the action of S on C is weakly separately continuous and weakly quasi-equicontinuous.

Problem 2.10. In any of the cases in the above result, does the converse also hold?

We call a semitopological semigroup S *extremely left amenable* (abbreviated ELA) if there is a left invariant mean m on $LUC(S)$ which is multiplicative, that is it satisfies further

$$m(fg) = m(f)m(g) \quad (f, g \in LUC(S)).$$

If S is a locally compact group, then S is ELA only when S is a singleton [3]. However, a non-trivial topological group which is not locally compact can be ELA. In fact, let S be the group of unitary operators on an infinite dimensional Hilbert space with the strong operator topology, then S is ELA [4]. It is shown in [15] that an F-algebra A is left amenable if and only if the semigroup $S = P_1(A)$ of normal positive functionals of norm 1 on A^* is ELA. For more examples we refer to [12].

In fact, Mitchell showed in [19] that a semitopological semigroup S is ELA if and only if it has the following fixed point property:

(fpp$_E$) Every jointly continuous representation of S on a nonempty compact Hausdorff space has a common fixed point for S.

Related to the fixed point property (fpp$_E$) we consider the following Schauder fixed point property for a semitopological semigroup S:

(fpp$_S$) Every jointly continuous representation of S on a nonempty compact convex subset C of a separated locally convex topological vector space has a common fixed point.

Of cause, every ELA semigroup has the fixed pont property (fpp_S). The well-known Schauder's Fixed Point Theorem can be stated as: the free commutative (discrete) semigroup on one generator has the fixed point property (fpp_S). Many other examples of S with (fpp_S) are discussed in [14, Section 4]. We raise the following problem.

Problem 2.11. What amenability property of a semitopological semigroup may be characterized by the Schauder fixed point property (fpp_S)?

For the representation of an ELA semigroup on a subset of a Banach space, the following is open.

Problem 2.12. Suppose that S is extremely left amenable and C is a weakly closed subset of a Banach space E, and suppose that $\mathcal{S} = \{T_s c : s \in S\}$ is a weakly continuous and norm nonexpansive representation of S on C such that $\{T_s c : s \in S\}$ is relatively weakly compact for some $c \in C$. Does C contain a common fixed point for S?

We know that the answer is "yes" when S is discrete. Indeed, in this case, for each finite subset σ of S there is $s_\sigma \in S$ such that $ss_\sigma = s_\sigma$ for all $s \in \sigma$ by a theorem of Granirer's [2] (see also [14, Theorem 4.2] for a short proof). Consider the net $\{s_\sigma c\}$. By the relative weak compactness of Sc, there is $z \in \overline{Sc}^{\text{wk}} \subset C$ such that (go to a subnet if necessary) $\text{wk-lim}_\sigma s_\sigma c = z$. Then, as readily checked, $T_s z = z$ for all $s \in S$ by the weak continuity of the S action on C.

More generally, the answer to Problem 2.12 is still affirmative (even without the norm nonexpansiveness assumption) if the representation is jointly continuous when C is equipped with the weak topology of E. This is indeed a consequence of [19, Theorem 1].

Problem 2.13. Let C be a nonempty closed convex subset of the sequence space c_0 and $\mathcal{S} = \{T_s c : s \in S\}$ be a representation of a commutative semigroup S as nonexpansive mappings on C. Suppose that $\{T_s c : s \in S\}$ is relatively weakly compact for some $c \in C$. Does C contain a common fixed point for S?

One may not drop the relative weak compactness condition on the orbit of c. For example, on the unit ball of c_0 define $T((x_i)) = (1, x_1, x_2, \cdots)$. Then T is nonexpansive, and obviously T has no fixed point in the unit ball.

Let E be a separated locally convex vector space and X a subset of E. Given an integer $n > 0$ we denote by $\mathcal{L}_n(X)$ the collection of all n-dimensional subspaces of E that are included in X. Let S be a semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a linear representation of S on E. We say that X is *n-consistent* with respect to S if $\mathcal{L}_n(X) \neq \emptyset$ and $T_s(L) \in \mathcal{L}_n(X)$ for all $s \in S$ whenever $L \in \mathcal{L}_n(X)$. We say that the representation $\mathcal{S}$ is *jointly continuous on compact sets* if the following is true: For each compact set $K \subset E$, if $(s_\alpha) \subset S$ and $(x_\alpha) \subset K$ are such that $s_\alpha \overset{\alpha}{\to} s \in S$, $x_\alpha \overset{\alpha}{\to} x \in K$ and $T_{s_\alpha}(x_\alpha) \in K$ for all α, then $T_{s_\alpha}(x_\alpha) \overset{\alpha}{\to} T_s(x)$. Obviously, if the mapping $(s, x) \mapsto T_s(x)$: $S \times E \to E$ is continuous, then $\mathcal{S}$ is jointly continuous on compact sets.

Let A be an F-algebra. It is shown in [15] that A is left amenable if and only if $S = P_1(A)$ has the following n-dimensional invariant subspace property for some (and then for all) $n > 0$:

(F_n) Let E be a separated locally convex vector space and $\mathcal{S} = \{T_s c : s \in S\}$ a linear representation of $S = P_1(A)$ on E such that the representation is jointly continuous on compact subsets of E . If X is a subset of E that is n-consistent with respect to S, and if there is a closed S-invariant subspace H of E with codimension n such that $(x + H) \cap X$ is compact for each $x \in E$, then there is $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ $(s \in S)$.

Let (F_n') denote the same property as (F_n) with "jointly continuous" replaced by "separately continuous" on compact subsets of E.

Problem 2.14. Let A be an F-algebra. Does (F_n) imply (F_n')?

Regard the F-algebra A as the Banach A-bimodule with the module multiplications given by the product of A. Then the dual space A^* is a Banach A-bimodule. We say that a subspace X of A^* is topologically left (resp. right) invariant if $a \cdot X \subset X$ (resp. $X \cdot a \subset X$) for each $a \in A$. We call X topologically invariant if it is both left and right topological invariant. An element f of A^* is almost periodic (resp. weakly almost periodic) if the map $a \mapsto f \cdot a$ from A into A^* is a compact (resp. weakly compact) operator. Let $AP(A)$ and $WAP(A)$ denote the collection of almost periodic and weakly almost periodic functionals on A respectively. Then $AP(A)$ and $WAP(A)$ are closed topologically invariant subspaces of A^*. Furthermore, $1 \in AP(A) \subset WAP(A)$. When G is a locally compact group and $A = L^1(G)$, then $AP(A) = AP(G)$ and $WAP(A) = WAP(G)$.

Let (F_n^A) denote the same property as (F_n) with joint continuity replaced by equicontinuity on compact subsets of E. It is known that if A satisfies (F_n^A) then $AP(A)$ has a TLIM (see [9]).

Problem 2.15. Does the existence of TLIM on $AP(A)$ imply (F_n^A) for all $n \geq 1$?

Let (F_n^W) denote the same property (F_n^A) with equicontinuity on compact subsets of E replaced by quasi-equicontinuity on compact subsets of E (which means that the closure of S in the product space E^K, for each compact set $K \subset E$, consists only of continuous maps from K to E). We have known that if A satisfies (F_n^W) for each $n \geq 1$ then $WAP(A)$ has a TLIM (see [9]).

Problem 2.16. Does the existence of TLIM on $WAP(A)$ imply (F_n^W) for all $n \geq 1$?

Let K be a subset of the dual space E^* of a Banach space E, and let T be a mapping from K into E^*. Denote the unit ball of E by E_1. We call T *pseudo weak*-nonexpansive* if, for each $\phi \in E_1$ and each $\varepsilon > 0$, there exists a finite set $\Lambda \subset E_1$ such that

$$|\langle \phi, Tx - Ty \rangle| \leq \max_{\phi' \in \Lambda} |\langle \phi', x - y \rangle| + \varepsilon$$

for all $x, y \in K$. It is readily seen that if T is pseudo weak*-nonexpansive, then is is norm nonexpansive. The converse is also true if K is weak* compact and T is weak* continuous [18, Lemma 3.4]. We wonder whether the converse is still true if the weak* continuity on T is removed.

Problem 2.17. Let T be a norm nonexpansive self mapping on a weak* compact subset of a dual Banach space. Must it be pseudo weak*-nonexpansive?

Any partially affirmative answer to this problem will considerably improve the main results of [17, 18].

It is shown in [17] that a left reversible semitopological semigroup S has the following fixed point property (see [18, Theorem 4.7] for a slight improvement of the result):

(fpp$_{*n}$) If K is a nonempty weak* compact convex subset of a dual Banach space and if K has the normal structure, then a norm nonexpansive and separately weak* continuous representation of S on K has a common fixed point in K.

It is also known that if S is left reversible, then $AP(S)$ has a LIM, and the converse is untrue [5].

Problem 2.18. Let S be a semitopological semigroup such that $AP(S)$ has a LIM. Does the above fixed point property (fpp$_{*n}$) hold?

If we asume further that the representation of S is weak* equicontinuous, then the answer to the problem is affirmative as given in [17, Corollary 4.18].

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