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Exact Solution of the Bagley-Torvik Equation Using Laplace Transform Method

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Abstract. In this paper, the exact solution of the Bagley-Torvik equation which has an important role in fractional order differential equations has been investigated by Laplace transformation method. The Bagley-Torvik equation is transformed into an algebraic equation with Laplace transform. This algebraic equation is solved and the unknown function is found with inverse Laplace transformation. Caputo fractional derivative is considered throughout this work. The examples presented demonstrate the validity and applicability of the Laplace transformation method used to find the exact solution of the Bagley-Torvik equation.

Keywords: Laplace transform method; Fractional order differential equation; Bagley-Torvik equation.

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1. Introduction

Fractional analysis and its applications have gained considerable popularity over the last thirty years as it is widely used in science and engineering fields such as control theory, signal processing, dynamical systems and heat conduction [9, 13, 15]. Physical and geometric interpretation of fractional differentation and fractional integration was investigated [16]. A new and more exact model for seepage flow in porous media with fractional derivatives has been proposed [7]. A generalization of the Lorenz dynamic system by using fractional derivatives was introduced [6]. Exact solutions of some fractional differential equations have been investigated by some authors [1, 3, 4, 5, 8, 11, 14, 21].

The Bagley-Torvik equation first proposed by Bagley and Torvik emerged as a result of the modeling of the motion of solid plates immersed in Newton's fluid [19]. The Bagley-Torvik Equation with Caputo derivative is given as the following form;

$$Ay''(t) + B\ {}^{c}D^{3/2}y(t) + Cy(t) = f(t), \qquad 0 \leq t \leq b \tag{1}$$

subject to the initial conditions

$$y(0) = \gamma_1, \quad y'(0) = \gamma_2, \tag{2}$$

where f(t) is function defined on the interval $0 \leq t \leq b$, ${}^{c}D^{3/2}$ is the derivative of y of order 3/2 in the sense of Caputo fractional differential operator, $A,\ B,\ C,\ b,\ \gamma_1,\ \gamma_2$ are real constants, $y(t)$ is an unknown function of the independent variable t. Exact solution of this equation has been investigated by some authors [2, 10, 17].

The main aim of this paper is to solve the Bagley-Torvik equation exactly by using Laplace transform method. The given method converts the mentioned equation to the algebraic equation. Since expressing this algebraic equation is solved and the unknown function is found with inverse Laplace transformation.

2. Fractional Calculus

In this section, some basic subjects of the fractional calculus which are used throught this paper are given.

Gamma Function: The Gamma function is defined by the improper integral [12]

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx, \qquad \Gamma : (0,\infty) \longrightarrow \mathbb{R} \tag{3}$$

Gamma function is

(i) convergent for $0 < p < \infty$
(ii) divergent for $p \leq 0$.

Beta Function: The Beta function is defined by the integral [12]

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \qquad \beta : (0,\infty)x(0,\infty) \longrightarrow \mathbb{R} \tag{4}$$

Beta function is

(i) convergent for $p > 0$ and $q > 0$

(ii) divergent for $p \leq 0$ and $q \leq 0$.

The Caputo fractional derivative: The Caputo fractional derivative of function f is defined by the integral

$$^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt \tag{5}$$

where f function can be continuously differentiable m times, α any positive integer and m is a positive integer such that $m \in \mathbb{N}$, $m-1 < \alpha < m$ [9].

3. Laplace Transform Method

Laplace transform is a method frequently employed in the engineering and science. The Laplace transform is particularly useful in solving linear ordinary differential equations. By applying the Laplace transform, one can change an ordinary differential equation into an algebraic equation, as algebraic equation is generally easier to deal with.

The Laplace transform is an integral transform. If $f(t)$ is defined over interval $[0,\infty)$, then the Laplace transform of $f(t)$, denoted as $F(s)$, is given as follow in [18, 20]:

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{6}$$

The Inverse Laplace Transform of $F(s)$ is defined as

$$f(t) = \mathcal{L}^{-1}[F(s)]. \tag{7}$$

The Laplace transform existence theorem states that, if $f(t)$ is piecewise continuous on every finite interval in $[0,\infty)$ satisfying

$$|f(t)| \leq Me^{at}$$

for all t in $[0,\infty)$, then $\mathcal{L}[f(t)]$ exists for all $s > a$.

The Laplace transform is also unique, in the sense that, given two functions $f_1(t)$ and $f_2(t)$ with the same transform so that

$$\mathcal{L}[f_1(t)] = \mathcal{L}[f_2(t)] = F(s).$$

The Laplace transform has many important properties.

Laplace and inverse Laplace transforms are linear:

$$\begin{aligned}\mathcal{L}[af(t)+bg(t)] &= a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)] = aF(s)+bG(s),\\ \mathcal{L}^{-1}[aF(s)+bG(s)] &= a\mathcal{L}^{-1}[F(s)]+b\mathcal{L}^{-1}G(s) = af(t)+bg(t).\end{aligned}$$

where the functions $f(t)$ and $g(t)$ are two separate functions which Laplace transforms can be taken and a, b are real constants.

Laplace transform of force function:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \tag{8}$$

$$\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n. \tag{9}$$

Laplace transform of integration:

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}, \tag{10}$$

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du. \tag{11}$$

Laplace transform of derivative:

$$\begin{aligned}\mathcal{L}[f'(t)] &= sF(s)-f(0)\\ \mathcal{L}[f''(t)] &= s^2F(s)-sf(0)-f'(0)\\ &\vdots\\ \mathcal{L}[f^{(n)}(t)] &= s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)\end{aligned}$$

Laplace transform of Caputo fractional derivative:

$$\mathcal{L}[{}^CD^{\alpha}f(t)] = \frac{s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)}{s^{n-\alpha}} \tag{12}$$

where $n\in\mathbb{N},\ n-1<\alpha\leq n$ (see [9, 15]).

4. Illustrative Examples

In this section, the exact solution of the Bagley-Torvik Equation was obtained for four test examples by applying Laplace transformation method.

Example 4.1. As the first example, we consider the following the Bagley-Torvik equation:

$$y''(t)+{}^CD^{3/2}y(t)+y(t) = t^3+5t+\frac{8t^{3/2}}{\sqrt{\pi}}$$

and initial conditions

$$y(0) = 0, \quad y'(0) = -1.$$

If Laplace transform is applied to this equation the algebraic equation is obtained as follows:

$$\begin{aligned}
\mathcal{L}\left[y''(t) +^{C} D^{3/2}y(t) + y(t)\right] &= \mathcal{L}\left[t^3 + 5t + \frac{8t^{3/2}}{\sqrt{\pi}}\right], \\
s^2Y(s) - sy(0) - y'(0) + \frac{s^2Y(s) - sy(0) - y'(0)}{s^{1/2}} + Y(s) &= \frac{6}{s^4} + \frac{5}{s^2} + \frac{6}{s^{5/2}}, \\
Y(s)\left(s^2 + s^{3/2} + 1\right) + s^{-1/2} + 1 &= \frac{6}{s^4} + \frac{5}{s^2} + \frac{6}{s^{5/2}}, \\
Y(s) &= \frac{6}{s^4} - \frac{1}{s^2}
\end{aligned}$$

where $\mathcal{L}[y(t)] = Y(s)$. If inverse Laplace transform of this algebraic equation is taken, exact solution of the initial value problem is found as follows:

$$\begin{aligned}
\mathcal{L}^{-1}\left[Y(s)\right] &= \mathcal{L}^{-1}\left[\frac{6}{s^4} - \frac{1}{s^2}\right], \\
y(t) &= t^3 - t.
\end{aligned}$$

Example 4.2. As the second example, the following form of the Bagley-Torvik equation has been studied:

$$^{C}D^{3/2}y(t) + y(t) = t^4 - 8t + \frac{64t^{5/2}}{5\sqrt{\pi}}$$

and initial conditions are as follows

$$y(0) = 0, \quad y'(0) = -8.$$

If Laplace transform is implemented to this equation the algebraic equation is found as follows:

$$\begin{aligned}
\mathcal{L}\left[^{C}D^{3/2}y(t) + y(t)\right] &= \mathcal{L}\left[t^4 - 8t + \frac{64t^{5/2}}{5\sqrt{\pi}}\right], \\
\frac{s^2Y(s) - sy(0) - y'(0)}{s^{1/2}} + Y(s) &= \frac{24}{s^5} - \frac{8}{s^2} + \frac{24}{s^{7/2}}, \\
Y(s)\left(s^{3/2} + 1\right) &= \frac{24 - 8s^3 + 24s^{3/2} - 8s^{9/2}}{s^5}, \\
Y(s) &= \frac{24}{s^5} - \frac{8}{s^2}
\end{aligned}$$

where $\mathcal{L}[y(t)] = Y(s)$. If inverse Laplace transform of this algebraic equation is received, exact solution of the initial value problem is obtained as follow:

$$\begin{aligned}
\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}[\frac{24}{s^5} - \frac{8}{s^2}], \\
y(t) &= t^4 - 8t.
\end{aligned}$$

Example 4.3. As the third example, the following the Bagley-Torvik equation is given:

$$^{C}D^{3/2}y(t) + y(t) = \frac{2t^{1/2}}{\Gamma(3/2)} + t^2 - t$$

and initial conditions

$$y(0) = 0, \quad y'(0) = -1.$$

The algebraic equation is determined if Laplace transform is applied to this equation as follows:

$$\begin{aligned}
\mathcal{L}\left[{}^{C}D^{3/2}y(t) + y(t)\right] &= \mathcal{L}\left[\frac{2t^{1/2}}{\Gamma(3/2)} + t^2 - t\right], \\
\frac{s^2Y(s) - sy(0) - y'(0)}{s^{1/2}} + Y(s) &= \frac{2}{s^{3/2}} + \frac{2}{s^3} - \frac{1}{s^2}, \\
Y(s)(s^{3/2} + 1) &= \frac{2s^{3/2} + 2 - s - s^{5/2}}{s^3}, \\
Y(s) &= \frac{2}{s^3} - \frac{1}{s^2}
\end{aligned}$$

where $\mathcal{L}[y(t)] = Y(s)$. If inverse Laplace transform of this algebraic equation is taken, exact solution of the initial value problem is found as follow:

$$\begin{aligned}
\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{2}{s^3} - \frac{1}{s^2}\right], \\
y(t) &= t^2 - t.
\end{aligned}$$

Example 4.4. As the fourth example, the following the Bagley-Torvik equation has been considered:

$$y''(t) +^{C} D^{3/2}y(t) + y(t) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}}$$

and initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

If Laplace transform is implemented to this equation the algebraic equation is found as follows:

$$\begin{aligned}
\mathcal{L}\left[y''(t) +^{C} D^{3/2}y(t) + y(t)\right] &= \mathcal{L}\left[t^2 + 2 + 4\sqrt{\frac{t}{\pi}}\right], \\
s^2Y(s) - sy(0) - y'(0) + \frac{s^2Y(s) - sy(0) - y'(0)}{s^{1/2}} + F(s) &= \frac{2}{s^3} + \frac{2}{s} + \frac{2}{s^{3/2}}, \\
Y(s)(s^2 + s^{3/2} + 1) &= \frac{2 + 2s^2 + 2s^{3/2}}{s^3}, \\
Y(s) &= \frac{2}{s^3}
\end{aligned}$$

where $\mathcal{L}[y(t)] = Y(s)$. Exact solution of the initial value problem is obtained if inverse Laplace transform of this algebraic equation is received as follows:

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{2}{s^3}\right],$$
$$y(t) = t^2.$$

5. Conclusion

In this paper, the exact solution of The Bagley-Torvik equation was obtained by applying Laplace and Inverse Laplace Transform. To demonstrate the applicability and efficiency of the proposed method four examples were examined. It was seen that The Laplace Transform Method was a remarkably successful technique for finding exact solution of the Bagley-Torvik equation.

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