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A Sharp Upper-Bound for the Norm of Positive Semi-Definite Block Matrices

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Dedicated to the memory of Professor Ky Fan (1914–2010)

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Abstract. This paper studies the positive semi-definite 2×2 block matrix $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$. A new sharp upper bound for $\|M\|$ is provided under the condition of X being normal, for any unitarily invariant norm $\|\cdot\|$. A special pattern among the eigenvalues of M when $A + B = kI$, for some $k > 0$, is explored.

Keywords: Positive semi-definite matrix; Block matrix; Unitarily invariant norm.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the space of $n \times n$ complex matrices. The identity matrix of appropriate size shall be denoted by I, and the group of $n \times n$ unitary matrices

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shall be denoted by $\mathbb{U}(n)$. A norm $\|\cdot\|$ over the space of matrices is unitarily invariant if $\|UXV\| = \|X\|$ for all $X \in \mathbb{C}^{n\times n}$ and $U, V \in \mathbb{U}(n)$. Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian. Let $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the largest eigenvalue and the smallest eigenvalue of A, respectively. We shall write $A \succeq 0$ ($A \succ 0$) if A is positive semi-definite (definite), and $A \preceq 0$ if A is negative semi-definite. $A \succeq B$ ($A \succ B$) shall indicate that $A - B \succeq 0$ ($A - B \succ 0$). We shall denote the spectral norm of X by $\|X\|_{\mathrm{sp}}$. Throughout this paper, we assume that M is the positive semi-definite block matrix in the form:

$$M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n},$$

where $A, B, X \in \mathbb{C}^{n\times n}$.

It was shown in [2] that

$$\|M\| \leq \|A + B\| \tag{1}$$

for any unitarily invariant norm, when the off-diagonal blocks of M are Hermitian. Note that the norm in the inequality (1) is defined on $\mathbb{C}^{2n\times 2n}$. For simplicity, we write $\|A + B\|$ to represent $\|(A + B) \oplus 0\|$.

Recently, in [3], the inequality (1) was extended to the form

$$\|M\| \leq \|A + B + \omega I\|. \tag{2}$$

Here, ω stands for the width of the smallest strip containing the numerical range of the matrix X. It was shown in [3] that the inequality (2) is sharp. When X is normal with collinear eigenvalues, the inequality (2) is reduced to (1) for any unitarily invariant norm since the numerical range of X is a line segment. In particular, (1) is true when A, B, and X are 2×2 complex matrices with X normal. However, in [4], it was shown that the inequality (1) doesn't hold in general for an arbitrary normal or unitary matrix X.

In this paper, we develop a sharp upper bound for $\|M\|$, which outperforms (2) in some cases under the assumption of X being normal. We derive the same eigenvalue relationship of [4, Proposition 2.1] when X commutes with A. This result is used to show that the new norm inequality is sharp. Finally, we provide a partial generalization of [6, Theorem 2.9].

2. Some New Norm Inequalities

We first investigate the eigenvalues of the matrix $M = \begin{bmatrix} A & X \\ X^* & kI - A \end{bmatrix} \succeq 0$, for some $k > 0$, under a special case.

Let us write the eigenvalues of M in increasing order:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n-1} \leq \lambda_{2n}.$$

Theorem 2.1. [4, Proposition 2.1] *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ *be positive semi-definite with* $A, B, X \in \mathbb{C}^{n\times n}$*, and* $A + B = kI$ *for some* $k > 0$*. If* $X^* = e^{i\theta} X$ *for some* $\theta \in \mathbb{R}$*. Then*

$$\lambda_j + \lambda_{2n+1-j} = k \quad \text{for} \quad j = 1, \ldots, n. \tag{3}$$

We obtain the same eigenvalue relationship when X commutes with A.

Theorem 2.2. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ *be positive semi-definite with* $A, B, X \in \mathbb{C}^{n\times n}$*, and* $A + B = kI$ *for some* $k > 0$*. If* $AX = XA$*, then*

$$\lambda_j + \lambda_{2n+1-j} = k \quad \text{for} \quad j = 1, \ldots, n. \tag{4}$$

Proof. We first express M as

$$\begin{bmatrix} A & X \\ X^* & kI - A \end{bmatrix} = \begin{bmatrix} A - \frac{k}{2}I & X \\ X^* & \frac{k}{2}I - A \end{bmatrix} + \begin{bmatrix} \frac{k}{2}I & 0 \\ 0 & \frac{k}{2}I \end{bmatrix}.$$

Now we write the eigenvalues of the Hermitian matrix

$$N = \begin{bmatrix} A - \frac{k}{2}I & X \\ X^* & \frac{k}{2}I - A \end{bmatrix}$$

in increasing order:

$$\mu_1 \le \mu_2 \le \ldots \le \mu_{2n-1} \le \mu_{2n}.$$

Clearly, $\lambda_i = \mu_i + \frac{k}{2}$ for $i = 1, \ldots, 2n$. Therefore, it suffices to show that if μ is a nonzero eigenvalue of N, then so is $-\mu$. In this case, if N has a zero eigenvalue, then it is repeated even number of times since the dimension of N is $2n \times 2n$.

Let us simply write $C = A - \frac{k}{2}I$, so that $N = \begin{bmatrix} C & X \\ X^* & -C \end{bmatrix}$. It follows by assumption that $CX = XC$. Assume that μ is a nonzero eigenvalue of N. By Schur determinant lemma, see [7, p. 4],

$$\begin{aligned}
\det(N - \mu I) &= \det\left(\begin{bmatrix} C - \mu I & X \\ X^* & -C - \mu I \end{bmatrix}\right) \\
&= \det\Big((C - \mu I)(-C - \mu I) - X^* X\Big) \\
&= \det\Big((C + \mu I)(\mu I - C) - X^* X\Big) \\
&= \det\left(\begin{bmatrix} C + \mu I & X \\ X^* & \mu I - C \end{bmatrix}\right) \\
&= \det(N + \mu I).
\end{aligned}$$

Therefore, $-\mu$ is also an eigenvalue of N and the result follows. ■

Next we present a sharp norm inequality involving M with normal off-diagonal blocks. The following lemmas play a key role in establishing our result.

Lemma 2.3. [6, Proposition 2.3] *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ *be positive semi-definite with* $A, B, X \in \mathbb{C}^{n\times n}$. *If* X^* *commutes with* A *or* B, *then*

$$\|M\| \leq \|A+B\|$$

for any unitarily invariant norm $\|\cdot\|$.

By celebrated Ky-Fan dominance theorem, we have the following result.

Lemma 2.4. [1, Lemma 2.1] *Let* $X, Y \in \mathbb{C}^{n\times n}$ *Hermitian such that* $Y \pm X \succeq 0$. *Then* $\|X\| \leq \|Y\|$ *for any unitarily invariant norm* $\|\cdot\|$.

Theorem 2.5. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ *be positive semi-definite with* $A, B, X \in \mathbb{C}^{n\times n}$ *and* X *be normal. Let*

$$\begin{aligned} s_1 &= \lambda_{max}(B - XX^*), \\ s_2 &= \lambda_{max}(A - XX^*), \\ s_3 &= \lambda_{max}(B - X - X^*), \\ s_4 &= \lambda_{max}(A - X - X^*). \end{aligned}$$

Then

$$\begin{aligned} \|M\| \leq \min\Big\{ & \|A + XX^* + s_1 I\|, \|B + XX^* + s_2 I\|, \\ & \|A + X + X^* + s_3 I\|, \|B + X + X^* + s_4 I\| \Big\}. \end{aligned} \tag{5}$$

Proof. First note that

$$\begin{bmatrix} A & X \\ X^* & XX^* + s_1 I \end{bmatrix} \succeq 0 \text{ and } \begin{bmatrix} A & X \\ X^* & X + X^* + s_3 I \end{bmatrix} \succeq 0$$

because they can be expressed as summations of two positive semi-definite matrices, i.e.,

$$\begin{bmatrix} A & X \\ X^* & XX^* + s_1 I \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s_1 I - (B - XX^*) \end{bmatrix}$$

and

$$\begin{bmatrix} A & X \\ X^* & X + X^* + s_3 I \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s_3 I - (B - X - X^*) \end{bmatrix}.$$

By Lemma 2.4, we see that

$$\|M\| \leq \left\| \begin{bmatrix} A & X \\ X^* & XX^* + s_1 I \end{bmatrix} \right\| \text{ and } \|M\| \leq \left\| \begin{bmatrix} A & X \\ X^* & X + X^* + s_3 I \end{bmatrix} \right\|.$$

Note from the normality of X that

$$X(XX^* + s_1 I_n) = XXX^* + s_1 X = XX^*X + s_1 X = (XX^* + s_1 I_n)X,$$

i.e., X commutes with $XX^* + s_1 I$, and

$$X(X+X^*+s_3 I) = XX+XX^*+s_3 X = XX+X^*X+s_3 X = (X+X^*+s_3 I)X,$$

i.e., X commutes with $X + X^* + s_3 I$. Thus, by Lemma 2.3, we get

$$\left\| \begin{bmatrix} A & X \\ X^* & XX^* + s_1 I \end{bmatrix} \right\| \leq \|A + XX^* + s_1 I\|$$

and

$$\left\| \begin{bmatrix} A & X \\ X^* & X + X^* + s_3 I \end{bmatrix} \right\| \leq \|A + X + X^* + s_3 I\|.$$

Therefore,

$$\|M\| \leq \|A + XX^* + s_1 I\| \text{ and } \|M\| \leq \|A + X + X^* + s_3 I\|.$$

Similarly we can show that

$$\|M\| \leq \|B + XX^* + s_2 I\| \text{ and } \|M\| \leq \|B + X + X^* + s_4 I\|.$$

Hence the result follows. ∎

We illustrate with an example that Theorem 2.5 improves (2) for some cases.

Example 2.6. Let $X = \operatorname{diag}(0, 1.25, i)$, $A = \begin{bmatrix} 1.25 & -0.25 \\ -0.25 & 1.25 \end{bmatrix} \oplus [2] \succeq 0$ and $B = [1] \oplus \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2 \end{bmatrix} \succeq 0$. We have $s_1 \approx 1.29, s_2 \approx 1.29, s_3 \approx 2.1, s_4 = 2$, and $\omega \approx 0.78$. Taking the spectral norm, we obtain

$$\begin{aligned} \|A + XX^* + s_1 I\|_{\text{sp}} &\approx 4.29, \\ \|B + XX^* + s_2 I\|_{\text{sp}} &\approx 5.14, \\ \|A + X + X^* + s_3 I\|_{\text{sp}} &\approx 5.87, \\ \|B + X + X^* + s_4 I\|_{\text{sp}} &\approx 6.6, \\ \|A + B + \omega I\|_{\text{sp}} &\approx 5.04. \end{aligned}$$

Therefore, $\|M\|_{\text{sp}} \leq 4.29$.

Now we show that the inequality (5) is sharp. Consider the block matrix in the form:

$$M_k = \begin{bmatrix} kI - XX^* & X \\ X^* & XX^* + kI \end{bmatrix} \in \mathbb{C}^{2n\times 2n},$$

where X is normal and $k > 0$. Observe that M_k is positive semi-definite for sufficiently large k. Let us fix k, so that $M_k \succeq 0$. Obviously, $kI - XX^*$ commutes with X. By Theorem 2.2,

$$\lambda_{2n} = 2k - \lambda_1,$$

where λ_1 and λ_{2n} are the smallest and the largest eigenvalues of M_k, respectively. Therefore,

$$M_k - \lambda_1 I = \begin{bmatrix} (k-\lambda_1)I - XX^* & X \\ X^* & XX^* + (k-\lambda_1)I \end{bmatrix} \succeq 0,$$

and $\|M_k - \lambda_1 I\|_{\mathrm{sp}} = 2k - 2\lambda_1$. Applying Theorem 2.5 for $M_k - \lambda_1 I$, we see that $s_1 = k - \lambda_1$ and

$$\|A + XX^* + s_1 I\|_{\mathrm{sp}} = \|(k-\lambda_1)I - XX^* + XX^* + s_1 I\|_{\mathrm{sp}} = 2k - 2\lambda_1.$$

One may wonder the conditions under which Theorem 2.5 outperforms (2). Next we present a sufficient condition to address this question.

Remark 2.7. For a given positive semi-definite block matrix M with X normal, the inequality

$$\begin{aligned} \|M\| \le \min \Big\{ & \|A + XX^* + s_1 I\|, \|B + XX^* + s_2 I\|, \\ & \|A + X + X^* + s_3 I\|, \|B + X + X^* + s_4 I\| \Big\} \\ \le \|A + B &+ \omega I\| \end{aligned} \tag{6}$$

holds if

$$\omega \ge \min\{s_1 - t_1, s_2 - t_2, s_3 - t_3, s_4 - t_4\}, \tag{7}$$

where

$$\begin{aligned} t_1 &= \lambda_{min}(B - XX^*), \\ t_2 &= \lambda_{min}(A - XX^*), \\ t_3 &= \lambda_{min}(B - X - X^*), \\ t_4 &= \lambda_{min}(A - X - X^*). \end{aligned}$$

Here is the explanation: Assume that $s_1 - t_1$ is the minimum of the set in (7). Suppose that $\omega \ge s_1 - t_1$. Then

$$\omega I \ge s_1 I - t_1 I,$$

which leads to

$$\omega I \ge s_1 I - (B - XX^*)I$$

and then

$$\omega I + A + B \geq s_1 I + A + XX^*)I.$$

Thus,

$$\|M\| \leq \|A + XX^* + s_1 I\| \leq \|A + B + \omega I\|.$$

Similarly, we can show that (6) holds if the minimum of the set in (7) is a different value.

We conclude this paper with a partial generalization of [6, Theorem 2.9]. Here we do not assume that X is normal.

Theorem 2.8. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$ *be positive semi-definite with* $A, B, X \in \mathbb{C}^{n\times n}$. *If* $M - C \succeq 0$ *for some Hermitian diagonal block matrix* $C = C_1 \oplus C_2$, *where* $C_1, C_2 \in \mathbb{C}^{n\times n}$. *Then we have*

$$\|M\| \leq 2\left\|A + B - \frac{C_1 + C_2}{2}\right\|$$

for any unitarily invariant norm $\|\cdot\|$.

Proof. Assume that $M - C \succeq 0$ for some Hermitian diagonal block matrix $C = C_1 \oplus C_2$. Let $P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$. Then, clearly,

$$P^T(M - C)P = \begin{bmatrix} B - C_2 & X^* \\ X & A - C_1 \end{bmatrix} \succeq 0.$$

Thus $M + P^T(M - C)P$ is positive partial transpose since $X + X^*$ is a Hermitian matrix and $M + P^T(M - C)P \succeq 0$. Therefore, by [5, Proposition 2.1],

$$\begin{aligned} \left\|M + P^T(M - C)P\right\| &= \left\|\begin{bmatrix} A + B - C_2 & X + X^* \\ X + X^* & A + B - C_1 \end{bmatrix}\right\| \\ &\leq 2\left\|A + B - \frac{C_1 + C_2}{2}\right\|, \end{aligned}$$

Observe now from Lemma 2.4 that $\|M\| \leq \left\|M + P^T(M - C)P\right\|$. Hence,

$$\|M\| \leq 2\left\|A + B - \frac{C_1 + C_2}{2}\right\|.$$

■

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