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Gelfand Spaces of Uniform Algebras on n-torus*

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Abstract. In this paper, we give the classification of the Gelfand space of uniform algebra $\mathcal{A}_\alpha$ on 2-torus. Moreover, we introduce the uniform algebra $\mathcal{A}_{\mathbb{S}}$ on n-torus $\mathbb{T}^n$. We give the Gelfand space of $\mathcal{A}_{\mathbb{S}}$.

Keywords: Uniform algebra; n-tours; Gelfand space.

1. Introduction

Let $\mathbb{T}^2$ denote the 2-torus $\mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle. Let $d\mu$ be normalized Lebesgue measure on $\mathbb{T}^2$, If $f : \mathbb{T}^2 \to \mathbb{C}$ is in L^2, the Fourier transform is a function on $\mathbb{Z}^2$ given by

$$\hat{f}(m,n) = \int_{\mathbb{T}^2} f(e^{is}, e^{it}) e^{-i(ms+nt)} d\mu.$$

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If α is a positive irrational number, define $\mathcal{A}_\alpha$ to be the set of continuous functions $f : \mathbb{T}^2 \to \mathbb{C}$ with the property that

$$\hat{f}(m,n) = 0 \quad \text{whenever } m + \alpha n < 0.$$

By the continuity of the Fourier transform, $\mathcal{A}_\alpha$ is a Banach space, and it is also a uniform algebra under the norm

$$\|f\| = \sup_{(z,w)\in\mathbb{T}^2} |f(z,w)|.$$

For more information, one can see [4]. It follows from [3, Corollary 15.18] that the C^*-enveloping $C_e^*(\mathcal{A}_\alpha)$ is $C(\mathfrak{S})$, where $\mathfrak{S}$ is the Shilov boundary of $\mathcal{A}_\alpha$. It follows from [6, Section 6] that $\mathfrak{S} = \mathbb{T}^2$.

In this paper, we will generalize $\mathcal{A}_\alpha$ to high dimension, that is, define the uniform algebra $\mathcal{A}_\mathbb{S}$ in n-torus $\mathbb{T}^n$. Then we give the Gelfand space of $\mathcal{A}_\mathbb{S}$. Furthermore, we will give the classfication of Gelfand space of $\mathcal{A}_\alpha$ in section 3.

2. Uniform Algebras on n-torus

For any $\mathfrak{m} = (m_0, m_1, \cdots, m_n) \in \mathbb{Z}^{n+1}$ and $\vartheta = (\theta_0, \theta_2, \cdots, \theta_n) \in [0, 2\pi]^{n+1}$, let

$$\mathfrak{m} \cdot \vartheta := m_0\theta_0 + m_1\theta_1 + \cdots + m_n\theta_n.$$

For every function $f \in C(\mathbb{T}^{n+1})$, its *Fourier coefficients* $\hat{f}(\mathfrak{m})$ defined by

$$\hat{f}(\mathfrak{m}) = \int_{[0,2\pi]} \cdots \int_{[0,2\pi]} f(\theta)e^{-im\cdot\theta} d\theta_0 d\theta_1 \cdots d\theta_n.$$

One can see [2] for more information about the Fourier analysis on n-torus $\mathbb{T}^n$.

Let $a_1, a_2, \cdots, a_n$ be positive irrational numbers such that $a_1, a_2, \cdots, a_n$ are linear independent in $\mathbb{Z}$. Let

$$\mathbb{S} = \{(m_0, m_1, \cdots, m_n) \in \mathbb{Z}^{n+1} : m_0 + m_1a_1 + \cdots + m_na_n \geq 0\}$$

and

$$\mathcal{A}_\mathbb{S} = \{f \in C(\mathbb{T}^{n+1}) : \hat{f}(\mathfrak{m}) = 0 \text{ if } \mathfrak{m} \notin \mathbb{S}\}.$$

By [2, Proposition 3.2.7], it is easy to see that $\mathcal{A}_\mathbb{S}$ is a uniform algebra. Moreover, $\mathcal{A}_\mathbb{S}$ is a Dirichlet algebra, that is, $\mathcal{A}_\mathbb{S} + \mathcal{A}_\mathbb{S}^*$ is dense in $C(\mathbb{T}^{n+1})$ (the reason is similar to the argument in [4, p. 91]).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}}$ denote the closure of $\mathbb{D}$ in $\mathbb{C}$. Put

$$\mathcal{A}(\mathbb{D}^{n+1}) = \{f \in C(\mathbb{T}^{n+1}) : \hat{f}(\mathfrak{m}) = 0 \text{ if } m_i \leq 0 \text{ for some } i\}.$$

Suppose that $X \subset \mathbb{C}^{n+1}$, recall that $h(X)$ is the set consisting of y such that $|p(y)| \leq \max_{x\in X} |p(x)|$ for any polynomial p (see [6, Definition 1.1]).

Let p be any polynomial and assume that

$$p(x_0, x_1, \cdots, x_n) = \sum_{j=0}^{k} b_j x_0^{m_0^j} x_1^{m_1^j} \cdots x_n^{m_n^j},$$

where $m_0^j, m_1^j, \cdots, m_n^j \in \mathbb{N}$. If the function $|p|$ attains its maximum on $\overline{\mathbb{D}}^{n+1}$ at $\tilde{\mathfrak{z}} = (z_0, z_1, \cdots, z_n)$ and there exists i such that $|z_i| < 1$. Without loss of generality, assume $|z_0| < 1$ and $|z_i| = 1$ for $1 \le i \le n$. Since

$$\max_{\mathfrak{z} \in \overline{\mathbb{D}}^{n+1}} |p(\mathfrak{z})| = |p(z_0, z_1, \cdots, z_n)| = |\sum_{j=0}^{k} b_j z_0^{m_0^j} z_1^{m_1^j} \cdots z_n^{m_n^j}|,$$

by Maximum Modulus Principle one can derive that

$$\max_{\mathfrak{z} \in \overline{\mathbb{D}}^{n+1}} |p(\mathfrak{z})| = |\sum_{j=0}^{k} (b_j z_1^{m_1^j} \cdots z_n^{m_n^j}) z_0^{m_0^j}| < |\sum_{j=0}^{k} (b_j z_1^{m_1^j} \cdots z_n^{m_n^j}) x'^{m_0^j}|,$$

where $|x'| = 1$. Therefore, we have that

$$\max_{\mathfrak{z} \in \overline{\mathbb{D}}^{n+1}} |p(\mathfrak{z})| = |p(z_0, z_1, \cdots, z_n)| < |p(x', z_1, \cdots, z_n)|,$$

which contradicts to that $|p|$ attain its maximum at $\tilde{\mathfrak{z}} = (z_0, z_1, \cdots, z_n) \in \overline{\mathbb{D}}^{n+1}$.

Theorem 2.1. *The Gelfand space for $\mathcal{A}(\mathbb{D}^{n+1})$ is $\overline{\mathbb{D}}^{n+1}$.*

Proof. The *Gelfand space* for $\mathcal{A}(\mathbb{D}^{n+1})$ is $h(\mathbb{T}^{n+1})$ (see [6, Theorem 1.1]). Moreover, $h(\mathbb{T}^{n+1})$ is $\overline{\mathbb{D}}^{n+1}$ by the above discription. ∎

Since $\mathcal{A}(\mathbb{D}^{n+1}) \subset \mathcal{A}_{\mathbb{S}} \subset C(\mathbb{T}^{n+1})$, we have that

$$M_{C(\mathbb{T}^{n+1})} \subset M_{\mathcal{A}_{\mathbb{S}}} \subset M_{\mathcal{A}(\mathbb{D}^{n+1})}.$$

Let

$$\mathbb{G} = \{(z_0, z_1, \cdots, z_n) \in \overline{\mathbb{D}}^{n+1} : |z_1| = |z_0|^{a_1}, |z_2| = |z_0|^{a_2}, \cdots, |z_n| = |z_0|^{a_n}\},$$

and we will prove that $\mathbb{G} = M_{\mathcal{A}_{\mathbb{S}}}$.

Theorem 2.2. *Let $\mathfrak{z} = (z_0, z_1, \cdots, z_n), \mathfrak{w} = (w_0, w_1, \cdots, w_n) \in \mathbb{G}$ be such that $|z_0|, |w_0| \neq 1$ and $z_0, w_0 \neq 0$. Then for any $\varepsilon > 0$, there exist $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ and $q, p_1, p_2, \cdots, p_n$ such that*

(i) $|x_1| = |x_i| < 1$ *for any $i = 1, 2, \cdots, n$;*
(ii) $|y_1| = |y_i| < 1$ *for any $i = 1, 2, \cdots, n$;*

(iii) $0 < a_i - \frac{p_i}{q} < \varepsilon$ *for any* $i = 1, 2, \cdots, n$;

(iv) $|x_1^q - z_0| + |x_1^{p_1} - z_1| + \cdots + |x_n^{p_n} - z_n| < \varepsilon$;

(v) $|y_1^q - w_0| + |y_1^{p_1} - w_1| + \cdots + |y_n^{p_n} - w_n| < \varepsilon$.

Proof. Let $k \in \mathbb{Z}$, $q = 2^k$ and $L = \{x \in \mathbb{C} : x^q = z_0\}$ and $J = \{y \in \mathbb{C} : y^q = z_0\}$. Then

$$
\begin{aligned}
|\,|x|^p - |z_i|| &= |\,|z_0|^{\frac{p}{q}} - |z_0|^{a_i}| \quad \forall\, x \in L;\\
|\,|y|^p - |w_i|| &= |\,|w_0|^{\frac{p}{q}} - |w_0|^{a_i}|, \quad \forall\, x \in L.
\end{aligned}
$$

One can choose odd numbers $p_1, p_2, \cdots, p_n$ such that $0 < a_i - \frac{p_i}{q} < \frac{1}{2^k}$ for each $i = 1, 2, \cdots, n$. Since the maximum common factor for p_i and 2^k is 1, there exists $x_i \in L$ and $y_i \in J$ such that

$$
\begin{aligned}
|\arg(x_i^{p_i}) - \arg(z_i)| &\le \frac{2\pi}{2^k}, \quad \forall\, i = 1, 2, \cdots, n;\\
|\arg(y_i^{p_i}) - \arg(w_i)| &\le \frac{2\pi}{2^k}, \quad \forall\, i = 1, 2, \cdots, n.
\end{aligned}
$$

Therefore, the proof is complete when k is sufficiently large. ■

Theorem 2.3. *Let* $f(x) = \sum_{j=0}^{k} c_j x_0^{m_0^j} x_1^{m_1^j} \cdots x_n^{m_n^j}$, *where* $m_0^j + m_1^j a_1 + m_2^j a_2 + \cdots + m_n^j a_n > 0$ *for every* $j = 0, 1, 2, \cdots, k$, *and if* $t_1, t_2, \cdots, t_n$ *satisfy*

(i) $t_i < a_i$ *for each* $i = 1, 2, \cdots, n$;

(ii) $m_0^j + m_1^j t_1 + m_2^j t_2 + \cdots + m_n^j t_n > 0$ *for each* $j = 0, 1, 2, \cdots, k$;

(iii) *for any subset* $I \subset \{1, 2, \cdots, n\}$, *we have that*

$$
j_0 + \sum_{i \in I} m_i^j a_i + \sum_{i \notin I} m_i^j t_i > 0.
$$

Then f *is continuous at* $\mathcal{Q} = \{\mathfrak{z} = (z_0, z_1, \cdots, z_n) \in \mathbb{D}^{n+1} : |z_0|^{a_i} \le |z_i| \le |z_0|^{t_i}$ *for all* $1 \le i \le n\}$.

Proof. We only need to show that f is continuous on 0, that is,

$$
\lim_{\mathfrak{z} \in \mathcal{Q}, \mathfrak{z} \to 0} z_0^{m_0^j} z_1^{m_1^j} \cdots z_n^{m_n^j} = 0
$$

for every $j = 0, 1, 2, \cdots, k$. We will divide into two cases.

Case 1: Suppose that $m_i^j \ge 0$ for every $i = 1, 2, \cdots, n$. Then we can derive that

$$
\begin{aligned}
|z_0^{m_0^j} z_1^{m_1^j} \cdots z_n^{m_n^j}| &= |z_0|^{m_0^j} |z_1|^{m_1^j} \cdots |z_n|^{m_n^j}\\
&\le |z_0|^{m_0^j + m_1^j t_1 + m_2^j t_2 + \cdots + m_n^j t_n} \to 0.
\end{aligned}
$$

Case 2: Suppose that there exists i such that $m_i^j < 0$. Without loss of generality, we can assume that $m_1^j < 0$ and the others are positive. Then we have that

$$|z_0^{m_0^j} z_1^{m_1^j} \cdots z_n^{m_n^j}| \leq |z_0|^{m_0^j + m_1^j a_1 + m_2^j t_2 + \cdots + m_n^j t_n} \to 0. \qquad \blacksquare$$

By Theorem 2.3, it is easy to prove the following result.

Theorem 2.4. *There exist* $t_1, t_2, \cdots, t_n$ *satisfy the conditions* (i)-(iii) *of Theorem 2.3, and for any* $t'_1, t'_2, \cdots, t'_n$ *with* $t_i \leq t'_i < a_i$ $(i = 1, 2, \cdots, n)$*, we have that* $t'_1, t'_2, \cdots, t'_n$ *also satisfy the conditions* (i)-(iii) *of Theorem 2.3.*

Theorem 2.5. *Let* $f(x) = \sum_{j=0}^{k} c_j x_0^{m_0^j} x_1^{m_1^j} \cdots x_n^{m_n^j}$ *satisfy* $m_0^j + m_1^j a_1 + m_2^j a_2 + \cdots + m_n^j a_n \geq 0$ *for every* $j = 0, 1, 2, \cdots, k$*. Then the maximum of* $|f|$ *on* $\mathbb{G}$ *is attained at some* $\tilde{\mathfrak{z}} = (z_0, z_1, \cdots, z_n)$ *with* $|z_0| = 1$.

Proof. For any $\mathfrak{z} = (z_0, z_1, \cdots, z_n) \in \mathbb{G}$ with $0 < |z_0| < 1$ and any $\varepsilon > 0$, it follows from Theorem 2.2 that there exist $q, p_1, p_2, \cdots, p_n, y_1, y_2, \cdots, y_n$ such that $|y_i| = |y_1|$ and $t_i < \frac{p_i}{q} < a_i$ for each $1 \leq i \leq n$ (that is, $(y_1^q, y_1^{p_1}, y_2^{p_2}, \cdots, y_n^{p_n}) \in \mathcal{Q}$) and

$$|z_0 - y_1^q| + |z_1 - y_1^{p_1}| + |z_2 - y_2^{p_2}| + \cdots + |z_n - y_n^{p_n}| \leq \varepsilon.$$

Here, $t_1, t_2, \cdots, t_n$ are defined in Theorem 2.4. Since $y_i = e^{i\theta_i} y_1$, one can derive that

$$\begin{aligned}
& f(y_1^q, y_1^{p_1}, y_2^{p_2}, \cdots, y_n^{p_n}) \\
= \ & f(y_1^q, y_1^{p_1}, (e^{i\theta_2} y_1)^{p_2}, \cdots, (e^{i\theta_n} y_1)^{p_n}) \\
= \ & \sum_{j=0}^{k} c_j (y_1^q)^{m_0^j} (y_1^{p_1})^{m_1^j} ((e^{i\theta_2} y_1)^{p_2})^{m_2^j} \cdots ((e^{i\theta_n} y_1)^{p_n})^{m_n^j} \\
= \ & \sum_{j=0}^{k} c'_j y_1^{q m_0^j + p_1 m_1^j + \cdots + p_n m_n^j}.
\end{aligned}$$

It follows from Theorem 2.4 that $q m_0^j + p_1 m_1^j + \cdots + p_n m_n^j \geq 0$ for any $j = 0, 1, 2, \cdots, k$. By the Maximum Modulus Principle, there exists t_0 with $|t_0| = 1$ such that

$$\begin{aligned}
& f(y_1^q, y_1^{p_1}, y_2^{p_2}, \cdots, y_n^{p_n}) \\
< \ & \sum_{j=0}^{k} c'_j t_0^{q j_0 + p_1 j_1 + \cdots + p_n j_n} \\
= \ & \sum_{j=0}^{k} c_j (t_0^q)^{j_0} (t_0^{p_1})^{j_1} ((e^{i\theta_2} t_0)^{p_2})^{j_2} \cdots ((e^{i\theta_n} t_0)^{p_n})^{j_n} \\
= \ & f(t_0^q, t_0^{p_1}, (e^{i\theta_2} t_0)^{p_2}, \cdots, (e^{i\theta_n} t_0)^{p_n}).
\end{aligned}$$

Note that $(t_0^q, t_0^{p_1}, (e^{i\theta_2}t_0)^{p_2}, \cdots, (e^{i\theta_n}t_0)^{p_n}) \in \mathbb{T}^{n+1}$, the proof is complete. ■

It follows from Theorem 2.5 that $\mathbb{G} \subset M_{\mathcal{A}_\mathbb{S}}$, and we will prove that $\mathbb{G} = M_{\mathcal{A}_\mathbb{S}}$.

Theorem 2.6. *Suppose that* $\mathfrak{z} = (z_0, z_1, \cdots, z_n) \in \overline{\mathbb{D}}^{n+1}$ *and* $\mathfrak{z} \notin \mathbb{G}$ *and there exist some* i *such that* $z_i = 0$*. Then* $\mathfrak{z} \notin M_{\mathcal{A}_\mathbb{S}}$*.*

Proof. Without loss of generality, assume that $\mathfrak{z} = (z_0, 0, z_2, \cdots, z_n)$ with $z_0 \neq 0$. Suppose on the contrary that $\mathfrak{z} \in M_{\mathcal{A}_\mathbb{S}}$, choose a function $f_0(x_0, x_1, \cdots, x_n) = x_0^{m_0} x_1^{m_1}$, where $m_0 > 0, m_1 < 0$ and $m_0 + a_1 m_1 > 0$. Then $f_0 \in \mathcal{A}_\mathbb{S}$. Put $g_0(x_0, x_1, \cdots, x_n) = x_1^{-m_1}$. Then we have that

$$z_0^{m_0} = \mathfrak{z}(f_0 g_0) = \mathfrak{z}(f_0)\mathfrak{z}(g_0) = 0,$$

which is a contradiction since $z_0^{m_0} \neq 0$. ■

Corollary 2.7. *Suppose that* $\mathfrak{z} = (z_0, z_1, \cdots, z_n) \in \overline{\mathbb{D}}^{n+1}, \mathfrak{z} \notin \mathbb{G}$ *and* $z_i \neq 0$ *for all* $0 \leq i \leq n$*. Then there exist* $m_0, m_1, \cdots, m_n$ *such that* $m_0 + a_1 m_1 + \cdots + a_n m_n \geq 0$ *and the function* $g(x_0, x_1, \cdots, x_n) = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n}$ *satisfies* $|g(\mathfrak{z})| > 1$*.*

Proof. Since $\mathfrak{z} \notin \mathbb{G}$, there exists i such that $|z_0|_i^a \neq |z_i|$. By [5, p. 12], if $m_0 + a_i m_i > 0$, the function $f(x_0, x_1, \cdots, x_n) = x_0^{m_0} x_i^{m_i}$ satisfies that $|f(\mathfrak{z})| > 1$, which implies that $\mathfrak{z} \notin M_{\mathcal{A}_\mathbb{S}}$. ■

Corollary 2.8. $M_{\mathcal{A}_\mathbb{S}} = \mathbb{G}$.

Proof. It follows from Theorem 2.6 and Corollary 2.7 that $M_{\mathcal{A}_S} \subset \mathbb{G}$. Therefore, $\mathbb{G} = M_{\mathcal{A}_\mathbb{S}}$. ■

3. Classifications of the Gelfand Space of $\mathcal{A}_\alpha$

In this section, let α be a positive irrational number, we will classify the Gelfand space M_α of $\mathcal{A}_\alpha$.

At first, we will define the equivalent relation in the Gelfand space $M_\mathcal{A}$ of the uniform algebra $\mathcal{A}$.

Definition 3.1. *Let* $\mathcal{A}$ *be a uniform algebra and* $\phi, \theta \in M_\mathcal{A}$*. We say that* $\phi \sim \psi$ *if exist* $c > 0$ *such that*

$$\frac{1}{c} < \frac{\mu(\theta)}{\mu(\phi)} < c$$

for all $\mu \in Re(\mathcal{A})$ *with* $\mu > 0$*. The relation* $\sim$ *is a equivalent relation, and the equivalent classes induced by* $\sim$ *are called parts (see* [1, p. 142]*).*

Remark 3.2. It follows from [6, p. 89] that $\{0\}$ is a singleton part.

Theorem 3.3. *Let $\theta = (z_0, z_0^\alpha), \phi = (w_0, w_0^\alpha) \in M_\alpha$. Suppose that $0 < z_0 < w_0 < 1$ and $\frac{2w_0}{3-w_0} < z_0 < w_0$. Then we have that $\theta \sim \phi$.*

Proof. Let $f(x) = \sum_{j=0}^{k} c_j x_0^{m_j^0} x_1^{m_j^1}$ be such that Re$f > 0$ and $m_j^0 + \alpha m_j^1 \geq 0$ for any $j = 0, 1, \cdots, k$. By Theorem 2.2, for any $\varepsilon > 0$, there exist $\tilde{z}, \tilde{w}, q, p$ such that

(a) the maximum common factor of p, q is 1,
(b) $0 < \alpha - \frac{p}{q} < \varepsilon$,
(c) $|\tilde{z}^q - z_0| + |\tilde{z}^p - z_1| < \varepsilon$,
(d) $|\tilde{w}^q - w_0| + |\tilde{w}^p - w_1| < \varepsilon$.

Note that $\tilde{z}, \tilde{w}$ are the qth roots of z, w, respectively. Choose m_j^0 and m_j^1 such that $m_j^0 q + m_j^1 p \geq 0$ for all $j = 0, 1, 2, \cdots, k$. Then one can derive that $g(t) := f(t^q, t^p)$ is a polynomial with respect to t. Let $u(t) = \text{Re}g(t)$. Then we have

$$\frac{\text{Re}f(\tilde{z}^q, \tilde{z}^p)}{\text{Re}f(\tilde{w}^q, \tilde{w}^p)} = \frac{\text{Re}g(\tilde{z})}{\text{Re}g(\tilde{w})} = \frac{u(\tilde{z})}{u(\tilde{w})}.$$

Since $t^{\frac{1}{n}-1}$ converge to t^{-1} uniformly on $[z_0, w_0]$, there exists q such that $t^{\frac{1}{q}-1} < \frac{2}{z_0}$ for all $t \in [z_0, w_0]$, which implies that

$$\tilde{w} - \tilde{z} = w_0^{\frac{1}{q}} - z_0^{\frac{1}{q}} = (w_0 - z_0)\frac{1}{q}\tilde{t}^{\frac{1}{q}-1} < (w_0 - z_0)\frac{2}{qz_0},$$

where $z_0 < \tilde{t} < w_0$. Note that $\frac{2w0}{3-w_0} \leq z_0 < w_0$, we have that

$$\tilde{w} - \tilde{z} < \frac{2w_0}{qz_0} - \frac{2}{q} \leq \frac{3 - w_0}{q} - \frac{2}{q} = \frac{1}{q}(1 - w_0),$$

and we also have

$$1 - \tilde{w} > \frac{1}{q}(1 - w_0).$$

Since $u(t)$ is harmonic on plane and positive on $\mathbb{D}$, by Harnack Inequality (see [7, Theorem 11.11]), one can derive that

$$\frac{\frac{1-w_0}{q} - (\tilde{w} - \tilde{z})}{\frac{1-w_0}{q} + \tilde{w} - \tilde{z}} u(\tilde{w}) \leq u(\tilde{z}) \leq \frac{\frac{1-w_0}{q} + \tilde{w} - \tilde{z}}{\frac{1-w_0}{q} - (\tilde{w} - \tilde{z})} u(\tilde{w}).$$

It follows from $\tilde{w} - \tilde{z} < \frac{2}{z_0}(w_0 - z_0) < \frac{1}{q}(1 - w_0)$ that

$$\frac{1 - w_0 - \frac{2}{z_0}(w_0 - z_0)}{1 - w_0 + \frac{2}{z_0}(w_0 - z_0)} u(\tilde{w}) \leq u(\tilde{z}) \leq \frac{1 - w_0 + \frac{2}{z_0}(w_0 - z_0)}{1 - w_0 - \frac{2}{z_0}(w_0 - z_0)} u(\tilde{w}).$$

Let

$$c = \frac{1 - w_0 - \frac{2}{z_0}(w_0 - z_0)}{1 - w_0 + \frac{2}{z_0}(w_0 - z_0)}.$$

Then one can derive that

$$c \leq \frac{f(\tilde{z}^q, \tilde{z}^p)}{f(\tilde{w}^q, \tilde{w}^p)} \leq \frac{1}{c}.$$

Let $q \to \infty$. We have that $c \leq \frac{f(z_0, z_1)}{f(w_0, w_1)} \leq \frac{1}{c}$.

For the general $g \in \mathrm{Re}\mathcal{A}_\alpha$ with $g > 0$, the Cesaro means g_n converges uniformly to g. Since for $n \in \mathbb{N}$, g_n satisfies that

$$c \leq \frac{g_n(z_0, z_1)}{g_n(w_0, w_1)} \leq \frac{1}{c},$$

one can derive that g satisfies that.

$$c \leq \frac{g(z_0, z_1)}{g(w_0, w_1)} \leq \frac{1}{c}.$$

■

Corollary 3.4. *Suppose that* $\theta = (z_0, z_0^\alpha), \phi = (w_0, w_0^\alpha)$ *are two elements in* M_α *such that* $0 < z_0 < w_0 < 1$. *Then* $\theta \sim \phi$.

Proof. By Theorem 3.3, if $0 < x < y < 1$ such that $\frac{2y}{3-y} < x \leq y$, then $(x, x^\alpha) \sim (y, y^\alpha)$. For the function defined by

$$\lambda(t) = t - \frac{2t}{3-t} = \frac{t - t^2}{3-t} \quad \forall\, t \in [0,1],$$

$\lambda(t)$ has a positive minimum δ on $[\frac{z_0}{2}, \frac{1-w_0}{2}]$. It follows from the transitivity of $\sim$ that $\theta \sim \phi$. ■

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