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Infimum of a Matrix Norm of A Induced by an Absolute Vector Norm

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Abstract. We characterize the infimum of a matrix norm of a square matrix A induced by an absolute norm, over the fields of real and complex numbers. Usually this infimum is greater than the spectral radius of A. If A is sign equivalent to a nonnegative matrix B then this infimum is the spectral radius of B.

Keywords: Absolute norm; Matrix norm; Nonnegative matrices; Spectral radius.

1. Introduction

Let $\mathbb{F}$ be the field of real or complex numbers $\mathbb{R}$ and $\mathbb{C}$ respectively. Denote by $\mathbb{F}^n$ and $\mathbb{F}^{n\times n}$ the vectors spaces column vectors and the matrices respectively. Assume that $\|\cdot\|_{\mathbb{F}} : \mathbb{F}^n \to [0, \infty)$ is a norm on $\mathbb{F}^n$. Then $\|\mathbf{x}\|_{\mathbb{F}}$ induces an operator norm on $\mathbb{F}^{n\times n}$, namely

$$\|A\|_{\mathbb{F}} = \max\{\|A\mathbf{x}\|_{\mathbb{F}}, \mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\|_{\mathbb{F}} \le 1\}.$$

For $A = [a_{ij}] \in \mathbb{F}^{n\times n}$ let $\rho(A)$ be the spectral radius of A, which is maximum modulus of all real and complex eigenvalues of A. The following inequality is well known over the complex numbers:

$$\rho(A) \le \|A\|_{\mathbb{F}} \text{ for } A \in \mathbb{F}^{n\times n}. \tag{1}$$

We will show that this inequality holds also over real numbers.

Denote by $\mathrm{GL}(n,\mathbb{F}) \subset \mathbb{F}^{n\times n}$ the group of invertible matrices. Hence the above inequality yields:

$$\rho(A) \le \inf\{\|PAP^{-1}\|_{\mathbb{F}},\ P \in \mathrm{GL}(n,\mathbb{F})\}. \tag{2}$$

(For $\mathbb{F} = \mathbb{C}$ this result is well known.)

The following result of the author characterizes the norms over $\mathbb{C}^n$ for which equality in (2) holds for all $A \in \mathbb{C}^{n\times n}$ [1]. A norm $\|\mathbf{x}\|_{\mathbb{F}}$ is called an absolute norm if $\|\mathbf{x}\|_{\mathbb{F}} = \||\mathbf{x}|\|_{\mathbb{F}}$, where $|(x_1,\dots,x_n)^\top| := (|x_1|,\dots,|x_n|)^\top$ and $|A| = [|a_{ij}|]$. Recall that an absolute norm is monotone: $\|\mathbf{x}\|_{\mathbb{F}} \le \|\mathbf{y}\|_{\mathbb{F}}$ if $|\mathbf{x}| \le |\mathbf{y}|$ [2, Theorem 7.1.3].

We call $\|A\|_{\mathbb{F}}$ a matrix absolute norm if $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm. A norm $\|\mathbf{x}\|_{\mathbb{F}}$ is called a transform absolute norm if there exists $P \in \mathrm{GL}(n,\mathbb{F})$ and an absolute norm $\nu(\mathbf{x})$ such that $\|\mathbf{x}\| = \nu(P\mathbf{x})$. Theorem 3 in [1] shows equality in (2) holds for all $A \in \mathbb{C}^{n\times n}$ if and only if the norm $\|\mathbf{x}\|_{\mathbb{C}}$ is an absolute transform norm. In particular we deduce the well known approximation result due to Householder [5, Theorem 4.4]: For each matrix $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ there exists a norm on $\mathbb{C}^n$ such that

$$\|A\|_{\mathbb{C}} \le \rho(A) + \varepsilon. \tag{3}$$

From the proof of this result in [5] or [6, (2), Section 2.3] it follows that $\|\cdot\|_{\mathbb{C}}$ can be chosen a transform absolute norm. (In [1] it was claimed erroneously that the proof of (3) yields that $\|\cdot\|_{\mathbb{C}}$ can be chosen an absolute norm.)

J. Najim asked if one can choose an absolute norm $\|\cdot\|_{\mathbb{C}}$ such that (3) holds. We show that this is false even for 2×2 real matrices.

The aim of this note to give a necessary and sufficient conditions when there exists an absolute norm on $\mathbb{F}^n$ such that for a given $A \in \mathbb{F}^{n\times n}$ and $\varepsilon > 0$ one has the inequality

$$\|A\|_{\mathbb{F}} \le \rho(A) + \varepsilon. \tag{4}$$

This condition can be stated as follows. Let $\|\mathbf{x}\|_2$ be the Euclidean norm on $\mathbb{F}^n$. Denote by $\mathcal{D}_n(\mathbb{F}) \subset \mathrm{GL}(n,\mathbb{F})$ the subgroup of diagonal matrices whose absolute value of diagonal elements is 1. For a positive integer m let $[m] = \{1,\dots,m\}$. Then (4) holds if and only if

$$\lim_{m\to\infty} \max\Big\{(\rho(A)+\varepsilon)^{-k}\|AD_1A\cdots D_{k-1}AD_k\mathbf{x}\|_2,\\ D_1,\dots,D_k \in \mathcal{D}_n(\mathbb{F}),\ \mathbf{x}\in\mathbb{F}^n,\ \|\mathbf{x}\|_2 = 1,\ k\in[m]\Big\} < \infty. \tag{5}$$

Equivalently, we characterize the following infimum: Let $\mathcal{N}(n,\mathbb{F})$ be a set of absolute norms on $\mathbb{F}^n$. Denote

$$\mu(A) = \limsup_{k\to\infty}\Big(\max\Big\{\|AD_1A\cdots D_{k-1}AD_k\mathbf{x}\|_2,\\ D_1,\dots,D_k\in\mathcal{D}_n(\mathbb{F}),\ \mathbf{x}\in\mathbb{F}^n,\ \|\mathbf{x}\|_2 = 1\Big\}\Big)^{1/k}. \tag{6}$$

Then

$$\mu(A) = \inf\{\|A\|_{\mathbb{F}},\ \|\cdot\|_{\mathbb{F}} \in \mathcal{N}(n,\mathbb{F})\}. \tag{7}$$

We now survey briefly the results of this paper. In Section 2 we prove the inequality (1). In Section 3 we prove characterization (7). In Section 4 we prove the inequality $\mu(A) \le \rho(|A|)$. Equality holds if A is sign equivalent to $|A|$.

2. Proof of Inequality (1)

Lemma 2.1. *Let $\|\mathbf{x}\|_{\mathbb{F}}$ be a norm on $\mathbb{F}^n$. Then the inequality* (1) *holds.*

Proof. Suppose first that $\mathbb{F} = \mathbb{C}$. Then there exists an eigenvalue of $\lambda \in \mathbb{C}$ of A such that $|\lambda| = \rho(A)$. Let $\mathbf{x} \in \mathbb{C}^n$ be the corresponding eigenvector satisfying $\|\mathbf{x}\|_{\mathbb{C}} = 1$. Then

$$\|A\|_{\mathbb{C}} \ge \|A\mathbf{x}\|_{\mathbb{C}} = \|\lambda\mathbf{x}\|_{\mathbb{C}} = \rho(A).$$

This shows (1).

Assume now that $A \in \mathbb{R}^{n\times n}$. Suppose first that A has a real eigenvalue λ such that $|\lambda| = \rho(A)$. Then the above arguments yield that $\|A\|_{\mathbb{R}} \ge \rho(A)$. Suppose that $\rho(A) > 0$ and $\lambda = \rho(A)\zeta$ where $\zeta = e^{2\pi\mathbf{i}\theta}$, $\theta \in [0,1)$. Suppose first that θ is a rational number. Then $\zeta^k = 1$ for some positive integer k. That is, λ^k is a positive eigenvalue of A^k. Recall that $\rho(A^k) = \rho(A)^k$. Using the previous result we obtain

$$\|A\|_{\mathbb{R}}^k \ge \|A^k\|_{\mathbb{R}} \ge \rho(A^k) = \rho(A)^k.$$

Hence (1) holds.

Suppose secondly that θ is irrational. Then the complex eigenvector of A corresponding to λ is $\mathbf{u} + \mathbf{i}\mathbf{v}$ can be chosen to satisfy

$$A^k(\mathbf{u}+\mathbf{i}\mathbf{v}) = \rho(A)^k e^{2\pi\mathbf{i}k\theta}(\mathbf{u}+\mathbf{i}\mathbf{v}), \quad \|\mathbf{u}\| = 1,\ \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\},\ k \in \mathbb{N}.$$

Hence

$$\begin{aligned}\|A^k\mathbf{u}\|_{\mathbb{R}} &= \rho(A)^k\|(\cos 2\pi k\theta)\mathbf{u} - (\sin 2\pi k\theta)\mathbf{v}\|_{\mathbb{R}}\\ &\ge \rho(A)^k\big(|\cos 2\pi k\theta|\|\mathbf{u}\|_{\mathbb{R}} - |\sin 2\pi k\theta|\|\mathbf{v}\|_{\mathbb{R}}\big).\end{aligned}$$

Recall that for a given $\delta \in (0,1)$ there exists an infinite sequence of positive integers k_l, m_l, for $l \in \mathbb{N}$, such that $|k_l\theta - m_l| \le \delta/(2\pi)$. Hence there exists an infinite sequence of positive integers k_l such that

$$\|A^{k_l}\mathbf{u}\|_{\mathbb{R}} \ge \frac{\rho(A)^{k_l}}{2}\|\mathbf{u}\|_{\mathbb{R}} = \frac{\rho(A)^{k_l}}{2}, \quad l \in \mathbb{N}.$$

Therefore

$$\|A\|_{\mathbb{R}} \ge (\|A^{k_l}\|_{\mathbb{R}})^{1/k_l} \ge (\|A^{k_l}\mathbf{u}\|_{\mathbb{R}})^{1/k_l} \ge \rho(A)2^{-1/k_l}.$$

Letting $l \to \infty$ we deduce that $\|A\|_{\mathbb{R}} \ge \rho(A)$. ∎

3. Proof of the Main Result

Lemma 3.1. *Let $\|\mathbf{x}\|_{\mathbb{F}}$ be an absolute norm on $\mathbb{F}^n$. Then for any $A \in \mathbb{F}^{n\times n}$, the following relations holds:*

$$\begin{aligned} \|D_1AD_2\|_{\mathbb{F}} = \|A\|_{\mathbb{F}} \text{ for all } D_1, D_2 \in \mathcal{D}_n(\mathbb{F}), \\ \|AD_1AD_2\cdots AD_k\|_{\mathbb{F}} \le \|A\|_{\mathbb{F}}^k, \quad k \in \mathbb{N}, \\ \|A\|_{\mathbb{F}} \ge \max\{\rho(DA),\, D \in \mathcal{D}_n(\mathbb{F})\}. \end{aligned} \tag{8}$$

Proof. Assume that $D_1, D_2 \in \mathcal{D}_n(\mathbb{F})$. Since $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm we obtain $\|D_2\mathbf{x}\|_{\mathbb{F}} = \|\mathbf{x}\|_{\mathbb{F}}$, and $\|D_2\|_{\mathbb{F}} = 1$. Hence

$$\frac{\|D_1AD_2\mathbf{x}\|_{\mathbb{F}}}{\|\mathbf{x}\|_{\mathbb{F}}} = \frac{\|AD_2\mathbf{x}\|_{\mathbb{F}}}{\|D_2\mathbf{x}\|_{\mathbb{F}}} \text{ for } \mathbf{x} \ne \mathbf{0}.$$

This proves the first equality of (8). Use the submultiplicativity of the norm $\|B\|_{\mathbb{F}}$ and the first equality (8) to deduce the second equality of (8).

The inequality (1) yields $\|A\|_{\mathbb{F}} = \|D_1A\|_{\mathbb{F}} \ge \rho(D_1A)$. This proves the third inequality of (8). ■

Corollary 3.2. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = DA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $\rho(A) = 0$, $\rho(B) = \|A\|_2 = 2$. Hence for any absolute norm on $\mathbb{R}^2$ we have the sharp inequality $\|A\|_{\mathbb{R}} \ge 2$.

Theorem 3.3. *Assume that $A \in \mathbb{F}^{n\times n}$. Let $\mu(A)$ be defined by* (6). *Then equality* (7) *holds.*

Proof. Assume that $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm. Recall that all norms on $\mathbb{F}^{n\times n}$ are equivalent. Hence for a given absolute norm $\|\mathbf{x}\|_{\mathbb{F}}$ one has an inequality

$$\frac{1}{K(\|\cdot\|_{\mathbb{F}})}\|\mathbf{x}\|_{\mathbb{F}} \le \|\mathbf{x}\|_2 \le K(\|\cdot\|_{\mathbb{F}})\|\mathbf{x}\|_{\mathbb{F}}, \quad \mathbf{x} \in \mathbb{F}^n,$$

for some $K(\|\cdot\|_{\mathbb{F}}) \ge 1$. Therefore in the definition of $\mu(A)$ given by (6) we can replace the norm $\|\mathbf{x}\|_2$ by an absolute norm $\|\mathbf{x}\|$, Use the second inequality of (8) to obtain

$$\mu(A) \le \|A\|_{\mathbb{F}}. \tag{9}$$

Let

$$\tilde{\mu}(A) = \inf\{\|A\|_{\mathbb{F}},\, \|\cdot\|_{\mathbb{F}} \in \mathcal{N}(n, \mathbb{F})\}.$$

The inequality (9) yields that $\mu(A) \le \tilde{\mu}(A)$.

We now show that for each $\varepsilon > 0$ there exists an absolute norm $\|\mathbf{x}\|_{\mathbb{F}}$ on $\mathbb{F}^n$ such that $\|A\|_{\mathbb{F}} \le \mu(A) + \varepsilon$. From the definition of $\mu(A)$ it is straightforward to show that

$$\begin{aligned}\lim_{m\to\infty} \max\{&(\mu(A)+\varepsilon)^{-m}\|AD_1A\cdots D_{m-1}AD_m\mathbf{x}\|_2,\\ &D_1,\ldots,D_m \in \mathcal{D}_n(\mathbb{F}),\, \mathbf{x}\in\mathbb{F}^n,\, \|\mathbf{x}\|_2 = 1\} = 0.\end{aligned} \tag{10}$$

Hence

$$\begin{aligned}\lim_{m\to\infty} \max\{&(\mu(A)+\varepsilon)^{-k}\|AD_1A\cdots D_{k-1}AD_k\mathbf{x}\|_2,\\ &D_1,\ldots,D_k \in \mathcal{D}_n(\mathbb{F}),\, \mathbf{x}\in\mathbb{F}^n,\, \|\mathbf{x}\|_2 = 1,\, k+1\in[m]\} < \infty.\end{aligned} \tag{11}$$

Define

$$\begin{aligned}\|\mathbf{x}\|_{\mathbb{F}} := \lim_{m\to\infty} \max\{&(\mu(A)+\varepsilon)^{-k}\|AD_1A\cdots D_{k-1}AD_k\mathbf{x}\|_2,\\ &D_1,\ldots,D_k \in \mathcal{D}_n(\mathbb{F}),\, \mathbf{x}\in\mathbb{F}^n,\, k+1\in[m]\} < \infty.\end{aligned} \tag{12}$$

(We let the values of $(\mu(A)+\varepsilon)^{-k}\|AD_1A\cdots D_{k-1}AD_k\mathbf{x}\|_2$ to be $\|\mathbf{x}\|_2$ and $\|AD_1\mathbf{x}\|_2$ for $k=0$ and $k=1$ respectively.) Clearly $\|\mathbf{0}\|_{\mathbb{F}} = 0$. The inequality (11) yields that $\|\mathbf{x}\|_{\mathbb{F}} < \infty$ for $\mathbf{x} \ne 0$. Clearly, $\|t\mathbf{x}\|_{\mathbb{F}} = |t|\|\mathbf{x}\|_{\mathbb{F}}$ for $t \in \mathbb{F}$. The inequality (10) yields that for each $\mathbf{x} \in \mathbb{F}$ there exists a nonnegative integer $k = k(\mathbf{x})$ and $D_1,\ldots,D_{k(\mathbf{x})} \in \mathcal{D}_n(\mathbb{F})$ such that

$$\|\mathbf{x}\|_{\mathbb{F}} = \begin{cases} \|\mathbf{x}\|_2 & \text{if } k(\mathbf{x}) = 0,\\ (\mu(A)+\varepsilon)^{-1}\|AD_1\mathbf{x}\|_2 & \text{if } k(\mathbf{x}) = 1,\\ (\mu(A)+\varepsilon)^{-k(\mathbf{x})}\|AD_1A\cdots D_{k(\mathbf{x})-1}AD_{k(\mathbf{x})}\mathbf{x}\|_2 & \text{if } k(\mathbf{x}) > 1.\end{cases}$$

The maximum definition of $\|\mathbf{x}\|_{\mathbb{F}}$ yields that $\|\mathbf{x}\|_{\mathbb{F}}$ satisfies the triangle inequality and the equality $\|D\mathbf{x}\|_{\mathbb{F}} = \|\mathbf{x}\|_{\mathbb{F}}$ for all $D \in \mathcal{D}_n(\mathbb{F})$. Hence $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm on $\mathbb{F}^n$. It is left to show that

$$\|A\mathbf{x}\|_{\mathbb{F}} \le (\mu(A)+\varepsilon)\|\mathbf{x}\|_{\mathbb{F}}, \quad \mathbf{x}\in\mathbb{F}^n. \tag{13}$$

Observe that

$$\begin{aligned}\|A\mathbf{x}\|_{\mathbb{F}} := (\mu(A)+\varepsilon)\lim_{m\to\infty} \max\Big\{&(\mu(A)+\varepsilon)^{-(k+1)}\|AD_1A\cdots D_{k-1}AD_kA\mathbf{x}\|_2,\\ &D_1,\ldots,D_k \in \mathcal{D}_n(\mathbb{F}),\, \mathbf{x}\in\mathbb{F}^n,\, k+1\in[m]\Big\}\\ <\ \infty.&\end{aligned}$$

Clearly

$$AD_1A\cdots D_{k-1}AD_kA\mathbf{x} = AD_1A\cdots D_{k-1}AD_kAD_{k+1}\mathbf{x}, \quad D_{k+1} = I_n.$$

Hence

$$\max\Big\{(\mu(A)+\varepsilon)^{-(k+1)}\|AD_1A\cdots D_{k-1}AD_kA\mathbf{x}\|_2,\\ D_1,\ldots,D_k\in\mathcal{D}_n(\mathbb{F}),\ \mathbf{x}\in\mathbb{F}^n,\ k+1\in[m]\Big\}\\ \le\ \max\Big\{(\mu(A)+\varepsilon)^{-q}\|AD_1A\cdots D_{q-1}AD_qA\mathbf{x}\|_2,\\ D_1,\ldots,D_q\in\mathcal{D}_n(\mathbb{F}),\ \mathbf{x}\in\mathbb{F}^n,\ q+1\in[m+1]\Big\}.$$

This establishes (13). Hence $\|A\|_{\mathbb{F}}\le\mu(A)+\varepsilon$. ■

Theorem 3.4. *Let $A\in\mathbb{F}^{n\times n}$. Then there exists an absolute norm $\|\mathbf{x}\|_{\mathbb{F}}$ and $\varepsilon>0$ such that the inequality* (4) *holds if and only if the condition* (5) *holds.*

Proof. Suppose first that there exists an absolute norm on $\mathbb{F}^n$ such that (4) holds. As in the proof of Theorem 3.3 we deduce that

$$(\rho(A)+\varepsilon)^{-k}\|AD_1\cdots D_{k-1}AD_k\mathbf{x}\|_{\mathbb{F}}\le(\rho(A)+\varepsilon)^{-k}\|A\|_{\mathbb{F}}^k\|\mathbf{x}\|_{\mathbb{F}}\le\|\mathbf{x}\|_{\mathbb{F}}.$$

In view of equivalence of norms $\|\mathbf{x}\|_{\mathbb{F}}$ and $\|\mathbf{x}\|_2$ we deduce

$$(\rho(A)+\varepsilon)^{-k}\|AD_1\cdots D_{k-1}AD_k\mathbf{x}\|_2\\ \le\ (\rho(A)+\varepsilon)^{-k}K(\|\cdot\|_{\mathbb{F}})\|AD_1\cdots D_{k-1}AD_k\mathbf{x}\|_{\mathbb{F}}\\ \le\ K(\|\cdot\|_{\mathbb{F}})\|\mathbf{x}\|_{\mathbb{F}}\le K(\|\cdot\|_{\mathbb{F}})^2\|\mathbf{x}\|_2.$$

Hence the condition (5) holds.

Assume now that the condition (5) holds. Define $\|\mathbf{x}\|_{\mathbb{F}}$ as in (12) by replacing $(\mu(A)+\varepsilon)$ with $(\rho(A)+\varepsilon)$. Then the arguments of the proof of Theorem 3.3 yield that $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm for which the inequality (4) holds. ■

4. Additional Results and Remarks

A matrix $A\in\mathbb{F}^{n\times n}$ is said to be sign equivalent to $B\in\mathbb{F}^{n\times n}$ if $A=D_1BD_2$ for some $D_1,D_2\in\mathcal{D}_n(\mathbb{F})$. A matrix $B\in\mathbb{R}^{n\times n}$ is called nonnegative if $B=|B|$. The following lemma generalizes the example in Corollary 3.2.

Lemma 4.1. *Let $A\in\mathbb{F}^{n\times n}$. Then*

$$\mu(A)\le\rho(|A|). \tag{14}$$

Equality holds if A is sign equivalent to $|A|$.

Proof. Let $B=|A|$ and assume that A is sign equivalent to B. Suppose first that B is an irreducible matrix. That is, $(I_n+B)^{n-1}$ is a positive matrix. Then

Perron-Frobenius theorem [2] yields that there exist positive eigenvector $\mathbf{u}$ and $\mathbf{v}$ of B and $B^\top$ respectively such that

$$B\mathbf{v} = \rho(B)\mathbf{u}, \quad B^\top\mathbf{u} = \rho(B)\mathbf{u}, \quad \|\mathbf{v}\|_2 = 1,\, \mathbf{v}^\top\mathbf{u} = 1,\, \rho(A) > 0.$$

For a positive vector $\mathbf{w} \in \mathbb{R}^n$ define an absolute norm on $\mathbb{F}^n$:

$$\nu(\mathbf{x}) = \mathbf{w}^\top|\mathbf{x}|. \tag{15}$$

Assume that $\mathbf{w} = \mathbf{v}$. Then

$$\begin{aligned} \nu(\mathbf{u}) &= \mathbf{v}^\top\mathbf{u} = 1, \quad \nu(B\mathbf{u}) = \rho(B)\nu(\mathbf{u}) = \rho(B), \\ \nu(B\mathbf{x}) &= \mathbf{v}^\top|B\mathbf{x}| \le \mathbf{v}^\top B|\mathbf{x}| = \rho(B)\mathbf{v}^\top|\mathbf{x}| = \rho(B)\nu(\mathbf{x}). \end{aligned}$$

Hence $\nu(B) = \rho(B)$. As ν is absolute we deduce from (8) that $\nu(A) = \nu(B) = \rho(B)$. Next observe that $B = D_1^{-1}AD_2^{-1}$ is similar to $D_2^{-1}D_1^{-1}A$. Hence $\rho(D_2^{-1}D_1^{-1}A) = \rho(B)$. Inequalities (8) yield that $\|A\|_{\mathbb{F}} \ge \rho(B)$ for any absolute norm. Hence $\mu(A) = \rho(|A|)$.

Assume that B is a reducible matrix and $\mathbf{w} > \mathbf{0}$. Let ν be defined by (15). Note that ν is a weighted ℓ_1 norm. It is straightforward to show that

$$\nu(B) = \max\left\{\frac{(B^\top\mathbf{w})_i}{w_i}, i \in [n]\right\}.$$

See [4, Example 5.6.4] for the case $\mathbf{w} = (1,\dots,1)^\top$. Recall the generalized Collatz-Wielandt characterization of the spectral radius of $\rho(B) = \rho(B^\top)$ [3, Part (1), Theorem 3.2]:

$$\rho(B) = \inf_{\mathbf{w}>\mathbf{0}} \max\left\{\frac{(B^\top\mathbf{w})_i}{w_i}, i \in [n]\right\}.$$

Hence for a given $\varepsilon > 0$ there exists $\mathbf{w} > 0$ such that

$$\rho(B) \le \nu(B) \le \rho(B) + \varepsilon.$$

Therefore $\mu(A) = \rho(|A|)$.

Assume now that A is not sign equivalent to $|A|$. Suppose that $\|\mathbf{x}\|_{\mathbb{F}}$ is an absolute norm. Observe that $|A\mathbf{x}| \le |A||\mathbf{x}|$. As $\|\mathbf{x}\|_{\mathbb{F}}$ is a monotone norm it follows that $\|A\mathbf{x}\|_{\mathbb{F}} \le \||A||\mathbf{x}|\|_{\mathbb{F}}$, and $\|A\|_{\mathbb{F}} \le \||A|\|_{\mathbb{F}}$. Hence inequality (14) holds. ∎

We close with section with a brief discussion of absolute norms on $\mathbb{R}^n$ and $\mathbb{C}^n$. Clearly if $\|\mathbf{x}\|_{\mathbb{C}}$ is an absolute norm on $\mathbb{C}^n$ then the restriction of this norm on $\mathbb{R}^n$ gives an absolute norm $\|\mathbf{x}\|_{\mathbb{R}}$. It is also well known that given an absolute norm $\|\mathbf{x}\|_{\mathbb{R}}$ it induces an absolute norm $\|\mathbf{x}\|_{\mathbb{C}}$ on $\mathbb{C}^n$ by the equality $\|\mathbf{x}\| = \||\mathbf{x}|\|_{\mathbb{R}}$. Indeed, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ we have that $|\mathbf{x}+\mathbf{y}| \le |\mathbf{x}|+|\mathbf{y}|$. The monotonicity of $\|\mathbf{x}\|_{\mathbb{R}}$ yields

$$\|\mathbf{x}+\mathbf{y}\|_{\mathbb{C}} = \||\mathbf{x}+\mathbf{y}|\|_{\mathbb{R}} \le \||\mathbf{x}|+|\mathbf{y}|\|_{\mathbb{R}} \le \||\mathbf{x}|\|_{\mathbb{R}} + \||\mathbf{y}|\|_{\mathbb{R}} = \|\mathbf{x}\|_{\mathbb{C}} + \|\mathbf{y}\|_{\mathbb{C}}.$$

Assume now that $\|\mathbf{x}\|_{\mathbb{R}}$ is an absolute norm on $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$. Then $\|A\|_{\mathbb{C}}$ is the induced norm by $\|\mathbf{x}\|_{\mathbb{C}}$. Clearly

$$\|A\|_{\mathbb{R}} \le \|A\|_{\mathbb{C}}. \tag{16}$$

It is not obvious to the author that one has always equality in the above inequality for a general absolute norm on $\mathbb{R}^n$.

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