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On (P, Q)-Outer Generalized Inverses and their Stability of Pseudo Spectrum

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Abstract. In this paper, we introduce some basic definitions, terminologies, and results which are related to (P, Q)-resolvent set and $(P, Q) - \varepsilon$-pseudo spectrum, and the major goal is to show the non-emptiness of these sets. Some properties of the outer generalized inverses. Additionally, we inquire additive findings for $(P, Q) - \varepsilon$-pseudo spectrum in Banach spaces.

Keywords: Outre generalized inverse; $(P, Q) - \varepsilon$-pseudo spectra; operator; Banach spaces.

1. Introduction

The concept of the generalized inverses was first introduced by I. Fredholm in 1903. He proposed a generalized inverse of an integral operator, called pseudoinverse. The generalized inverses of differential operators were brought up in D. Hilbert's discussion of the generalized Green's functions in 1904. For a history of the generalized inverses of differential operators, the reader is referred to W. Reid's in 1931.

The iterative methods for solving nonlinear equations [6, 12] and implementation to statistics [9]. In specific, outer generalized inverses move a remarkable part in stable approximations of ill-posed problems and in nonlinear and linear problems inclosed rank-deficient generalized inverses [13]. The researchers have determined much numerical procedure for computing the $T^2_{(P,Q)}$ generalized inverse in the literature (see [8, 16, 21]).

It is worth recalling the notion of pseudo spectrum in the complex setting which was addressed by several authors. This notion was first introduced in 1967 by Varah [20]. Trefethen [18, 19] applied pseudo spectrum to evoked approximate eigenvalues and operators, and matrices. Sundry authors have scrupulous this notion such as Wolff [22] and Davies [7] who first came up with the notion of approximation pseudo spectrum for linear operators. Furthermore, inspired by the notion of pseudo spectrum, A. Ammar, A. Jeribi and B. Saadaoui in their works [1, 2, 3, 4, 5, 14], could provide these consequence for the essential pseudo spectra of multivaued linear operator. Let X be a Banach spaces. Let $\mathcal{B}(X)$ denotes the set of all bounded linear operators defined on the Banach space X. Let P the projection operator in $\mathcal{B}(X)$. For $T \in X$, it is proved that the interior of the level set of $(P, I-P)-\varepsilon$-pseudo spectrum of T is empty in the unbounded component of $(P, I-P)$-resolvent set of T. The main aim of this paper is to investigate and classify the possible cases, when the stability of the $(P,Q)-\varepsilon$-pseudo resolvent map is not constant in an open connected subset of the $(P,Q)-\varepsilon$-pseudo resolvent set. We introduce some basic definitions, terminologies, and results which are related to $(P, I-P)$-resolvent set. Consider two idempotent elements $P, Q \in \mathcal{B}(X)$ i.e. $P^2 = P$ and $Q^2 = Q$.

Definition 1.1. *Let $T \in \mathcal{B}(X)$. An operator $S \in \mathcal{B}(X)$ satisfying,*

$$STS = S, \quad ST = P \quad \text{and} \quad I - TS = Q$$

will be called a (P,Q) outer generalized inverse of T and it is denoted by $T^{(2)}_{P,Q}$.

Definition 1.2. *For an element $T \in \mathcal{B}(X)$, the (P,Q)-resolvent set is defined as*

$$\rho^{(2)}_{(P,Q)}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \text{ exist}\Big\}.$$

The complement of the set $\rho^{(2)}_{(P,Q)}(T)$ is called (P,Q)-spectrum and it is denoted by $\sigma^{(2)}_{(P,Q)}(T)$.

From now onwards, we consider the idempotent $P \neq 0$ and $P \neq I$ and we fix the operator $Q = I - P$. If $\lambda \in \rho^{(2)}_{(P,Q)}(T)$, then we denote $(\lambda - T)^{(2)}_{P,Q}$ by $R_T(\lambda)$. For given $T \in \mathcal{B}(X)$, if $R_T(\lambda)$ exists for some $\lambda \in \mathbb{C}$, then from Definition 1.2,

$$R_T(\lambda)(\lambda - T) = P \quad \text{and} \quad (\lambda - T)R_T(\lambda) = P \tag{1}$$

By Eq. (1), $TP = PT$. Consequently, if $TP \neq PT$ then $\sigma^{(2)}_{(P,Q)}(T) = \mathbb{C}$. Because of this reason, in the rest of the paper we assume $TP = PT$.

Our aim in this work is to show some properties of $(P, Q) - \varepsilon$-pseudo spectrum of linear operators in Banach spaces. For the geometric understanding of(P, Q)-outer generalized inverse, because of the inequalities in $(P, Q) - \varepsilon$-pseudo spectrum and in order to understand it and show the relationship between their $(P, Q) - \varepsilon$-pseudo spectrum and the (P, Q)-spectrum. An example is constructed to show that Theorem 3.7 is verify (Example 3.8).

The rest of this paper is organized as follows. In Section 2, we establish a unified representation Theorem for outer generalized inverses of operators on Banach spaces. The main aim of Section 3 is to characterize the $(P, Q) - \varepsilon$-pseudo spectrum of linear operator and investigate some properties.

2. (P, Q)-outer Generalized Inverse

This section contains auxiliary results and properties of (P, Q)-outer generalized inverse that will be needed in the sequel.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$ and $P, Q \in \mathcal{B}(X)$. Then, the following statements are equivalent:*

(i) $(\lambda - T)^{(2)}_{P,Q}$ *exists.*
(ii) $(I - Q)(\lambda - T) = (I - Q)(\lambda - T)P$ *and there exists some* $S \in \mathcal{B}(X)$ *such that* $PS = S$, $SQ = 0$, $S(\lambda - T)P = P$ *and* $(\lambda - T)S = I - Q$. *Moreover, if* $(\lambda - T)^{(2)}_{P,Q}$ *exists, then it is unique.*

Proof. (i)$\Longrightarrow$(ii) If $(\lambda - T)^{(2)}_{P,Q}$ exists, then we can take $S = (\lambda - T)^{(2)}_{P,Q}$.

(ii)$\Longrightarrow$(i) We calculation in the following way:

$$S(\lambda - T) = PS(I - Q)(\lambda - T) = PS(I - Q)(\lambda - T)P = S(\lambda - T)P = P$$

and

$$S(\lambda - T)S = PS = S.$$

To demonstrate the singularity, suppose that there exist two literal generalized inverses $S, R \in \mathcal{B}(X)$ of $(\lambda - T)$ such that $S(\lambda - T) = R(\lambda - T) = P$, $(\lambda - T)S = R = I - Q$. Then, we have

$$R = R(\lambda - T)R = R(\lambda - T)S = S.$$

∎

Theorem 2.2. *Suppose that* $T^{(2)}_{P,Q}$ *and* $R^{(2)}_{P,Q}$ *exist. If*

$$T^{(2)}_{P,Q}R + R^{(2)}_{P,Q}T + I = 0, \qquad TR^{(2)}_{P,Q} + RT^{(2)}_{P,Q} + I = 0,$$

then $(T + R)^{(2)}_{P,Q}$ *exists and* $(T + R)^{(2)}_{P,Q} = T^{(2)}_{P,Q} + R^{(2)}_{P,Q}$.

Proof. Using the fact that $T^{(2)}_{P,Q}$ and $R^{(2)}_{P,Q}$ exists, we have

$$\begin{aligned}
&(T^{(2)}_{P,Q} + R^{(2)}_{P,Q})(T+R)(T^{(2)}_{P,Q} + R^{(2)}_{P,Q}) \\
&= T^{(2)}_{P,Q} + PR^{(2)}_{P,Q} + T^{(2)}_{P,Q}RT^{(2)}_{P,Q} + T^{(2)}_{P,Q}(I-Q) + R^{(2)}_{P,Q}(I-Q) + R^{(2)}_{P,Q}TR^{(2)}_{P,Q} \\
&\quad + PT^{(2)}_{P,Q} + R^{(2)}_{P,Q} \\
&= T^{(2)}_{P,Q} + R^{(2)}_{P,Q} + T^{(2)}_{P,Q}RT^{(2)}_{P,Q} + T^{(2)}_{P,Q} + R^{(2)}_{P,Q} + R^{(2)}_{P,Q} + R^{(2)}_{P,Q}TR^{(2)}_{P,Q} \\
&\quad + T^{(2)}_{P,Q} + R^{(2)}_{P,Q} \\
&= T^{(2)}_{P,Q} + R^{(2)}_{P,Q} + T^{(2)}_{P,Q}(RT^{(2)}_{P,Q} + I) + R^{(2)}_{P,Q}(I + TR^{(2)}_{P,Q}) + T^{(2)}_{P,Q} + R^{(2)}_{P,Q} \\
&= T^{(2)}_{P,Q} + R^{(2)}_{P,Q} + T^{(2)}_{P,Q}(-TR^{(2)}_{P,Q}) + R(-RT^{(2)}_{P,Q}) + T^{(2)}_{P,Q} + R^{(2)}_{P,Q} \\
&= T^{(2)}_{P,Q} + R^{(2)}_{P,Q} - PR^{(2)}_{P,Q} - PT^{(2)}_{P,Q} + T^{(2)}_{P,Q} + R^{(2)}_{P,Q} \\
&= T^{(2)}_{P,Q} + R^{(2)}_{P,Q},
\end{aligned}$$

$$\begin{aligned}
&(T^{(2)}_{P,Q} + R^{(2)}_{P,Q})(T+R) \\
&= T^{(2)}_{P,Q}T + T^{(2)}_{P,Q}R + R^{(2)}_{P,Q}T + R^{(2)}_{P,Q}R \\
&= P + PT^{(2)}_{P,Q}R + PR^{(2)}_{P,Q}T + P \\
&= P + P(T^{(2)}_{P,Q}R + R^{(2)}_{P,Q}T + I) \\
&= P
\end{aligned}$$

and also

$$\begin{aligned}
&(T+R)(T^{(2)}_{P,Q} + R^{(2)}_{P,Q}) \\
&= TT^{(2)}_{P,Q} + RT^{(2)}_{P,Q} + TR^{(2)}_{P,Q} + RR^{(2)}_{P,Q} \\
&= (I-Q) + RT^{(2)}_{P,Q} + TR^{(2)}_{P,Q} + (I-Q) \\
&= (I-Q) + RT^{(2)}_{P,Q}(I-Q) + TR^{(2)}_{P,Q}(I-Q) + (I-Q) \\
&= (I-Q) + (RT^{(2)}_{P,Q} + TR^{(2)}_{P,Q} + I)(I-Q) \\
&= (I-Q).
\end{aligned}$$

Thus, we proved $(T+R)^{(2)}_{P,Q} = T^{(2)}_{P,Q} + R^{(2)}_{P,Q}$. ■

Theorem 2.3. *Let $T \in \mathcal{B}(X)$ and let $P, Q \in \mathcal{B}(X)$ be such that $(\lambda - T)^{(2)}_{P,Q}$ exists. Then, for $S \in \mathcal{B}(X)$ the following statements are equivalent:*

(i) *There exists the generalized inverse $S^{(2)}_{P,Q}$.*

(ii) *$S(\lambda - T)^{(2)}_{P,Q}(\lambda - T) = (\lambda - T)(\lambda - T)^{(2)}_{P,Q}S$ and there exists the generalized inverse $(S(\lambda - T)^{(2)}_{P,Q}(\lambda - T))^{(2)}_{P,Q}$*

(iii) *$S(\lambda - T)^{(2)}_{P,Q}(\lambda - T) = (\lambda - T)(\lambda - T)^{(2)}_{P,Q}S$ and $I + (\lambda - T)^{(2)}_{P,Q}(S - (\lambda - T))$ is invertible.*

(iv) $S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)=(\lambda-T)(\lambda-T)^{(2)}_{P,Q}S$ *and* $I+(S-(\lambda-T))(\lambda-T)^{(2)}_{P,Q}$ *is invertible.*

Moreover, if previous statements are valid, then

$$\begin{aligned} S^{(2)}_{P,Q} &= \Big[I+(\lambda-T)^{(2)}_{P,Q}(S-(\lambda-T))\Big]^{-1}(\lambda-T)^{(2)}_{P,Q} \\ &= (\lambda-T)^{(2)}_{P,Q}\Big[I+(\lambda-T)^{(2)}_{P,Q}(S-(\lambda-T))\Big]^{-1}. \end{aligned}$$

Proof. (i)$\Longrightarrow$(ii) We immediately obtain $S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)=SS^{(2)}_{P,Q}S=(\lambda-T)(\lambda-T)^{(2)}_{P,Q}S$. We also have

$$S^{(2)}_{P,Q}(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))S^{(2)}_{P,Q}=S^{(2)}_{P,Q}(SS^{(2)}_{P,Q}S)S^{(2)}_{P,Q}=S^{(2)}_{P,Q}.$$

Since

$$S^{(2)}_{P,Q}(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))=S^{(2)}_{P,Q}S^{(2)}_{P,Q}S=S^{(2)}_{P,Q}S=P$$

and

$$(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))S^{(2)}_{P,Q}=SS^{(2)}_{P,Q}SS^{(2)}_{P,Q}=I-Q,$$

we conclude that the equality $(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))^{(2)}_{P,Q}=S^{(2)}_{P,Q}$.

(ii)$\Longrightarrow$(i) Let $R=(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))^{(2)}_{P,Q}$. We have $RS(\lambda-T)^{(2)}_{P,Q}(\lambda-T)=P=(\lambda-T)^{(2)}_{P,Q}(\lambda-T)$ and $S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R=I-Q=(\lambda-T)(\lambda-T)^{(2)}_{P,Q}=(\lambda-T)(\lambda-T)^{(2)}_{P,Q}SR$. Hence, we obtain

$$\begin{aligned} RSR &= R(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R)SR(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R) \\ &= R(\lambda-T)(\lambda-T)^{(2)}_{P,Q}(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R)=R(\lambda-T)(\lambda-T)^{(2)}_{P,Q} \\ &= R(\lambda-T)(\lambda-T)^{(2)}_{P,Q}SR=R. \end{aligned}$$

Also, we have

$$\begin{aligned} RS &= RS(\lambda-T)^{(2)}_{P,Q}(\lambda-T)RS=R(\lambda-T)(\lambda-T)^{(2)}_{P,Q}S \\ &= RS(\lambda-T)^{(2)}_{P,Q}(\lambda-T)=P \end{aligned}$$

and

$$SR=SRS(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R=S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)R=I-Q.$$

It follows that $S^{(2)}_{P,Q}=(S(\lambda-T)^{(2)}_{P,Q}(\lambda-T))^{(2)}_{P,Q}$.

(i) and (ii)$\Longrightarrow$(iii) We compute

$$\begin{aligned}
&(I+(\lambda-T)^{(2)}_{P,Q}S-(\lambda-T)^{(2)}_{P,Q}(\lambda-T))(S^{(2)}_{P,Q}(\lambda-T)+I\\
&-(\lambda-T)^{(2)}_{P,Q}(\lambda-T))\\
=\ &S^{(2)}_{P,Q}(\lambda-T)+I-(\lambda-T)^{(2)}_{P,Q}(\lambda-T)+(\lambda-T)^{(2)}_{P,Q}SS^{(2)}_{P,Q}(\lambda-T)\\
&+(\lambda-T)^{(2)}_{P,Q}S-(\lambda-T)^{(2)}_{P,Q}S(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\\
&-(\lambda-T)^{(2)}_{P,Q}(\lambda-T)S^{(2)}_{P,Q}(\lambda-T)-(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\\
&+(\lambda-T)^{(2)}_{P,Q}(\lambda-T)(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\\
=\ &I
\end{aligned}$$

Analogously,

$$(S^{(2)}_{P,Q}(\lambda-T)+I-(\lambda-T)^{(2)}_{P,Q}(\lambda-T))(I+(\lambda-T)^{(2)}_{P,Q}S-(\lambda-T)^{(2)}_{P,Q}(\lambda-T))=I$$

and consequently

$$(I+(\lambda-T)^{(2)}_{P,Q}(S-(\lambda-T)))^{-1}=S^{(2)}_{P,Q}(\lambda-T)+I-(\lambda-T)^{(2)}_{P,Q}(\lambda-T).$$

(iii)$\Longrightarrow$(i) Notice that we have

$$\begin{aligned}
R&=I+(\lambda-T)^{(2)}_{P,Q}(S-(\lambda-T))\\
&=I-(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\\
&\quad+(\lambda-T)^{(2)}_{P,Q}(\lambda-T)[(\lambda-T)^{(2)}_{P,Q}S](\lambda-T)^{(2)}_{P,Q}(\lambda-T),
\end{aligned}$$

since we know that $(\lambda-T)^{(2)}_{P,Q}(\lambda-T)[(\lambda-T)^{(2)}_{P,Q}S](\lambda-T)^{(2)}_{P,Q}(\lambda-T)=(\lambda-T)^{(2)}_{P,Q}S$ is invertible. We notice

$$\begin{aligned}
K&=R^{-1}(\lambda-T)^{(2)}_{P,Q}\\
&=\Big[I-(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\\
&\quad+\Big((\lambda-T)^{(2)}_{P,Q}(\lambda-T)[(\lambda-T)^{(2)}_{P,Q}S](\lambda-T)^{(2)}_{P,Q}(\lambda-T)\Big)^{-1}\Big](\lambda-T)^{(2)}_{P,Q}\\
&=\Big[(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\big(\lambda-T)^{(2)}_{P,Q}S\big)(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\Big]^{-1}(\lambda-T)^{(2)}_{P,Q}.
\end{aligned}$$

We prove $K=S^{(2)}_{P,Q}$. Notice that

$$\begin{aligned}
KSK&=\Big((\lambda-T)^{(2)}_{P,Q}S\Big)^{-1}(\lambda-T)^{(2)}_{P,Q}S\\
&\quad\cdot\Big[(\lambda-T)^{(2)}_{P,Q}(\lambda-T)((\lambda-T)^{(2)}_{P,Q}S)(\lambda-T)^{(2)}_{P,Q}(\lambda-T)\Big]^{-1}(\lambda-T)^{(2)}_{P,Q}\\
&=K.
\end{aligned}$$

Also,

$$KS = (\lambda - T)^{(2)}_{P,Q}(\lambda - T)$$

and

$$\begin{aligned} SK &= S\Big((\lambda - T)^{(2)}_{P,Q}S\Big)^{-1}(\lambda - T)^{(2)}_{P,Q} \\ &= (\lambda - T)(\lambda - T)^{(2)}_{P,Q}S\Big((\lambda - T)^{(2)}_{P,Q}S\Big)^{-1}(\lambda - T)^{(2)}_{P,Q} \\ &= (\lambda - S)(\lambda - T)^{(2)}_{P,Q}. \end{aligned}$$

Finally, if (i), (ii) and (iii) are satisfied, from the part (iii)$\Longrightarrow$(i), it follows that

$$\Big(I + (\lambda - T)^{(2)}_{P,Q}(S - (\lambda - T))\Big)^{-1}(\lambda - T)^{(2)}_{P,Q} = S^{(2)}_{P,Q}.$$

The proof of all cases involving the part (iv) are similar. ∎

Lemma 2.4. *Let $T \in \mathcal{B}(X)$. If $\lambda \in \rho(T)$, then $\lambda \in \rho^{(2)}_{(P,Q)}(T)$.*

Proof. It is easy to see that $R_T(\lambda) = P(\lambda - T)^{-1}$ for any $\lambda \in \rho^{(2)}_{(P,Q)}(T)$. ∎

Theorem 2.5. *The set $\rho^{(2)}_{(P,Q)}(T)$ is a nonempty open subset of $\mathbb{C}$.*

Proof. By Lemma 2.4, $\rho^{(2)}_{(P,Q)}(T)$ is nonempty. Take $\mu \in \rho^{(2)}_{(P,Q)}(T)$, for any $\lambda \in \mathbb{C}$ satisfies

$$\Big|\mu - \lambda\Big| < \frac{1}{\|R_T(\mu)\|}$$

we have $I - R_T(\mu)(\mu - \lambda)$ is invertible. Then $I - R_T(\mu)((\mu - T) - (\lambda - T))$ is invertible. Since

$$(\lambda - T)R_T(\mu)(\mu - T) = (\lambda - T)P = P(\lambda - T) = (\mu - T)R_T(\mu)(\lambda - T),$$

hence by Theorem 2.3, we get $\lambda \in \rho^{(2)}_{(P,Q)}(T)$. ∎

3. Stability of $(P,Q) - \varepsilon$-pseudo Spectrum

The $(P,Q)-\varepsilon$-pseudo spectrum were studied in [11, 17]. Let $\varepsilon > 0$ and $T \in \mathcal{B}(X)$. The $(P,Q) - \varepsilon$-pseudo spectrum is defined as

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \text{ does not exist or } \Big\|(\lambda - T)^{(2)}_{P,Q}\Big\| \geq \varepsilon\Big\}.$$

By convention, we write $\|R_T(\lambda)\| = \infty$ if $R_T(\lambda)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma^{(2)}_{(P,Q)}(T)$. It is well known that $\rho^{(2)}_{(P,Q)-\varepsilon}(T)$ for any

$T \in \mathcal{B}(X)$ is a nonempty open subset, the following remark prove the same for $(P,Q)-\varepsilon$-pseudo resolvent set.

Remark 3.1. Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then we have the following statements:

(i) If $\varepsilon_1 < \varepsilon_2$, then $\sigma^{(2)}_{(P,Q)-\varepsilon_1}(T) \subset \sigma^{(2)}_{(P,Q)-\varepsilon_2}(T)$.

(ii) $\sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \sigma^{(2)}_{(P,Q)}(T) \cup \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\varepsilon}\right\}$.

Proposition 3.2. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then $(P,Q)-\varepsilon$-pseudo spectra verifies the following properties:*

(i) $\sigma^{(2)}_{(P,Q)-\varepsilon}(T) \neq \emptyset$.

(ii) $\bigcap_{\varepsilon>0} \sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \sigma^{(2)}_{(P,Q)}(T)$.

Proof. (i) We argue by contradiction. Suppose that $\sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \emptyset$. Then

$$\rho^{(2)}_{(P,Q)-\varepsilon}(T) = \mathbb{C}.$$

It means exactly that

$$\rho^{(2)}_{(P,Q)}(T) \bigcap \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| \leq \frac{1}{\varepsilon}\right\} = \mathbb{C}$$

and $\rho^{(2)}_{(P,Q)}(T) = \mathbb{C}$. Hence,

$$\left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| \leq \frac{1}{\varepsilon}\right\} = \mathbb{C}.$$

Consider the function

$$\begin{aligned} \psi : \mathbb{C} &\longrightarrow \mathcal{B}(X) \\ \lambda &\longmapsto (\lambda - T)^{(2)}_{(P,Q)}. \end{aligned}$$

Since ψ is analytic on $\mathbb{C}$ and for every $\lambda \in \mathbb{C}$, we have

$$\|\psi(\lambda)\| = \|R_T(\lambda)\| \leq \frac{1}{\varepsilon}.$$

Then ψ is an entire bounded function. Therefore, using Liouville theorem, we obtain that ψ is constant. It follows that $R_T(\lambda)$ is null, which is a contradiction.

(ii) Let $\bigcap_{\varepsilon>0} \sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \bigcap_{\varepsilon>0} \left(\sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\varepsilon}\right\}\right)$. Then

$$\bigcap_{\varepsilon>0} \sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \sigma^{(2)}_{(P,Q)}(T) \bigcup \left(\bigcap_{\varepsilon>0} \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\varepsilon}\right\}\right).$$

For the inclusion in the direct sense: Let $\lambda \in \mathbb{C}$ such that for all $\varepsilon > 0$

$$\|R_T(\lambda)\| > \frac{1}{\varepsilon}.$$

Consequently,

$$\lim_{\varepsilon \longrightarrow 0^+} \|R_T(\lambda)\| = \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\varepsilon} = +\infty.$$

For the other inclusion: Let $\lambda \in \mathbb{C}$ such that

$$\|R_T(\lambda)\| = +\infty > \frac{1}{\varepsilon} \text{ for all } \varepsilon > 0.$$

We deduce that,

$$\begin{aligned}\bigcap_{\varepsilon>0} \sigma^{(2)}_{(P,Q)-\varepsilon}(T) &= \sigma^{(2)}_{(P,Q)}(T) \bigcup \left(\bigcap_{\varepsilon>0} \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| = +\infty\} \right) \\ &= \sigma^{(2)}_{(P,Q)}(T).\end{aligned}$$

■

In the last theorem, we study results of stability and properties of $(P,Q)-\varepsilon$-pseudo spectra.

Proposition 3.3. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then for any α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ we have, $\sigma^{(2)}_{(P,Q)-\varepsilon}(\alpha I + \beta T) = \alpha + \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{|\beta|}}(T)\,\beta$.*

Proof. Let α, $\beta \in \mathbb{C}$ such that $\beta \neq 0$. Then $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(\alpha I + \beta T)$ if, and only if,

$$\lambda \in \sigma^{(2)}_{(P,Q)}(\alpha I + \beta T) \text{ or } \|R_{\alpha I+\beta T}(\lambda)\| > \frac{1}{\varepsilon}.$$

It is easy to see that $R_{\alpha I+\beta T}(\lambda) = P((\lambda-\alpha)I - \beta T)^{-1}$. Then it is easy to verify that $\lambda \in \sigma^{(2)}_{(P,Q)}(\alpha I + \beta T)$ or $\|P((\lambda-\alpha)I - \beta T)^{-1}\| > \frac{1}{\varepsilon}$ if, and only if, $((\lambda-\alpha)I - \beta T)^{(2)}_{(P,Q)}$ exists or $\|P((\lambda-\alpha)I + \beta\tilde{T})^{-1}\| > \frac{1}{\varepsilon}$ if, and only if, $(\beta(\beta^{-1}(\lambda-\alpha)I - T))^{(2)}_{(P,Q)}$ exists or $\|P(\beta^{-1}(\lambda-\alpha) - \tilde{T})^{-1}\| = |\beta|\,\|P((\lambda-\alpha) - \beta T)^{-1}\| > \frac{|\beta|}{\varepsilon}$. One may verify that the latter is equivalent to

$$\beta^{-1}(\lambda-\alpha) \in \sigma^{(2)}_{(P,Q)}(T) \text{ or } \|R_T(\beta^{-1}(\lambda-\alpha))\| > \frac{|\beta|}{\varepsilon}.$$

Then, on a par with $\beta^{-1}(\lambda-\alpha) \in \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{|\beta|}}(T)$, this tantamount to $\lambda \in \alpha + \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{|\beta|}}(T)\,\beta$. ■

Proposition 3.4. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then for $\delta > 0$ we have*

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq D_\delta + \sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq \sigma^{(2)}_{(P,Q)-\varepsilon+\delta}(T),$$

where

$$D_\delta = \left\{ \lambda \in \mathbb{C} : |\lambda|\|P^{-1}\| \leq \delta, \quad \begin{array}{c} T^{(2)}_{P,Q}\lambda + (\lambda)^{(2)}_{P,Q}T + I = 0, \quad \textit{and} \\ T(\lambda)^{(2)}_{P,Q} + \lambda T^{(2)}_{P,Q} + I = 0 \end{array} \right\}.$$

Proof. Let $\lambda \in D_\delta + \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. Then there exist $\lambda_1 \in D_\delta$ and $\lambda_2 \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ such that $\lambda = \lambda_1 + \lambda_2$. Presume that $\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon+\delta}(T)$. Then $\lambda_1 + \lambda_2 \in \rho^{(2)}_{(P,Q)}(T)$ and $\|R_T(\lambda_1 + \lambda_2)\| \leq \frac{1}{\varepsilon+\delta}$. Therefore $(\lambda_1 + \lambda_2 - T)^{(2)}_{(P,Q)}$ exists and $\|P(\lambda_1 + \lambda_2 - T)^{-1}\|^{-1} \geq \varepsilon + \delta > \|\lambda_1 I\|$ as $\lambda_1 \in D_\delta$. By using Theorem 2.2 we get,

$$(\lambda_1 + \lambda_2 - T)^{(2)}_{(P,Q)} - (\lambda_1)^{(2)}_{(P,Q)} = (\lambda_2 - T)^{(2)}_{(P,Q)} \quad \text{exists.}$$

Therefore, $\lambda_2 \in \rho^{(2)}_{(P,Q)}(T)$. On the other hand,

$$\begin{aligned}
\|R_T(\lambda_2)\|^{-1} &= \|P(\lambda_2 - T)^{-1}\|^{-1} \\
&= \|P(T - \lambda_2)^{-1}\|^{-1} \\
&= \|P(T - \lambda_2 - \lambda_1 + \lambda_1)^{-1}\|^{-1} \\
&\geq \|P(T - \lambda_2 - \lambda_1)^{-1}\|^{-1} - |\lambda_1|\|P^{-1}\| \\
&\geq \varepsilon + \delta - |\lambda_1|\|P^{-1}\| \\
&\geq \varepsilon.
\end{aligned}$$

Hence

$$\|R_T(\lambda_2)\| \leq \frac{1}{\varepsilon}.$$

We conclude that $\lambda_2 \notin \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. This is a contradiction. ∎

Proposition 3.5. *Let $T, S \in \mathcal{B}(X)$. If $S^{(2)}_{P,Q}$ exists and*

$$T^{(2)}_{P,Q}S + S^{(2)}_{P,Q}T + I = 0, \qquad TS^{(2)}_{P,Q} + ST^{(2)}_{P,Q} + I = 0,$$

then for any $\varepsilon > 0$ and $\|SP^{-1}\| < \varepsilon$,

$$\sigma^{(2)}_{(P,Q)-\varepsilon-\|SP^{-1}\|}(T) \subseteq \sigma^{(2)}_{(P,Q)-\varepsilon}(T+S) \subseteq \sigma^{(2)}_{(P,Q)-\varepsilon+\|SP^{-1}\|}(T).$$

Proof. For the second inclusion, let $\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon+\|SP^{-1}\|}(T)$. Then $\lambda \in \rho^{(2)}_{(P,Q)}(T)$ and $\|R_T(\lambda)\| \leq \frac{1}{\varepsilon+\|SP^{-1}\|}$. Hence $(\lambda - T)^{(2)}_{(P,Q)}$ exists and $\|R_T(\lambda)\|^{-1} \geq \varepsilon + \|SP^{-1}\|$. Using Theorem 2.2, $(\lambda - T - S)^{(2)}_{(P,Q)}$ exists. Then, $\lambda \in \rho^{(2)}_{(P,Q)}(T+S)$. On the other hand, we have

$$\begin{aligned}
\|R_{T+S}(\lambda)\|^{-1} &= \|P(\lambda - T - S)^{-1}\|^{-1} \\
&\geq \|P(\lambda - T)^{-1}\|^{-1} - \|SP^{-1}\| \\
&\geq \varepsilon + \|SP^{-1}\| - \|SP^{-1}\| \\
&\geq \varepsilon.
\end{aligned}$$

We obtain

$$\|R_{T+S}(\lambda)\| \leq \frac{1}{\varepsilon}.$$

Hence

$$\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon}(T+S).$$

For the first inclusion, let $\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon}(T+S)$. Then $\lambda \in \rho^{(2)}_{(P,Q)}(T+S)$ and $\|R_{T+S}(\lambda)\| \leq \frac{1}{\varepsilon}$. Hence $(\lambda - T - S)^{(2)}_{(P,Q)}$ exists and $\|R_{T+S}(\lambda)\|^{-1} \geq \varepsilon > \|SP^{-1}\|$. Using Theorem 2.2, $(\lambda - T - S)^{(2)}_{(P,Q)} + S^{(2)}_{(P,Q)} = (\lambda - T)^{(2)}_{(P,Q)}$ exists. So, $\lambda \in \rho^{(2)}_{(P,Q)}(T+S)(T)$. On the other hand, we have

$$\begin{aligned}\|R_T(\lambda)\|^{-1} &= \|R_{T+S-S}(\lambda)\|^{-1} \\ &\geq \|R_{T+S}(\lambda)\|^{-1} - \|SP^{-1}\| \\ &\geq \varepsilon - \|SP^{-1}\|.\end{aligned}$$

We deduce

$$\|R_T(\lambda)\| \leq \frac{1}{\varepsilon - \|SP^{-1}\|}.$$

We conclude that $\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon-\|SP^{-1}\|}(T)$. ∎

Theorem 3.6. *Let $T, V \in \mathcal{B}(X)$ such that $0 \in \rho(V)$ with V and V^{-1} commutes with $(\lambda - T)^{(2)}_{(P,Q)}$ and P for all $\lambda \in \mathbb{C}$. Let $k = \|V\|\|V^{-1}\|$. Let $S = VTV^{-1}$. Then we have the following statements:*

(i) $\sigma^{(2)}_{(P,Q)}(S) \subseteq \sigma^{(2)}_{(P,Q)}(T)$.

(ii) *For $\varepsilon > 0$ we have*

$$\sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k}}(S) \subseteq \sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k^2}}(T).$$

Proof. (i) On the other hand,

$$\begin{aligned}\lambda - S &= \lambda - VTV^{-1}, \\ V^{-1}(\lambda - S)V &= (\lambda V^{-1}V - V^{-1}VT), \\ V^{-1}(\lambda - S)V &= (\lambda - T), \\ (\lambda - S) &= V(\lambda - T)V^{-1}.\end{aligned}$$

Now, if $\lambda \in \rho^{(2)}_{(P,Q)}(T)$ then $(\lambda - T)^{(2)}_{(P,Q)}$ exists. We have

$$\begin{aligned}&(\lambda - T)^{(2)}_{(P,Q)} V(\lambda - T)V^{-1}(\lambda - T)^{(2)}_{(P,Q)} \\ &= V(\lambda - T)^{(2)}_{(P,Q)}(\lambda - T)(\lambda - T)^{(2)}_{(P,Q)}V^{-1} \\ &= V(\lambda - T)^{(2)}_{(P,Q)}V^{-1} \\ &= (\lambda - T)^{(2)}_{(P,Q)}VV^{-1} = (\lambda - T)^{(2)}_{(P,Q)},\end{aligned}$$

$$\begin{aligned}(\lambda - T)^{(2)}_{(P,Q)}V(\lambda - T)V^{-1} &= V(\lambda - T)^{(2)}_{(P,Q)}(\lambda - T)V^{-1} \\ &= VPV^{-1} \\ &= PVV^{-1} = P\end{aligned}$$

and also

$$\begin{aligned} V(\lambda-T)V^{-1}(\lambda-T)^{(2)}_{(P,Q)} &= V(\lambda-T)(\lambda-T)^{(2)}_{(P,Q)}V^{-1} \\ &= V(I-Q)V^{-1} \\ &= (I-Q)VV^{-1} = (I-Q). \end{aligned}$$

Then we conclude $V(\lambda-T)V^{-1} = \lambda - S$ exists. Hence $\lambda \in \rho^{(2)}_{(P,Q)}(S)$.

(ii) Now, we have $V^{-1}(\lambda-S)V = (\lambda-T)$ and $(\lambda-S) = V(\lambda-T)V^{-1}$. Then $V^{-1}(\lambda-S)^{-1}V = (\lambda-T)^{-1}$ and $(\lambda-S)^{-1} = V(\lambda-T)^{-1}V^{-1}$. Thus

$$\begin{aligned} \|R_S(\lambda)\| &= \|P(\lambda-S)^{-1}\| = \|PV(\lambda-T)^{-1}V^{-1}\| \\ &\leq \|VP(\lambda-T)^{-1}\|\,\|V^{-1}\| \\ &\leq \|V\|\,\|P(\lambda-T)^{-1}\|\,\|V^{-1}\| \\ &\leq k\,\|P(\lambda-T)^{-1}\| \\ &\leq k\,\|R_T(\lambda)\|. \end{aligned}$$

In the same way, $\|R_S(\lambda)\| \leq k\|R_T(\lambda)\|$. For $\lambda \in \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k}}(S)$, then $\lambda \in \sigma^{(2)}_{(P,Q)}(S)$ or $\|R_S(\lambda)\| > \frac{\varepsilon}{k}$. Then, $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$ or $\|R_T(\lambda)\| \geq \frac{1}{k}\ \|R_S(\lambda)\| > \frac{1}{\varepsilon}$. Hence $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. Therefore $\sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k}}(S) \subseteq \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. On the other hand, for $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ or } \|R_T(\lambda)\| > \frac{1}{\varepsilon}.$$

Then,

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \text{ or } \|R_T(\lambda)\| \geq \frac{1}{k}\ \|R_S(\lambda)\| > \frac{\varepsilon}{k^2}.$$

Hence

$$\lambda \in \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k^2}}(T).$$

We conclude that $\sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq \sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{k^2}}(T)$. ■

We accord several further consequences on the station of the pseudospectra. We begin with the next general outcome. Although the outcome is fully recognized, we contain the evidence. For a subset $\Omega \in \mathbb{C}$ we set as usual

$$d(\lambda,\Omega) = \inf\Big\{|z-\lambda| : z \in \Omega\Big\},$$

and note that if Ω is compact, then the infimum is attained for some point in Ω.

Theorem 3.7. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then we have the following statements:*

(i) *If* $\lambda \notin \sigma^{(2)}_{(P,Q)}(T)$, *then*

$$\|R_T(\lambda)\| \geq \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T))}.$$

(ii) *If* $\lambda \notin \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, *then* $\|R_T(\lambda)\| \geq \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T))+\varepsilon}$.

Proof. (i) Let $\lambda \in \rho^{(2)}_{(P,Q)}(T)$. We Have $\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T)) = \inf\{|z - \lambda|$ such that $z \in \sigma^{(2)}_{(P,Q)}(T)\}$. Then for all $\eta > 0$ there exists $z_\eta \in \sigma^{(2)}_{(P,Q)}(T)$ such that

$$|\lambda - z_\eta| < \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T)) + \eta.$$

If $|\lambda - z_\eta| < R_T(\lambda)^{-1}$, then by Theorem 2.5 we have $(\lambda - T + z_\eta - \lambda)^{(2)}_{(P,Q)} = (z_\eta - T)^{(2)}_{(P,Q)}$ does not exist, since $(\lambda - T)^{(2)}_{(P,Q)}$ does not exist and $|(z_\eta - \lambda)I| = |\lambda - z_\eta| < \|R_T(\lambda)\|^{-1}$. Hence $z_\eta \in \rho^{(2)}_{(P,Q)}(T)$. This is a contradiction. Therefore $|\lambda - z_\eta| \geq \|R_T(\lambda)\|^{-1}$ for all $\eta > 0$. Thus

$$\begin{aligned}\|R_T(\lambda)\|^{-1} &\leq |\lambda - z_\eta| \\ &< \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T)) + \eta \quad \text{for all } \eta > 0.\end{aligned}$$

So

$$\|R_T(\lambda)\|^{-1} \leq \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T)).$$

Hence $\|R_T(\lambda)\| \geq \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T))}$.

(ii) Let $\lambda \in \rho^{(2)}_{(P,Q)-\varepsilon}(T)$. Assume that $\|R_T(\lambda)\| < \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T))+\varepsilon}$, i.e,

$$\|R_T(\lambda)\|^{-1} > \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) + \varepsilon.$$

Since $\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) = \inf\{|z - \lambda|$ such that $z \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)\}$, for η, $0 < \eta \leq \varepsilon$, there exists $z_\eta \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ such that $|\lambda - z_\eta| < \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) + \eta$. Therefore

$$\begin{aligned}|\lambda - z_\eta| &< \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) + \varepsilon. \\ &< \|R_T(\lambda)\|^{-1}\end{aligned}$$

But $(\lambda - T)^{(2)}_{(P,Q)-\varepsilon}$ exists (as $\lambda \in \rho^{(2)}_{(P,Q)-\varepsilon}(T)$). Using Theorem 2.5, we have $(\lambda - T + z_\eta - \lambda)^{(2)}_{(P,Q)} = (z_\eta - T)^{(2)}_{(P,Q)}$ exists. Therefore $z_\eta \in \rho^{(2)}_{(P,Q)-\varepsilon}(T)$. But $z_\eta \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then $\|R_T(z_\eta)\| > \frac{1}{\varepsilon}$, i.e., $\|R_T(z_\eta)\|^{-1} < \varepsilon$. On the other hand,

$$\begin{aligned}\|R_T(z_\eta)\|^{-1} &= \|P(\lambda - T + z_\eta - \lambda)^{-1}\|^{-1} \\ &\geq \|R_T(\lambda)\|^{-1} - |z_\eta - \lambda|.\end{aligned}$$

Therefore

$$\begin{aligned}\|R_T(\lambda)\|^{-1} &\leq \|R_T(z_\eta)\|^{-1} + |z_\eta - \lambda| \\ &< \varepsilon + \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) + \eta \quad \text{for all} \quad 0 < \eta < \varepsilon.\end{aligned}$$

Thus $\|R_T(\lambda)\|^{-1} \leq \varepsilon + \mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T))$. This is a contradiction. We conclude that $\|R_T(\lambda)\| \geq \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)-\varepsilon}(T)) + \varepsilon}$. ■

Example 3.8. Consider the Banach space $l^\infty(\mathbb{N})$ with norm

$$\|x\|_* = |x| + \sup_{n \neq 0} |x_n| \quad \text{where } x = \Big(x_0, x_1, x_2, \cdots\Big).$$

and the box represents the zeroth coordinate of an element in $l^\infty(\mathbb{N})$. For $M > \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(A))}$, take an operator $A \in B(l^\infty(\mathbb{N}))$ such that

$$A\Big(x_0, x_1, x_2, x_3, \cdots\Big) = \Big(x_0, \frac{x_1}{M}, x_2, x_3, \cdots\Big).$$

Take $r := \min\{\frac{1}{M}; \frac{1}{2} - \frac{1}{M}\}$ and from [15, Theorem 3.1], we know that

$$\Big\|(A - \lambda)^{-1}\Big\| = \Big\|(A - \lambda)^{-1}(e_0)\Big\| = M$$

where $e_0 = \Big(1, 0, 0, 0, \cdots\Big)$ and $\lambda \in \mathbb{C}$ such that $|\lambda| < r$. Consider the Banach space $X = l^\infty(\mathbb{N}) \oplus l^\infty(\mathbb{N})$ with norm $\|(x, y)\| = (\|x\|^2 \|y\|^2)^{\frac{1}{2}}$. By [10, Theorem 1.8.6], X is a Banach space. We take the following operators

$$T : X \longrightarrow X \quad \text{defined by} \quad T(x, y) = (A(x), A(y))$$

and we defined

$$P : X \longrightarrow X \quad \text{defined by} \quad P(x, y) = (x, 0).$$

It is easy to see that $P^2 = P$ and $PT = TP$. By [10, Theorem 1.8.6],

$$\sigma^{(2)}_{(P,Q)}(T) = \sigma^{(2)}_{(P,Q)}(A)$$

and so we get,

$$R_T(\lambda) = (\lambda - T)^{-1} P \quad \text{for all} \quad \lambda \in \{\lambda \in \mathbb{C} : |\lambda| < r\}.$$

From [17, Example 4], we know that $\|R_T(\lambda)\| = M$. We deduce that

$$\|R_T(\lambda)\| \geq \frac{1}{\mathrm{d}(\lambda, \sigma^{(2)}_{(P,Q)}(T))}.$$

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