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JB-Algebras and Analytic Loos Symmetric Cones

Jimmie Lawson
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
Email: lawson@math.lsu.edu

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Abstract. In earlier papers the author has developed the theory of Loos symmetric cones, which are Loos symmetric spaces on an open cone Ω in a Banach space, and developed the foundations of a geometric theory based on (modified) Loos axioms for such cones. Such cones present an abstract version of the cone of positive elements in a C^*-algebra. In this paper the close connection between Loos symmetric cones and a special class of Jordan algebras called JB-algebras is developed. Recent work of C.-H. Chu is applied to show that every Loos symmetric cone arises as the cone of positive invertible elements of a unital JB-algebra and conversely that this JB-algebra is unique and can be constructed on the Banach space containing the Loos symmetric cone.

Keywords: Loos symmetric cone; Normal cone; Symmetric space; JB-algebra.

1. Introduction

In a series of three papers [12, 11, 13] and a two-volume monograph [14] O. Loos defined and launched the study of what have come to be called *Loos symmetric spaces*. He set out four basic axioms for his objects of study, sometimes referred to as the *Loos axioms*. We take the four axioms of Loos, strengthen the fourth axiom, assume that they are satisfied for an open cone in a Banach space and add two elementary axiomatic properties that connect the Loos structure with the linear structure of the Banach space and with the partial order determined by the cone and call the resulting structure on the cone a *Loos symmetric cone*. Y. Lim and the author began the study of such structures in [10]. Later in [6] the author, inspired by earlier work of E. Andruchow, G. Corach, and D. Stojanoff [1, 2] on positive cones in C^*-algebras, demonstrated how the geometric structure of Loos symmetric cones can be effective for insight into and the study and derivation of various operator-like inequalities involving the more general Loos symmetric cones. Exponential and log functions that exhibit many desirable features reminiscent of those of the exponential function from the space of self-adjoint elements to the cone of positive elements in a unital C^*-algebra were also introduced. It was also shown that the Thompson metric arises as the distance function for a natural Finsler structure on Ω and its minimal geodesics agree with the geodesics of the spray (connection) arising from the Loos structure.

Inspired by the recent work of C.-H. Chu [3], in this paper we apply his work to show that given any Loos symmetric cone C, an open cone in a Banach space V, one can construct a unital Jordan algebra on V that is a JB-algebra and that has the given Loos symmetric cone as its cone of positive elements and conversely. This result is a variant of one of Chu's main results. It is a generalization of the classical work of E.B. Vinberg [22, 23] and M. Koecher [5], who independently established the one-to-one correspondence between finite-dimensional linearly homogeneous self-dual cones and unital euclidean Jordan algebras.

In the following material we rather leisurely section by section put together the ingredients, much of which is already known, that are needed to establish our results.

2. Loos Systems

We recall from [7] the underlying algebraic structure with which we work and basic properties thereof. A *Loos system* consists of a binary algebraic system $(X, \bullet)$, with left translation $S_a b := a \bullet b$ axiomatized to represent the geometric notion of reflecting through the point a for each a. The point symmetry (reflection) of b through a, satisfies for all $a, b, c \in X$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_X)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(S4) the equation $x \bullet a = b$ ($S_x a = b$) has a unique solution $x \in X$, called the *midpoint* or *geometric mean* of a and b, and denoted by $a\#b$.

Systems satisfying only Axioms (S1)-(S3) are called *symmetric sets*. Axiom (S4) asserts the unique existence of a "midpoint of symmetry" $a\#b$ that reflects a to b and (by (S2)) also reflects b to a.

A *pointed Loos system* is a triple $(X, \bullet, \varepsilon)$, where $(X, \bullet)$ is a Loos system and $\varepsilon \in X$ is some distinguished point, called the *base point*. In this setting we define

$$x^{-1} := S_\varepsilon x, \quad x^2 := S_x \varepsilon, \quad x^{1/2} := \varepsilon\#x \tag{1}$$

and inductively from these definitions all dyadic powers are defined (see [9, Section 2]) so that the following rules are satisfied:

$$(x^r)^s = x^{rs}, \quad x^r \bullet x^s = x^{2r-s}, \quad x^r \# x^s = x^{\frac{(r+s)}{2}}. \tag{2}$$

If we consider the dyadic rationals $\mathbb{D}$ endowed with the structure $a \bullet b = 2a - b$ (the reflection of b through a), then $a\#b = (a+b)/2$, the usual midpoint, and the map $t \mapsto x^t : \mathbb{D} \to X$ is both a $\bullet$-homomorphism and $\#$-homomorphism. From this fact the preceding rules (and others) easily follow.

A *twisted subgroup* P of a group G is a subset containing the identity e that is closed with respect to the operation $x \bullet y = xy^{-1}x$. The system $(P, \bullet)$ satisfies (S1), (S2), and (S3), and further satisfies (S4) if and only if P is uniquely 2-divisible (every element of P has a unique square root in P); see [7, Prop. 4.2]. It is natural to choose e for the distinguished point of P. As a special case, consider the group $(\mathbb{R}, +)$ of additive real numbers, which is a uniquely 2-divisible twisted subgroup of itself. Then $(\mathbb{R}, \bullet, 0)$ is a pointed Loos system, where $s \bullet t = 2s - t$.

A Loos system X has a symmetry group called the *displacement group* $G(X)$ (also called the transvection group). The group is generated under the composition by all transformations of the form $S_x S_y$, $x, y \in X$. It follows from Axioms (S2) and (S3) that these are automorphisms and thus there is a group action $(g, x) \mapsto g.x : G(X) \times X \to X$ with $G(X)$ acting as automorphisms. If X is pointed with base point ε, then $G(X)$ is generated by all displacements $S_x S_\varepsilon$ and X embeds into $G(X)$ as a twisted subgroup (closed under $g \bullet h = gh^{-1}g$) via the *quadratic representation* $Q : X \to G(X)$ defined by $Q(x) = S_x S_\varepsilon$. The image $Q(X)$ is a Loos system under the preceding $\bullet$-operation and the quadratic representation is an isomorphism between X and $Q(X)$. In particular, $Q(x\#y) = Q(x)\#Q(y)$ and $Q(x^{1/2}) = Q(x)^{1/2}$, indeed $Q(x^r) = Q(x)^r$ for all dyadic rationals r. For $x, y \in X$, we write interchangeably as convenient $Q(x)y$ or $Q(x)(y)$.

Remark 2.1. The following useful calculation rules can be found in [14, Chapter II, Lemma 1.1] or [17] or can easily be derived by the methods there:

(i) $Q(Q(x)y) = Q(x)Q(y)Q(x)$.
(ii) $(Q(x))^{-1} = Q(x^{-1})$.
(iii) $(Q(x)y)^{-1} = Q(x^{-1})y^{-1}$.
(iv) $Q(x^r)x^s = x^{2r+s}$.

The following is sometimes useful for checking that a system is a Loos system (see [7]).

Proposition 2.2. *Let* $(X, \bullet, \varepsilon)$ *satisfy* (S1)*,* (S2)*, and* (S3)*. Then* (S4) *is also satisfied if the equation* $x \bullet \varepsilon = b$ *has a unique solution for all* $x \in X$*.*

3. Jordan Algebras

A Jordan algebra is an algebra V equipped with a commutative multiplication xy such that $x(x^2y) = x^2(xy)$ holds for all x, y. We consider only Jordan algebras over a field not of characteristic 2 and equipped with a multiplicative identity e.

For $x \in V$, we define the multiplication operator $L(x) : V \to V$ by $L(x)y = xy$. The *quadratic representation* of the Jordan algebra V is defined by $P(x) = 2L(x)^2 - L(x^2)$ for each $x \in X$. We note that $L(e)$ and $P(e)$ are the identity maps. We frequently write $P(x)y$ for $P(x)(y)$.

An element $u \in V$ is called *invertible* if there exists an element v such that $uv = e$ and $u^2v = u$. We recall [21, Proposition 19.18].

Proposition 3.1. *An element $u \in V$ is invertible if and only if $P(u)$ is invertible. In this case there is a uniquely determined inverse $v = u^{-1}$ given by the formula $v = P(u)^{-1}(u)$. Furthermore, $P(u^{-1}) = P(u)^{-1}$ and $P(u)u^{-1} = u$.*

The basic properties

(i) $(P(w)z)^{-1} = P(w^{-1})z^{-1}$ for w, z invertible,

(ii) $P(P(z)w) = P(z)P(w)P(z)$ (the "fundamental identity")

of the quadratic representation are well-known.

We define a binary operation on $V^{[-1]}$, the invertible elements of V, by $w \bullet z = P(w)(z^{-1})$.

Lemma 3.2. *Properties* (S1), (S2), *and* (S3) *of a Loos system are satisfied by* $(V^{[-1]}, \bullet)$.

Remark 3.3. We note that for a pointed symmetric system $(X, \bullet, e)$ we defined $x^{-1} = S_e x = e \bullet x$. In the Jordan algebra setting this translates to $e \bullet x = P(e)(x^{-1}) = x^{-1}$. Thus inverses in the symmetric system agree with inverses in the Jordan algebra. One notes that $Q(x)(y) = S_x S_e(y) = S_x(y^{-1}) = P(x)(y)$, so $Q(x) = P(x)$. In a similar fashion one sees that the notion of x^n agrees in both systems for all $n \in \mathbb{Z}$ and indeed for all dyadic powers of invertible elements if the Jordan algebra has unique square roots.

4. Generalities on Cones

Let V be a Banach space and let Ω be an open convex cone of V for which $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$. The openness of Ω implies $\Omega - \Omega = V$. We define a partial order $x \leq y$ if $y - x \in \overline{\Omega}$, the closure of Ω. We assume that Ω is a normal cone: that is, there exists a constant K with $||x|| \leq K||y||$ for all $x, y \in \Omega$ with $x \leq y$. For a normal cone Ω, the relation

$$x \leq y \text{ if and only if } y - x \in \overline{\Omega}$$

is a partial order. We write $x < y$ if $y - x \in \Omega$.

A map $F : \Omega \to \Omega$ is *linear* if $F(rx + y) = rF(x) + F(y)$ for all $x, y \in \Omega$ and $r > 0$. Any (continuous) linear map F on Ω extends uniquely to a (continuous) linear map on V; we simply define $F(y - x) = F(y) - F(x)$.

Any member ε of Ω is an order unit for the ordered space $(V, \leq)$. The order unit norm is defined by

$$\|x\|_\varepsilon = \inf\{r > 0 : -r\varepsilon \leq x \leq r\varepsilon\}. \tag{3}$$

The cone Ω is normal if and only if the order unit norm determined by ε is compatible, i.e., determines the topology of V. In this case $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$ with respect to the order unit norm, that is, we may assume without loss of generality that $K = 1$.

A.C. Thompson [20] (cf. [18, 19]) has proved that Ω is a complete metric space with respect to the Thompson part metric defined by

$$d(x, y) = \max\{\log M(x/y), \log M(y/x)\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\}$. He also showed that the metric topology is equal to the relative norm topology.

The Thompson metric on Ω can be alternatively realized as an appropriately defined Finsler length metric, as has appeared in the work of R. Nussbaum [19]. For $x \in \Omega$ and $v \in V = T_x\Omega$, we define the *fundamental Finsler metric* by

$$|v|_x := \inf\{t > 0 : -tx \leq v \leq tx\}. \tag{4}$$

The following theorem is a result of R. Nussbaum [19, Theorem 1.1].

Theorem 4.1. *The Thompson metric $d(x, y)$ agrees with the Finsler distance from x to y for the fundamental Finsler metric,*

$$d(x, y) = \inf\left\{\int_0^1 |\gamma'(t)|_{\gamma(t)} dt : \gamma \in S, \gamma(0) = x, \gamma(1) = y\right\} \tag{5}$$

where S denotes the set of piecewise C^1 maps $\gamma : [0, 1] \to \Omega$

Remark 4.2. Let $\mathrm{Aut}(\Omega)$ denote all invertible continuous linear maps that carry Ω onto itself. Then any $F \in \mathrm{Aut}(\Omega)$ is order-preserving, and hence one sees directly from (4) that the fundamental Finsler metric is preserved, and hence the Finsler distance, which is equal to the Thompson metric, is preserved by F.

5. Analytic Loos Cones

We come now to our main object of interest, analytic Loos symmetric cones. We need the following notion.

Definition 5.1. *A one-parameter Loos homomorphism is a (continuous) map $\alpha : (\mathbb{R}, \bullet) \to (X, \bullet)$ satisfying $\alpha(s \bullet t) = \alpha(s) \bullet \alpha(t)$ for all $s, t \in \mathbb{R}$, where the operation $s \bullet t = 2s - t$ makes $(\mathbb{R}, \bullet)$ a Loos system and $(X, \bullet)$ is a Loos system. If for $x, y \in X$, there exists a unique one-parameter Loos homomorphism α with $\alpha(0) = x$ and $\alpha(1) = y$, then we set $x \#_t y = \alpha(t)$. We note that $\alpha(1/2) = x \#_{1/2} y = x \# y$.*

One-parameter groups play an important role in the theory of Lie groups and one-parameter Loos homomorphisms from the Loos system $(R, \bullet)$ are likewise important in the theory of Loos symmetric spaces and cones.

The following is a slight modification of axioms of a Loos symmetric cone that appeared in [10, 6]. We note in the following that (v) connects the symmetric structure to the linear structure and (vi) to the order structure.

Definition 5.2. *An* analytic pointed Loos symmetric cone *is a triple* $(\Omega, \bullet, e)$ *satisfying*

(i) *Ω is an open cone in a Banach space V for which $\overline{\Omega}$ is a normal cone and $e \in \Omega$.*
(ii) *$(\Omega, \bullet, e)$ is a Loos symmetric system satisfying* (S1)-(S4)*.*
(iii) *the maps $(x, y) \mapsto x \bullet y$ and $(x, y) \mapsto x \# y$ are analytic from $\Omega \times \Omega \to \Omega$.*
(iv) *There exists an analytic diffeomorphism* $\exp : V \to \Omega$ *for which $t \mapsto \exp(tx)$ is a one-parameter Loos homomorphism from $(\mathbb{R}, \bullet) \to (\Omega, \bullet)$ for all $x \neq 0$ in V. All Loos one-parameter homomorphisms in which 0 is mapped to e arise in this way.*
(v) *every basic displacement $Q(x) = S_x S_\varepsilon$ on ω extends to a continuous linear map on V.*
(vi) *$x^{1/2} \leq (\varepsilon + x)/2$ for all $x \in \Omega$.*

We briefly summarize some key results from [6], which treats smooth Loos symmetric cones. Nearly all the results of [6] carry over directly to the analytic setting.

Theorem 5.3. *Let Ω be an open cone in a Banach space V for which $\overline{\Omega}$ is a normal cone and fix $e \in \Omega$. Assume that $(\Omega, \bullet, e)$ is a Loos system satisfying* (S1)-(S4)*, that $(x, y) \mapsto x \bullet y$ is analytic, and that* (5) *and* (6) *are satisfied. Then the triple $(\Omega, \bullet, e)$ is an analytic pointed Loos symmetric cone. Furthermore, the following are satisfied:*

(i) *the map $(t, x, y) \mapsto x \#_t y$ exists and is analytic;*
(ii) *for $x, y \in \Omega$, $x \# y$ is a metric midpoint of x and y with respect to the Thompson metric d, and thus $d(x, y \bullet x) = 2d(x, y)$. Furthermore, $t \mapsto x \#_t y$ for $0 \leq t \leq 1$ is a minimal metric geodesic and has length $d(x, y)$.*

Proof. The Loos operation on a smooth Loos symmetric cone Ω gives rise to a spray (a variant of a connection), which in turn leads to a smooth exponential function $\exp : T\Omega \to \Omega$, which we restrict to $T_e\Omega = V$; see [16] and [6, Thm. 4.2]. This yields the exponential map, in our setting an analytic diffeomorphism $\exp : V \to \Omega$, that appears in (iv) of Definition 5.2. Furthermore, the mapping $t \mapsto \exp(t \log x) = e\#_t x$ for $x \in \Omega$ is an analytic one-parameter Loos homomorphism containing x and e. We also note that $x \mapsto \exp(\frac{1}{2}\log x) = e\#x$ is analytic. From [6, Remark 6.2] we conclude (i).

For (ii), by [6, Corollary 6.6] $x\#y$ is a metric midpoint and $t \mapsto x\#_t y$, $0 \leq t \leq 1$, is a minimal metric geodesic of length $d(x, y)$. Since y reflects x to $y \bullet x$, we conclude $y = x\#(y \bullet x)$. By the metric midpoint property, we have

$$d(x, y \bullet x) = 2d(x, x\#(y \bullet x)) = 2d(x, y).$$

■

We turn to some observations on the displacement group $G(\Omega)$ generated by all maps of the form $S_x S_y$. Since the displacements uniquely extend from Ω to linear maps of V, it is convenient to consider $G(\Omega)$ as maps a group of maps on Ω or V as the context dictates.

Proposition 5.4. *Let $(\Omega, \bullet, e)$ be an analytic pointed Loos symmetric cone. Then $G(\Omega)$ acts transitively on Ω and the fundamental Finsler metric is $G(\Omega)$ invariant.*

Proof. Since for $x, y \in \Omega$, $S_{x\#y}S_x(x) = y$ the action is transitive. The invariance of the fundamental Finsler metric follows from Remark 4.2 (It is also shown in [3, pp. 12]). ■

Remark 5.5. The fact that $G(\Omega)$ acts transitively and its members can be extended to (continuous) linear maps on V is expressed by saying that the cone Ω is *linearly homogeneous.*

6. Uniqueness of Symmetries

We continue with an analytic pointed Loos symmetric cone $(\Omega, \bullet, e)$ equipped with the Thompson metric. By Theorem 4.2 and Steps 1,2, and 5 of its proof in [10] each $Q(x)$ for $x \in \Omega$ is an order-preserving invertible map on V and hence its restriction $Q(x)|_\Omega = S_x S_e$ is a Thompson isometry and the map S_e sending x to x^{-1} is order-reversing and a Thompson isometry. Hence $S_x = (S_x S_e)S_e = Q(x)S_e$ is also order-reversing and a Thompson isometry.

For each $a \in \Omega$, we have the symmetry $S_a b = a \bullet b$ and for each a, b we have the midpoint of symmetry $a\#b$, which is also a metric midpoint (Theorem 5.3(ii)). If we take any three points a, b, c, then $a\#b$ is a metric midpoint and the distance $d(c, (a\#b) \bullet c)$ is equal to $2d(c, (a\#b) \bullet c)$ from Theorem 5.3 (ii). The key point here is that midpoints reflect all other points c so that the distance between c and its reflection is twice the distance between c and $a\#b$. We note also that each S_a is bijective (with inverse S_a), is order-reversing, and is an isometry for the Thompson metric by the preceding paragraph.

Proposition 6.1. *Let $\theta : \Omega \to \Omega$ be an isometry with respect to the Thompson metric. Then θ preserves midpoints, i.e., $\theta(a\#b) = \theta(a)\#\theta(b)$ for all $a, b \in \Omega$.*

Proof. The proof of a more general result appears as a result entitled "Lemma" on page 3852 of [15]. The paragraphs before this proposition establish that the hypotheses of that Lemma are satisfied. ∎

Theorem 6.2. *Let $F : \Omega_1 \to \Omega_2$ be a smooth map that is a Loos homomorphism and suppose $F(e_1) = e_2$, i.e. F preserves the pointings. Let $dF : V_1 \to V_2$ be the derivative map from $T_{e_1}\Omega_1 = V_1$ to $T_{e_2}\Omega_2 = V_2$. Then the following diagram is communtative.*

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{F} & \Omega_2 \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ V_1 & \xrightarrow{dF} & V_2 \end{array}$$

Proof. The proof is essentially the same as in the Lie group/Lie algebra case. The fact F is a Loos homomorphism means that it also preserves $x\#_t y$ for all t, and hence one-parameter Loos maps. The latter through e at $t = 0$ are parameterized by the tangent vectors at the distinguished point with $v \in V$ corresponding to $t \mapsto \exp(tv)$. The one-parameter Loos map $t \mapsto F(\exp(tv))$ has derivative at 0 equal to $dF(v)$, so $F(\exp(tv) = \exp(tdF(v)) = \exp(dF(tv))$, which yields the commutativity of the diagram. ∎

Corollary 6.3. *Let $(\Omega, \bullet, e)$ be an analytic pointed Loos symmetric cone. Let $\theta : \Omega \to \Omega$ be a smooth involution which is an isometry with respect to the Thompson metric and with $\theta(e) = e$, $d\theta_e : V \to V = -1$. Then θ is uniquely determined as $\theta(a) = a^{-1}$ for all $a \in \Omega$.*

Proof. From Proposition 6.1 and the preceding commutative diagram we have $\theta(v) = \exp(d\theta(\log v)) = \exp(-\log v) = v^{-1}$ for $v \in \Omega$. ∎

Definition 6.4. *A map $\theta : \Omega \to \Omega$ is said to be a point symmetry at a if $\theta(a) = a$, $d\theta(a) : T_a\Omega \to T_a\Omega = -1$, $\theta \circ \theta = 1_\Omega$, and θ is a metric isometry with respect to the Thompson metric.*

By the linear homogeneity of Ω via isometries, in particular the fact the $Q(a^{1/2})(e) = a$, one can transfer the uniqueness of point symmetries to all points of Ω.

Corollary 6.5. *The point symmetries $\{S_a : a \in \Omega\}$ are uniquely determined.*

In [3] Chu considers open cones that are linearly homogeneous and have a unique point symmetry at each point. His notion of a point symmetry is close to, but slightly stronger, than ours.

7. From JB-algebras to Symmetric Cones

We recall some basic facts about JB-algebras. Good resources for these and much more about JB-algebras and related material can be found in [4, Chapter 3], [21] (particularly Sections 21 and 22), [16, Example VI.6], and [3, Section 2]. Also the material of this section overlaps considerably with short treatments of symmetric cones in JB-algebras given in [10, 6].

In the following we consider only Jordan algebras over $\mathbb{R}$ equipped with a multiplicative identity e.

A JB-algebra V is a real Jordan algebra with unit e endowed with a complete norm $||\cdot||$ such that

$$\begin{aligned} ||zw|| &\leq ||z||\,||w||, \\ ||z^2|| &= ||z||^2, \\ ||z||^2 &\leq ||z^2+w^2||. \end{aligned}$$

For $z \in V$, a JB-algebra, let $C(z)$ denote the smallest closed Jordan subalgebra of V containing $\{e, z\}$. Then $C(z)$ is an associative (and, of course, commutative) algebra [4, Section 3.2]. The *spectrum* $\mathrm{Spec}(z)$ of z is the set of all $\lambda \in \mathbb{R}$ such that $z - \lambda e$ does not have an inverse in $C(z)$. Then $V^+ = \{z^2 : z \in V\} = \{z \in V\colon\ \mathrm{Spec}(z) \subseteq [0,\infty)\}$ is closed proper cone inducing a closed partial order on V, e is an order unit, and its order unit norm agrees with the original JB-algebra norm

$$|z|_e := \inf\{t > 0 : te \pm z \geq 0\}.$$

Hence the cone V^+, the cone of *positive elements*, is normal. See [4, Section 3.3].

The set $\Omega = \{z \in V\colon\ \mathrm{Spec}(z) \subseteq]0,\infty)\}$ is the interior of V^+. It is an open cone (closed under addition and multiplication by positive scalars), dense in V^+, and consists of the invertible elements of V^+. The JB-algebra V has an exponential function exp defined by the usual exponential power series (this depends on the associativity of each $C(z)$), and Ω can be alternatively realized as

$$\Omega = \exp(V) := \{\exp(z) : z \in V\}.$$

The mapping $\exp : V \to \Omega$ is an analytic diffeomorphism with inverse $\log : \Omega \to V$. See [21, Corollary 21.22].

We recall for $w \in V$ the multiplication operator $L(w)$ defined by $L(w)(z) = wz$ and the quadratic representation defined by $P(z) = 2L(z)^2 - L(z^2)$ for each $z \in V$. Since Ω consists of the invertible elements of V^+ and since (from Section 3) $P(w)$ preserves invertible elements if w is invertible, we have for $w \in \Omega$

that $P(w)(\Omega)$ consists of invertible elements, is a connected set and contains $P(w)e = w^2 = \exp(2\log w) \in \Omega$, it must be contained in the connected component Ω in the invertible elements. Similarly $P(w)^{-1} = P(w^{-1})$ carries Ω into itself, so $P(w)$ is a linear automorphism of V preserving Ω (and V^+). It then follows from Lemma 3.2 that the Loos operation $w \bullet z = P(w)z^{-1}$ satisfies (S1), (S2), and (S3). We note that any $z \in \Omega$ has a unique square root in Ω, namely $z^{1/2} = \exp(\frac{1}{2}\log z)$. In light of Proposition 2.2 and Remark 3.3 we conclude that axiom (S4) of a Loos system holds, and thus $(\Omega, \bullet, e)$, where $w \bullet z = P(w)(z^{-1})$, is a Loos system.

We note that $z \to z^{-1} = \exp(-\log z)$ is analytic on Ω since exp and log are analytic. The Loos operation on Ω is given by

$$w \bullet z = 2w(wz^{-1}) - w^2 z^{-1}$$

is then analytic since Jordan multiplication is continuous and bilinear. By (S3) $P(w) = S_w S_e$ is an automorphism of $(\Omega, \bullet)$. Using this fact one sees that $w\#z = P(w^{1/2})(P(w^{-1/2})z)^{1/2}$, and hence is also analytic.

For the Loos system Ω we have by Remark 3.3 that $Q(z) = P(z)|_\Omega$, and so $Q(x)$ is the restriction of a linear map.

Finally we need to check the condition $z^{1/2} \leq \frac{1}{2}(e+z)$. Working inside communtative, associative subalgebra $C(z)$, the smallest closed subalgebra containing $\{e, z\}$, we have $0 \leq (z^{1/2} - e)^2 = z - 2z^{1/2} + e$, from which the desired inequality immediately follows.

From the above combined with Remark 3.3 we obtain the following theorem.

Theorem 7.1. *Let V be a unital JB-algebra. Then V induces on the open cone Ω of invertible positive elements the structure of an analytic Loos symmetric cone. Furthermore, the Thompson metric on Ω is the Finsler distance for the fundamental Finsler metric and is invariant under* $\mathrm{Aut}(\Omega)$*. The Loos system of the cone and the multiplication of the Jordan algebra are related by $Q(z) = P(z)$ for all $z \in \Omega$.*

8. From Loos Symmetric Cones to JB-algebras

There exists a well-known one-to-one correspondence between finite dimensional Euclidean Jordan algebras with identity and finite-dimensional symmetric cones (ones that are linearly homogeneous and self-dual) that was established independently by E.B. Vinberg [22, 23] and M. Koecher [5]. The generalization of the construction of the Jordan algebra from the symmetric cone in the Banach setting has been obtained in recent work by C.-H. Chu [3]. In his paper Chu gives a construction for a JB-algebra from a linearly homogeneous Finsler symmetric cone. For Chu, a Finsler symmetric cone Ω is a normal cone in which there is for each p a unique $s_p : \Omega \to \Omega$ which is a ν-isomorphism for a given Finsler metric ν on Ω, satisfies $(s_p)^2$ is the identity map on Ω, $s_p(p) = p$ and p is an isolated fixed-point. We have seen in the sections on Analytic Loos Cones and Uniqueness of Symmetries that the analytic Loos symmetric cones we have been considering in this paper are Finsler symmetric cones with respect to the fundamental Finsler metric. We give a mildly modified construction for the JB-algebra associated with $(\Omega, \bullet, e)$, an analytic pointed Loos symmetric cone. The approach leans heavily on Chu's work [3, Theorem 3.2] and earlier work of Neeb [16, Example III.9].

Let $(\Omega, \bullet, e)$, be an analytic pointed Loos symmetric cone, where Ω is an open cone in V. For each $v \in V$ identified with $T_e\Omega$, we define a linear vector field X_v on V as follows. Let $\alpha_v : \mathbb{R} \to \Omega$ be defined by $\alpha_v(t) = \exp(tv)$, a one-parameter Loos homomorphism. Then $\alpha_v(0) = e$ and $\alpha_v'(0) = v$. We define a vector field X_v on V by

$$X_v(p) := \frac{d}{dt}|_{t=0} Q(\exp(\frac{1}{2}tv))(p).$$

We can define this on all of V, not just Ω, since $Q(x)$ extends to a linear map on V for all $x \in \Omega$. We note for $p = e$, $\alpha_v(t) = Q(\exp(\frac{1}{2}tv))(e)$, so $X_v(e) = v$.

We set $\mathfrak{p} = \{X_v : v \in V\}$ and let $\mathfrak{g}$ be the Lie algebra generated by $\mathfrak{p}$ within the Lie algebra of all analytic vector fields. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$ consisting of all analytic vector fields Y with $Y_e = 0$. Furthermore, one has that

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \qquad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}.$$

We can now define the Jordan multiplication: $uv = X_u(v)$. We obtain the following theorem.

Theorem 8.1. *Let $(\Omega, \bullet, e)$ be an analytic pointed Loos symmetric cone. Then one can define a Jordan multiplication on the containing Banach space V so that a JB-algebra results with unit e and with $\overline{C} = \{u^2 : u \in V\}$.*

We refer to [3, Theorem 3.2] for this result, where the proof there needs mild modifications to our setting. We do not go through the proof again in detail, since we are primarily interested in applying Chu's work to the theory of Loos symmetric cones.

9. Uniqueness of Constructions

From preceding section we know that for the JB-algebra $(V, \cdot)$ constructed from an analytic pointed Loos symmetric cone $(C, \bullet)$, its set of squares is $\overline{C}$. It must then be the case that the Loos cone of V is $(C, *)$ for some Loos operation $*$. By Corollary 6.2 the point symmetries S_a, $a \in C$ of C are uniquely determined, so they must agree with those originally given on C. Hence $a * b = S_a b = a \bullet b$.

Conversely suppose that we start with a JB-algebra $(V, \cdot)$ with unit. By Theorem 7.1 the cone $C = \exp V$ becomes an analytic pointed Loos symmetric cone with distinguished point e and $Q(z) = P(z)$ for all $z \in \Omega$. For $z \in C$, we have

$$Q(z)(e) = P(z)(e) = 2z(ze) - z^2 e = z^2$$

so that the squaring map $z \to z^2$ is determined by the Loos quadratic map $Q(z)$ applied to e. (Alternatively this says that the same z^2 is obtained whether computed in the Loos system or the Jordan algebra.) But once one knows the squaring map, the whole Jordan multiplication (if it exists) is determined since first for $a, b \in C$,

$$ab = \frac{1}{2}((a+b)^2 - a^2 - b^2),$$

and then for any $u = a - b$ and $v = c - d$ for $a, b, c, d \in C$,

$$uv = ac - ad - bc + ad.$$

Next we consider the JB-algebra $(V, \circ)$ arising from C. Then $(V, \circ)$ gives rise to a Loos system on C and it must be the same one as the one giving rise to $(V, \circ)$ by the first paragraph. Thus the Jordan algebra of $(V, \circ)$, which is uniquely determined by the Loos system C from the preceding paragraph, must be the same in both cases. Thus we obtain a one-to-one correspondence between the Jordan algebras of unital JB-algebras and analytic pointed Loos symmetric cones.

10. Closing Comments

Substantial progress has already been achieved in the study of Loos symmetric cones [10, 6]. Sample results include the Loewner-Heinz inequality, the harmonic-geometric-arithmetic mean inequality, the expansiveness of the exponential map, and considerable more. In light of this paper one can consider Loos symmetric cones as the positive cones of unital JB-algebras. This adds a concrete exponential function and the machinery of JB-algebras to bring to bear on the theory and should be a significant aid to further development of the theory. In the converse direction, the more detailed description of the positive cone of a unital JB-algebra could provide additional useful tools to their study.

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