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Best Approximation and Fixed Points

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Dedicated to the memory of Professor Ky Fan (1914–2010)

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Abstract. Picard sequences in the context of metric spaces are used to prove several results for the presence of a fixed point. We further extend Brosowski-Meinardus type results on invariant approximation in the case of normed linear spaces as an application of the findings obtained. Some examples are also given to show how the obtained results might be put to use. The findings presented improve on several previously published findings.

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1. Introduction and Preliminaries

Because of its wide variety of applications, fixed point theory has sparked a lot of interest in solving issues arising from nonlinear differential equations, nonlinear

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integral equations, game theory, mathematical economics, control theory, and so on. Various investigators have studied fixed point theorems for various forms of nonlinear contractive maps (see [4]-[29]) and references cited therein).

Let M be a nonempty set and $T : M \to M$. A sequence $\{u_n\}$ defined by $u_n = T^n u_0$ is called a *Picard sequence* based at the point $u_0 \in M$. T is said to be a *Picard operator* if it has a unique fixed point $z \in M$ and $z = \lim_{n\to\infty} T^n u$ for all $u \in M$. Various classes of Picard operators exist in the literature (see, for example, [1, 2, 3, 25, 26]).

In normed linear spaces, Meinardus [16] proposed the concept of invariant approximation. In 1969, Fan [10] gave the classical best approximation theorem and Brosowski [4] demonstrated certain conclusions on invariant approximation using fixed point theory, generalising Meinardus' work. Various generalisations of Ky-Fan's and Brosowski's findings arose in the literature after that.

Singh [28] further attempted to show that the corresponding theorem of Singh [27] remains true if T is supposed to be nonexpansive only on $P_C(x) \cup \{x\}$. Many results have been proved since then in this direction (see Chandok et al. [7], Chandok and Narang [8, 9], Mukherjee and Som [17], Narang and Chandok [19, 20, 21], Rao and Mariadoss [23] and references cited therein). In this study, we show several similar types of results for the set of best approximation on T-invariant points. We use Hardy-Roger type contraction mappings in the setting of metric spaces to show some novel conclusions for the existence of Picard operators. We obtain some interesting Brosowski-Meinardus type results on invariant approximation in the framework of normed linear spaces as an application of the results proved in the second section. There are also some non-trivial examples presented.

Definition 1.1. *Let M be a nonempty subset of real normed linear space E and x an element of E, not in the closure of M. The set of best M-approximants to x consists of those $g_0 \in M$ satisfying $||x - g_0|| = \inf\{||x - g|| : g \in M\}$ and it is denoted by $P_M(x)$ (see* [29]*).*

Let T be a self mapping defined on a subset M of a normed linear space E. A best approximant y in M to an element x_0 in E is an invariant approximation in E to x_0 if $Ty = y$.

Example 1.2. (see [28]) Let $E = \mathbb{R}$ and $M = [0, \frac{1}{2}] \subset E$. Define $T : E \to E$ as

$$Tx = \begin{cases} x - 1 & \text{if } x < 0, \\ x & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{x+1}{2} & \text{if } x > \frac{1}{2}. \end{cases}$$

Clearly, $T(M) = M$ and $T(1) = 1$. Also, $P_M(1) = \{\frac{1}{2}\}$. Hence T has a fixed point in E which is a best approximation to 1 in M. Thus, $\frac{1}{2}$ is an invariant approximation.

Definition 1.3. *An element $g_0 \in G$ is said to be best copproximation (see* [12, 18]*) to $x \in E$ if $||g_0 - g|| \leq ||x - g||$, for all $g \in G$ where G is a nonempty subset of a normed linear space E. The set of all best coapproximants to x in G is denoted by $R_G(x)$.*

Definition 1.4. *A nonempty subset M of E is said to be starshaped if there exists some $z \in M$ such that $\lambda x + (1-\lambda)z \in M$ for each $x \in M$, $\lambda \in [0,1]$.*

The point z is called star-center of the set M.

It is clear that every convex subset is starshaped, but a starshaped set need not be convex.

Definition 1.5. *Let M be a nonempty subset of a metric space (X,d) and $T : M \to M$ be a self map. Then T is said to be asymptotically regular (see,* [5]*) if for all $x \in M$, $d(T^n(x), T^{n+1}(x)) \to 0$ as $n \to \infty$.*

2. Fixed Points

We demonstrate some fixed point results for generalized contraction mappings in metric spaces in this section.

We start with the following result which will be needed in the sequel.

Proposition 2.1. *Let (X,d) be a metric space and T be self mapping on X such that for all $x, y \in X$, we have*

$$\begin{aligned} d(Tx,Ty) \leq & \alpha_1 \frac{[1+d(y,Ty)]d(x,Tx)}{1+d(x,y)} + \alpha_2 \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_3(d(x,y)) \\ & + \alpha_4(d(x,Tx)+d(y,Ty)) + \alpha_5(d(x,Ty)+d(y,Tx)), \end{aligned} \tag{1}$$

where $\alpha_i \in [0,1)$, $i = \{1,2,3,4,5\}$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$ and $\alpha_4 + \alpha_5 < 1$. Then T is asymptotically regular.

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ be the Picard sequence in X such that $x_{n+1} = Tx_n = T^n x_0$, for every $n \geq 0$. So, from (1), we have

$$\begin{aligned} & d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n) \\ \leq\ & \alpha_1 \frac{[1+d(x_n,Tx_n)]d(x_{n+1},Tx_{n+1})}{1+d(x_{n+1},x_n)} + \alpha_2 \frac{d(x_{n+1},Tx_{n+1})d(x_n,Tx_n)}{1+d(x_{n+1},x_n)} \\ & +\alpha_3(d(x_{n+1},x_n)) + \alpha_4(d(x_{n+1},Tx_{n+1}) + d(x_n,Tx_n)) \\ & +\alpha_5(d(x_{n+1},Tx_n) + d(x_n,Tx_{n+1})) \\ =\ & \alpha_1 \frac{[1+d(x_n,x_{n+1})]d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_n)} + \alpha_2 \frac{d(x_{n+1},x_{n+2})d(x_n,x_{n+1})}{1+d(x_{n+1},x_n)} \\ & +\alpha_3(d(x_{n+1},x_n)) + \alpha_4(d(x_{n+1},x_{n+2}) + d(x_n,x_{n+1})) \\ & +\alpha_5(d(x_{n+1},x_{n+1}) + d(x_n,x_{n+2})) \end{aligned}$$

$$\begin{aligned}
&\leq [\alpha_1+\alpha_2]d(x_{n+1},x_{n+2})+\alpha_3(d(x_{n+1},x_n))\\
&\quad +\alpha_4\ (d(x_{n+1},x_{n+2})+d(x_n,x_{n+1}))+\alpha_5(d(x_n,x_{n+1})+d(x_{n+1},x_{n+2}))\\
&= (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(x_{n+1},x_{n+2})+(\alpha_3+\alpha_4+\alpha_5)d(x_n,x_{n+1}),
\end{aligned}$$

which implies that

$$d(x_{n+2},x_{n+1}) \leq \frac{\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)}d(x_{n+1},x_n). \tag{2}$$

Take $k=\frac{\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)}<1$. Hence sequence $\{d(T^nx_0,T^{n+1}x_0)\}$ is a decreasing sequence. Using mathematical induction, we have

$$d(x_{n+2},x_{n+1}) \leq (k)^{n+1}\,d(x_1,x_0). \tag{3}$$

Taking the limit $n\to\infty$, we have $d(x_{n+2},x_{n+1})\to 0$, that is, $d(T^nx_0,T^{n+1}x_0)\to 0$. Hence the result. ∎

Using Proposition 2.1, we prove our results.

Theorem 2.2. *If T is asymptotically regular and satisfies* (1) *on a complete metric space (X,d), then T is a Picard operator.*

Proof. Using Proposition 2.1, we get the sequence $\{d(T^nx_0,T^{n+1}x_0)\}$ is decreasing and $d(T^nx_0,T^{n+1}x_0)\to 0$ as $n\to\infty$ for all $x_0\in X$. We claim that $\{x_n\}$ is a Cauchy sequence. For $m>n$, and $k=\frac{\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)}$, we have

$$\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{m-1},x_m)\\
&\leq (k^n+k^{n+1}+\ldots+k^{m-1})d(x_0,x_1)\\
&\leq \frac{k^n(1-k^{m-n})}{1-k}d(x_0,x_1).
\end{aligned}$$

Therefore, $d(x_m,x_n)\to 0$, when $m,n\to\infty$. Thus $\{x_n\}$ is a Cauchy sequence in a complete metric space X and so there exists $u\in X$ such that $\lim_{n\to\infty}x_n=u$.

We shall show that the point u is a fixed point of T. Suppose that $Tu\neq u$. Then $d(u,Tu)>0$. Consider

$$\begin{aligned}
&d(x_{n+1},Tu)=d(Tx_n,Tu)\\
&\leq \alpha_1\frac{d(x_n,Tx_n)[1+d(u,Tu)]}{1+d(x_n,u)}+\alpha_2\frac{d(x_n,Tx_n)d(u,Tu)}{1+d(x_n,u)}+\alpha_3(d(x_n,u))\\
&\quad +\alpha_4(d(x_n,Tx_n)+d(u,Tu))+\alpha_5(d(x_n,Tu)+d(u,Tx_n))\\
&= \alpha_1\frac{d(x_n,x_{n+1})[1+d(u,Tu)]}{1+d(x_n,u)}+\alpha_2\frac{d(x_n,x_{n+1})d(u,Tu)}{1+d(x_n,u)}+\alpha_3(d(x_n,u))\\
&\quad +\alpha_4(d(x_n,x_{n+1})+d(u,Tu))+\alpha_5(d(x_n,Tu)+d(u,x_{n+1})).
\end{aligned}$$

Taking $n\to\infty$, we have $d(u,Tu)\leq(\alpha_4+\alpha_5)d(u,Tu)$. This implies $(1-(\alpha_4+\alpha_5))d(u,Tu)\leq 0$, which is a contradiction. Thus $d(u,Tu)=0$. Hence u is a fixed point of T. Here, we can see that every Picard sequence converges to the fixed point of T. It is easy to show the uniqueness of fixed point. Therefore, T is a Picard operator. ■

Example 2.3. Let $X = [0,1]$ and d be a usual metric on X.

Define $T : X \to X$ as $Tx = \begin{cases} \frac{2}{5} & \text{if } x \in [0, \frac{2}{3}), \\ \frac{1}{5} & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$

Suppose $\alpha_1 = \frac{1}{7}, \alpha_2 = \frac{1}{7}, \alpha_3 = \frac{1}{7}, \alpha_4 = \frac{1}{14}, \alpha_5 = \frac{1}{14} \in [0,1)$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 = \frac{10}{14} < 1$. Thus using Theorem 2.2, T is a Picard operator. Notice that $0.4 \in X$ is a fixed point of T.

It is easy to see that if we choose $x = 0$, $y = 1$, T is not a Banach contraction.

Theorem 2.4. *Let T be a self mapping from a complete metric space (X, d) into itself satisfying (1). Suppose that for some positive integer n, T^n is continuous. Then T is a Picard operator.*

Proof. On the same lines of Theorem 2.2, we define a sequence $\{x_n\}$ convergent to $u \in X$. Therefore, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, converging to u. By the continuity of T^n, we have

$$\begin{aligned} T^n(u) &= T^n(\lim_{i\to\infty} x_{n_i}) \\ &= \lim_{i\to\infty} T^n(x_{n_i}) \\ &= \lim_{i\to\infty} x_{n_i+1} \\ &= u. \end{aligned}$$

Hence u is a fixed point of T^n.

Now, we show that $Tu = u$. Let m be the smallest positive integer such that $T^m(u) = u$ and $T^p(u) \neq u$, for $p = 1, 2, ..., m-1$. If $m = 1$, we have the result. Consider $m > 1$ and using inequality (1), we have

$$\begin{aligned} & d(Tu, u) = d(Tu, T^m(u)) \\ \le\ & \alpha_1 \frac{d(u,Tu)[1 + d(T^{m-1}u, T(T^{m-1}u))]}{1 + d(u, T^{m-1}u)} + \alpha_2 \frac{d(u,Tu)d(T^{m-1}u, T(T^{m-1}u))}{1 + d(u, T^{m-1}u)} \\ & +\alpha_3(d(u, T^{m-1}u)) + \alpha_4(d(u,Tu) + d(T^{m-1}u, T(T^{m-1}u))) \\ & +\alpha_5(d(u, T(T^{m-1}u)) + d(T^{m-1}u, Tu)) \\ <\ & (\alpha_1 + \alpha_2)d(u,Tu) + \alpha_3(d(u, T^{m-1}u)) + \alpha_4(d(u,Tu) + d(T^{m-1}u, u)) \\ & +\alpha_5(d(T^{m-1}u, Tu)) \\ \le\ & (\alpha_1 + \alpha_2)d(u,Tu) + \alpha_3(d(u, T^{m-1}u)) + \alpha_4(d(u,Tu) + d(T^{m-1}u, u)) \\ & +\alpha_5(d(T^{m-1}u, u) + d(u, Tu)). \end{aligned}$$

This implies that $d(u, Tu) < kd(u, T^{m-1}u)$, where $k = \frac{\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)}$. Now, inductively, we get

$$\begin{aligned}d(u,T^{m-1}(u)) &= d(T^m u,T^{m-1}(u))\\ &\leq kd(T^{m-1}u,T^{m-2}(u))\\ &\leq k^2 d(T^{m-2}u,T^{m-3}(u)) \leq ... \leq k^{m-1}d(Tu,u),\end{aligned}$$

where $k = \frac{\alpha_3+\alpha_4+\alpha_5}{1-(\alpha_1+\alpha_2+\alpha_4+\alpha_5)}$. Notice that $k < 1$. Therefore

$$d(u,Tu) \leq k^m d(u,Tu) < d(u,Tu),$$

which is a contradiction. Hence $Tu = u$. The uniqueness follows easily. Hence the result. ■

Theorem 2.5. *Let (X,d) be a complete metric space and T be self mapping on X such that for all $x \in X$ and $Tx \neq T^2x$, we have*

$$\begin{aligned}d(Tx,T^2x) \leq &\alpha_1\frac{[1+d(x,Tx)]d(Tx,T^2x)}{1+d(x,Tx)} + \alpha_2\frac{d(x,Tx)d(Tx,T^2x)}{1+d(x,Tx)}\\ &+ \alpha_3 d(x,Tx) + \alpha_4(d(x,Tx)+d(Tx,T^2x)) + \alpha_5 d(x,T^2x),\end{aligned} \tag{4}$$

where $\alpha_i \in [0,1)$ $(i = \{1,2,3,4,5\})$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5 < 1$ and $\alpha_4+\alpha_5 < 1$.

Then T is a Picard operator.

Proof. Let x_0 be an arbitrary point in X. We assume that $x_0 \neq Tx_0$ for all $x_0 \in X$. Let $\{x_n\}$ be the Picard sequence in X such that $x_{n+1} = Tx_n = T^n x_0$, and $b_n = d(x_n,x_{n+1}) = d(T^n x_0,T^{n+1}x_0)$ for every $n \geq 0$. So, from (4), we have

$$\begin{aligned}b_{n+1} &= d(x_{n+1},x_{n+2}) = d(T^n x_0,T^{n+1}x_0)\\ &\leq \alpha_1\frac{[1+b_n]b_{n+1}}{1+b_n} + \alpha_2\frac{b_n b_{n+1}}{1+b_n} + \alpha_3 b_n + \alpha_4(b_n+b_{n+1})\\ &\quad +\alpha_5(d(x_n,Tx_{n+1}))\\ &\leq (\alpha_1+\alpha_2)b_{n+1} + \alpha_2 b_n + \alpha_3(b_n+b_{n+1}) + \alpha_4(b_n+b_{n+1})\\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)b_{n+1} + (\alpha_2+\alpha_3+\alpha_4)b_n,\end{aligned}$$

which implies that

$$b_{n+1} \leq \frac{\alpha_2+\alpha_3+\alpha_4}{1-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}b_n. \tag{5}$$

Here, $k = \frac{\alpha_2+\alpha_3+\alpha_4}{1-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} < 1$. Hence sequence $\{d(T^n x_0,T^{n+1}x_0)\}$ is a decreasing sequence. Using mathematical induction, we have

$$b_{n+1} \leq (k)^{n+1}\, b_0. \tag{6}$$

Taking the limit $n \to \infty$, we have $b_{n+1} \to 0$.

We claim that $\{x_n\}$ is a Cauchy sequence. For $m > n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leq (k^{m-1} + k^{m-2} + \ldots + k^n) d(x_1, x_0) \\ &\leq \frac{k^n(1-k^{m-n})}{1-k} d(x_1, x_0). \end{aligned}$$

Therefore, $d(x_m, x_n) \to 0$, when $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in a complete metric space X and so there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

If $u = Tu$, we have the result. Assume that $u \neq Tu$. If $Tx_n = Tu$ for infinite values of $n \geq 0$, then the sequence $\{x_n\}$ has a subsequence that converges to Tu and uniqueness of limit implies $u = Tu$. Then we can assume that $Tx_n \neq Tu$ for all $n \geq 0$. Consider

$$\begin{aligned} & d(Tu, x_{n+1}) = d(Tu, T^2 x_{n-1}) \\ \leq\ & \alpha_1 \frac{d(Tx_{n-1}, T^2x_{n-1})[1 + d(u, Tu)]}{1 + d(u, Tx_{n-1})} + \alpha_2 \frac{d(Tx_{n-1}, T^2x_{n-1}) d(u, Tu)}{1 + d(u, Tx_{n-1})} \\ & + \alpha_3(d(u, Tx_{n-1})) + \alpha_4(d(Tx_{n-1}, T^2x_{n-1}) + d(u, Tu)) \\ & + \alpha_5(d(Tx_{n-1}, Tu) + d(u, T^2x_{n-1})) \\ =\ & \alpha_1 \frac{d(x_n, x_{n+1})[1 + d(u, Tu)]}{1 + d(x_n, u)} + \alpha_2 \frac{d(x_n, x_{n+1}) d(u, Tu)}{1 + d(x_n, u)} + \alpha_3(d(x_n, u)) \\ & + \alpha_4(d(x_n, x_{n+1}) + d(u, Tu)) + \alpha_5(d(x_n, Tu) + d(u, x_{n+1})). \end{aligned}$$

Taking $n \to \infty$, we have $d(u, Tu) \leq (\alpha_4 + \alpha_5) d(u, Tu)$. This implies $(1 - (\alpha_4 + \alpha_5))\ d(u, Tu) \leq 0$, which is a contradiction. Thus $d(u, Tu) = 0$. Hence u is a fixed point of T. It is easy to verify that u is unique fixed point of T. ■

Theorem 2.6. *Let (X, d) be a complete metric space and T be a self mapping defined on X. Suppose that for some positive integer m, T satisfies the following condition*

$$\begin{aligned} d(T^m x, T^m y) \leq & \alpha_1 \frac{d(x, T^m x)[1 + d(y, T^m y)]}{1 + d(x, y)} + \alpha_2 \frac{d(x, T^m x) d(y, T^m y)}{1 + d(x, y)} \\ & + \alpha_3(d(x, y)) + \alpha_4(d(x, T^m x) + d(y, T^m y)) \\ & + \alpha_5(d(x, T^m y) + d(y, T^m x)), \end{aligned} \tag{7}$$

for all $x, y \in X$ and for some $\alpha_i \in [0, 1)$ $(i = \{1, 2, 3, 4, 5\})$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$ and $\alpha_4 + \alpha_5 < 1$. If T^m is continuous, then T is a Picard operator.

Proof. By Theorem 2.2, we conclude that T^m has a unique fixed point, say $u \in X$. Consider

$$Tu = T(T^m u) = T^m(Tu).$$

Thus Tu is also a fixed point of T^m. But by Theorem 2.2, we know that T^m has a unique fixed point u. It follows that $u = Tu$. Hence the result. ■

Example 2.7. Let $X = [0, 1]$ and d be a usual metric on X. Define $T : X \to X$ as

$$Tx = \begin{cases} 0 & \text{if } \; x \in [0, \frac{1}{3}], \\ \frac{1}{3} & \text{if } \; x \in (\frac{1}{3}, 1]. \end{cases}$$

Then choosing appropriately $\alpha_i \in [0, 1)$ $(i = \{1, 2, 3, 4, 5\})$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$, inequality (7) of Theorem 2.6 is satisfied for all $x, y \in X$. Thus using Theorem 2.6, $T^2x = 0$ for all $x \in [0, 1]$ and 0 is a fixed point of T^2 and hence of T.

Haghi et al. [14], in 2011, proved the following lemma by using the axiom of choice.

Lemma 2.8. *Let X be a nonempty set and $T : X \to X$ a function. Then there exist a set $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

By using the above lemma and Theorem 2.2, we prove the following common fixed point theorem for two self maps.

Theorem 2.9. *Let (X, d) be a complete metric space and T, S be two self maps on X. Suppose that there exist $\alpha_i \in [0, 1)$ $(i = \{1, 2, 3, 4, 5\})$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$ and $\alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$, we have*

$$\begin{aligned} d(Tx, Ty) \leq & \alpha_1 \frac{d(Sx, Tx)[1 + d(Sy, Ty)]}{1 + d(Sx, Sy)} + \alpha_2 \frac{d(Sx, Tx)d(Sy, Ty)}{1 + d(Sx, Sy)} \\ & + \alpha_3(d(Sx, Sy)) + \alpha_4(d(Sx, Tx) + d(Sy, Ty)) \\ & + \alpha_5(d(Sx, Ty) + d(Sy, Tx)). \end{aligned} \tag{8}$$

satisfying If $T(X) \subseteq S(X)$ and $S(X)$ is a complete subset of X then T and S have a unique common fixed point in X.

Proof. By using Lemma 2.8, there exist $E \subseteq X$ such that $S(E) = S(X)$ and $S : E \to X$ is one-to-one. Define $h : S(E) \to S(E)$ by $h(Su) = Tu$. Clearly, h is well defined as S is one-to-one on E. Also, $d(h(Su), h(Sv)) \leq \alpha_1 \frac{d(Sx,Tx)[1+d(Sy,Ty)]}{1+d(Sx,Sy)} + \alpha_2 \frac{d(Sx,Tx)d(Sy,Ty)}{1+d(Sx,Sy)} + \alpha_3(d(Sx, Sy)) + \alpha_4(d(Sx, Tx) + d(Sy, Ty)) + \alpha_5(d(Sx, Ty) + d(Sy, Tx))$ for all $Sx, Sy \in S(E)$. Since $S(E) = S(X)$ is complete, by using Theorem 2.2, we can easily prove that T and S have a unique common fixed point in X. ∎

3. Ordered Metric Spaces

Fixed point theory for self mappings on partially ordered sets has been initiated by Ran and Reurings [22], in dealing with matrix equations, and continued by many mathematicians, particularly in dealing with differential equations.

Let (X, d) be a metric space and $(X, \leq)$ be a partially ordered non-empty set. The triplet $(X, d, \leq)$ is called a metric space endowed with partial order. Moreover, two elements $x, y \in X$ are called comparable if $x \leq y$ or $y \leq x$ holds. A self mapping T on a partially ordered set $(X, \leq)$ is called nondecreasing if $Tx \leq Ty$ whenever $x \leq y$ for all $x, y \in X$. Also, a metric space endowed with a partial order $(X, d, \leq)$ is called regular if for every nondecreasing sequence $\{x_n\}$ in X, convergent to some $x \in X$, we get $x_n \leq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 3.1. *Let $(X, d, \leq)$ be a complete metric space endowed with a partial order and T be a nondecreasing self mapping on X. Assume that there exist $\alpha_i \in [0, 1)$ $(i = \{1, 2, 3, 4, 5\})$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5 < 1$, $\alpha_1+\alpha_4+\alpha_5 < 1$ and $\alpha_3 + 2\alpha_5 < 1$ such that for all comparable $x, y \in X$, we have*

$$\begin{aligned} d(Tx, Ty) \leq\ & \alpha_1 \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \alpha_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \alpha_3(d(x, y)) \\ & +\alpha_4(d(x, Tx) + d(y, Ty)) + \alpha_5(d(x, Ty) + d(y, Tx)). \end{aligned} \tag{9}$$

Further if there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ and X is regular, then T has a fixed point. Moreover, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X such that $x_0 \leq Tx_0$ and $\{x_n\}$ be the Picard sequence in X such that $x_{n+1} = Tx_n = T^n x_0$, for every $n \geq 0$. As T is nondecreasing, we deduce that

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots, \tag{10}$$

that is, x_n and x_{n+1} are comparable and $Tx_n \neq Tx_{n+1}$ for all $n \in \mathbb{N}$.

Proceeding as in the Proposition 2.1 and Theorem 2.2, we obtain that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

If $u = Tu$, we have the result. Assume that $u \neq Tu$. Since X is regular, from (10) we deduce that x_n and u are comparable and $Tx_n \neq Tu$ for all $n \in \mathbb{N}\cup\{0\}$. Consider

$$\begin{aligned} & d(x_{n+1}, Tu) = d(Tx_n, Tu) \\ \leq\ & \alpha_1 \frac{[1 + d(x_n, Tx_n)]d(u, Tu)}{1 + d(x_n, u)} + \alpha_2 \frac{d(x_n, Tx_n)d(u, Tu)}{1 + d(x_n, u)} + \alpha_3(d(x_n, u)) \\ & +\alpha_4(d(x_n, Tx_n) + d(u, Tu)) + \alpha_5(d(x_n, Tu) + d(u, Tx_n)) \\ =\ & \alpha_1 \frac{[1 + d(x_n, x_{n+1})]d(u, Tu)}{1 + d(x_n, u)} + \alpha_2 \frac{d(x_n, x_{n+1})d(u, Tu)}{1 + d(x_n, u)} + \alpha_3(d(x_n, u)) \\ & +\alpha_4(d(x_n, x_{n+1}) + d(u, Tu)) + \alpha_5(d(x_n, Tu) + d(u, x_{n+1})). \end{aligned}$$

Taking $n \to \infty$, we have $d(u, Tu) \leq (\alpha_1 + \alpha_4 + \alpha_5)d(u, Tu)$. This implies $(1 - (\alpha_1 + \alpha_4 + \alpha_5))d(u, Tu) \leq 0$, which is a contradiction. Thus $d(u, Tu) = 0$. Hence u is a fixed point of T.

Next, we assume that the set of fixed points of T is well-ordered. We claim that the fixed point of T is unique. Assume on the contrary that there exists another fixed point w in X such that $u \neq w$. Then, by using the condition (9) with $x = u$ and $y = w$, we get

$$\begin{aligned} d(Tu, Tw) \leq\ & \alpha_1 \frac{[1 + d(u, Tu)]d(w, Tw)}{1 + d(u, w)} + \alpha_2 \frac{d(u, Tu)d(w, Tw)}{1 + d(u, w)} + \alpha_3(d(u, w)) \\ & + \alpha_4(d(u, Tu) + d(w, Tw)) + \alpha_5(d(u, Tw) + d(w, Tu)) \\ =\ & (\alpha_3 + 2\alpha_5)\ d(u, w). \end{aligned}$$

This implies $(1 - (\alpha_3 + 2\alpha_5))d(u, w) \leq 0$, which is a contradiction. Hence $u = w$. Conversely, if T has a unique fixed point, then the set of fixed points of T, being a singleton, is well-ordered. ∎

4. Best Approximation

As an application of results proved in the previous sections, we prove some results on the set of best approximation.

Here, $F(T)$ denotes the set of all fixed points of T, clA denotes the closure of set A and $dist(x, A)$ denotes distance of set A from a point x.

Theorem 4.1. *Let T be a self mapping of a normed linear space E with $x \in F(T)$. If $C \subseteq E$, $D = P_C(x)$ is nonempty, closed and starshaped with star-center p, $clT(D) \subseteq D$, $clT(D)$ is compact, T is continuous on D and*

$$\begin{aligned} ||Tx - Ty|| \leq & \alpha_1 \frac{dist(x, [Tx, p])[1 + dist(y, [Ty, p])]}{1 + ||x - y||} \\ & + \alpha_2 \frac{dist(x, [Tx, p])dist(y, [Ty, p])}{1 + ||x - y||} \\ & + \alpha_3||x - y|| + \alpha_4(dist(x, [Tx, p]) + dist(y, [Ty, p])) \\ & + \alpha_5(dist(x, [Ty, p]) + dist(y, [Tx, p])), \end{aligned} \tag{11}$$

for all $x, y \in D$, where $\alpha_i \in [0, 1), (i = \{1, 2, 3, 4, 5\})$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$ and $\alpha_4 + \alpha_5 < 1$. Then $D \cap F(T) \neq \emptyset$.

Proof. Since D is nonempty and starshaped, there exists a star-center p in D such that $\lambda p + (1 - \lambda)z \in D$, for all $z \in D$, $0 \leq \lambda \leq 1$. Define $T_n : D \to D$ as $T_n z = \lambda_n Tz + (1 - \lambda_n)p$, $z \in D$ where $\{\lambda_n\}$ is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Also, from (11), we have

$$\begin{aligned} & ||T_n z - T_n y|| = \lambda_n ||Tz - Ty|| \\ \leq\ & \lambda_n [\alpha_1 \frac{dist(z, [Tz, p])[1 + dist(y, [Ty, p])]}{1 + ||z - y||} + \alpha_2 \frac{dist(z, [Tz, p])dist(y, [Ty, p])}{1 + ||z - y||} \\ & + \alpha_3||z - y|| + \alpha_4(dist(z, [Tz, p]) + dist(y, [Ty, p])) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_5(dist(z,[Ty,p])+dist(y,[Tz,p]))] \\
\le\ &\lambda_n[\alpha_1\frac{||z-T_nz||[1+||y-T_ny||]}{1+||z-y||}+\alpha_2\frac{||z-T_nz||\,||y-T_ny||}{1+||z-y||} \\
&+\alpha_3||z-y||+\alpha_4(||z-T_nz||+||y-T_ny||) \\
&+\alpha_5(||z-T_ny||+||y-T_nz||)],
\end{aligned}
$$

where $\lambda_n[\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5]<1$. Therefore by Theorem 2.2, each T_n has a unique fixed point z_n in D. Since $clT(D)$ is compact, there is a subsequence $\{Tz_{n_i}\}$ of $\{Tz_n\}$ such that $Tz_{n_i}\to z_0\in D$. We claim that $Tz_0=z_0$. As T is continuous, we have

$$z_{n_i}=T_{n_i}z_{n_i}=\lambda_{n_i}Tz_{n_i}+(1-\lambda_{n_i})p\to Tz_0.$$

Thus $z_{n_i}\to Tz_0$ and consequently, $Tz_0=z_0$ i.e. $z_0\in D$ is a T-invariant point. ∎

Let G_0 denote the class of closed convex subsets of a normed linear space E containing 0. For $C\in G_0$ and $u\in E$, let $C_u=\{x\in C: ||x||\le 2||u||\}$. Then $P_C(u)\subset C_u\in G_0$.

Theorem 4.2. *Let T be a continuous self mapping of a normed linear space E with $u\in F(T)$ and $C\in G_0$ such that $T(C_u)\subset C$. Suppose that $clT(C_u)$ is compact, and $||Tx-u||\le||x-u||$ for all $x\in C_u$. Then we have the following statements:*

(i) *$P_C(u)$ is nonempty, closed and convex;*
(ii) *$T(P_C(u))\subseteq P_C(u)$;*
(iii) *$P_C(u)\cap F(T)\ne\emptyset$, provided that T satisfies (11) for some $p\in P_C(u)$.*

Proof. If $u\in C$ then the results are obvious. So assume that $u\notin C$. If $x\in C-C_u$ then $||x||>2||u||$. Therefore, $||x-u||\ge||x||-||u||>2||u||-||u||=||u||\ge dist(u,C)$. Since $cl(T(C_u))$ is compact, and by the continuity of the norm, there exists $z\in cl(T(C_u))$ such that $dist(u,cl(T(C_u)))=||z-u||$. Hence

$$
\begin{aligned}
dist(u,C)&\le dist(u,cl(T(C_u)))\ \text{ as }\ T(C_u)\subseteq C\implies clT(C_u)\subseteq C\\
&\le dist(u,T(C_u))\\
&\le ||u-Tx||\\
&\le ||u-x||,
\end{aligned}
$$

for all $x\in C_u$. Therefore, $||z-u||\le dist(u,C_u)=dist(u,C)$ and $||z-u||=dist(u,C)$, i.e.

$$dist(u,C)=dist(u,clT(C_u))=||z-u||.$$

Hence $z\in P_C(u)$ and so $P_C(u)$ is nonempty. The closedness and convexity follow from that of C. This proves (i).

Now to prove $T(P_C(u)) \subseteq P_C(u)$, let $y \in T(P_C(u))$. Then $y = Tz$, for some $z \in P_C(u) \subset C_u$. Consider

$$||u - y|| = ||u - Tz|| \leq ||u - z|| = dist(u, C),$$

and so $y \in P_C(z)$ as $P_C(u) \subset C_u$ implies that $T(P_C(u)) \subset C$, that is, $y \in C$ and $T(P_C(u)) \subseteq P_C(u)$.

Using the similar arguments as in Theorem 4.1, it is easy to show that $P_C(u) \cap F(T) \neq \emptyset$. ■

We now prove a result for T-invariant points from the set of best coapproximations.

Theorem 4.3. *Let T be a continuous self mapping satisfying condition $||Tx - y|| \leq ||x - y||$, for all $x, y \in E$ and inequality* (11) *on a normed linear space E, G a subset of E such that $R_G(x)$ is compact and starshaped. Then $R_G(x)$ contains a T-invariant point.*

Proof. Let $g_0 \in R_G(x)$. Consider

$$||Tg_0 - g|| \leq ||g_0 - g|| \leq ||x - g||,$$

for all $g \in G$ and so $Tg_0 \in R_G(x)$, i.e. $T : R_G(x) \to R_G(x)$. Since $R_G(x)$ is starshaped, there exists $p \in R_G(x)$ such that $\lambda z + (1 - \lambda)p \in R_G(x)$ for all $z \in R_G(x)$, $\lambda \in [0, 1)$.

Let $\{\lambda_n\}$, $0 \leq \lambda_n < 1$, be a sequence of real numbers such that $\lambda_n \to 1$ as $n \to \infty$. Define $T_n : R_G(x) \to R_G(x)$ as $T_n z = \lambda_n Tz + (1 - \lambda_n)p$, $z \in R_G(x)$. Since T is a self map on $R_G(x)$ and $R_G(x)$ is starshaped, each T_n is a well defined and maps $R_G(x)$ into $R_G(x)$. Following the similar lines of Theorem 4.1, each T_n has a unique fixed point x_n in $R_G(x)$ i.e. $T_n x_n = x_n$ for each n. Since $R_G(x)$ is compact, $\{x_n\}$ has a subsequence $x_{n_i} \to x \in R_G(x)$.

Now, we claim that $Tx = x$. As T is continuous, we have

$$x_{n_i} = T_{n_i} x_{n_i} = \lambda_{n_i} T x_{n_i} + (1 - \lambda_{n_i})p \to Tx.$$

Thus $x_{n_i} \to Tx$ and consequently, $Tx = x$ i.e. $x \in R_G(x)$ is a T-invariant point. ■

Remark 4.4.

(i) When $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$, in Theorem 2.2, we have a famous Banach contraction principle.

(ii) When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = 0$, in Theorem 2.2, we have a Kannan contraction mapping (see [15]).

(iii) When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, in Theorem 2.2, we have a Fisher contraction mapping (see [11]). A similar conclusion was also obtained by Chatterjea (see [6]).

(iv) Contraction maaping (1) in Theorem 2.2, is also an extension of Reich type contraction mapping (see [24]).

(v) For different variants of inequality (1), we have many interesting results by appropriately choosing $\alpha_i, i = \{1, 2, 3, 4, 5\}$.

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