Southeast Asian Bulletin of Mathematics (2022) 46: 737–748

Southeast Asian
Bulletin of
Mathematics

On the Largest Singular Value of a Matrix and Generalized Inverses

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Received 20 March 2021
Accepted 17 March 2022

Communicated by Pan Shun Lau

Dedicated to the memory of Professor Ky Fan (1914–2010)

AMS Mathematics Subject Classification(2020): 15A60, 39B42, 26D15, 06F20

Abstract. In this paper we study properties of the largest singular value $s_1(\cdot)$ of matrices viewed as a norm on the space of complex matrices. We give a refinement of the submultiplicativity inequality characterizing $s_1(\cdot)$. In our approach we use the equality case of the inequality. We introduce a corresponding preorder on the matrix space and show the monotonicity of a certain functional induced by $s_1(\cdot)$. We also provide some inequalities by using generalized inverses of matrices.

Keywords: Complex (real) matrix; Largest singular value; Monotone functional; Generalized inverse of matrix.

1. Preliminaries

Throughout $\mathbb{M}_n$ denotes the set of all $n \times n$ matrices over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\mathbb{U}_n$ denotes the set of all $n \times n$ unitary matrices. As usual, I_n stands for the $n \times n$ identity matrix. The symbol $(\cdot)^*$ means the conjugate transpose of a matrix. The elements of $\mathbb{F}^n$ are viewed as $n \times 1$ column vectors. The norm on $\mathbb{F}^n$ is given by $|x| = (x^*x)^{1/2}$ for $x \in \mathbb{F}^n$.

For a given hermitian matrix $A \in \mathbb{M}_n$, the *eigenvalues* of A stated in decreasing order are denoted by $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$.

By Spectral Decomposition, each hermitian matrix A has the form $A = U(\operatorname{diag}\lambda(A))U^*$ for some unitary matrix U of order n, where the symbol $\operatorname{diag} a$ means the diagonal matrix with the entries of the vector $a \in \mathbb{F}^n$ on the main diagonal.

For a given matrix $A \in \mathbb{M}_n$, by $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ are denoted the *singular values* of A arranged in decreasing order. So, $s_i(A) = \lambda_i(AA^*)^{1/2}$ for $i = 1, 2, \ldots, n$.

By Singular Value Decomposition, each matrix $A \in \mathbb{M}_n$ has the form $A = U_1 \operatorname{diag} s(A) U_2^*$ for some unitary matrices U_1, U_2 of order n.

It is well known that the map $s_1(\cdot) : X \mapsto s_1(X)$ for $X \in \mathbb{M}_n$ is a norm on $\mathbb{M}_n$. In addition, $s_1(\cdot)$ is *unitarily invariant* (abbreviated as *u.i.*), that is, $s_1(UXV) = s_1(X)$ for all $X \in \mathbb{M}_n$ and $U, V \in \mathbb{U}_n$ (see [1, 2, 4]).

Furthermore, $s_1(\cdot)$ is *submultiplicative*, as follows:

$$s_1(XY) \leq s_1(X)s_1(Y) \quad \text{for all } X, Y \in \mathbb{M}_n \tag{1}$$

(see [1, 2]).

As usual, hermitian $A, B \in \mathbb{M}_n$ are called *simultaneously diagonalizable* if there exists $n \times n$ unitary matrix U such that $A = U(\operatorname{diag}\lambda(A))U^*$ and $B = U(\operatorname{diag}\lambda(B))U^*$.

In this paper, matrices $A, B \in \mathbb{M}_n$ are called *simultaneously diagonalizable via singular values*, if there exist $n \times n$ unitary matrices U_1, U_2 such that $A = U_1(\operatorname{diag} s(A))U_2^*$ and $B = U_1(\operatorname{diag} s(B))U_2^*$.

Likewise, matrices $A, B \in \mathbb{M}_n$ are called *semi-simultaneously diagonalizable via singular values*, if there exist $n \times n$ unitary matrices U_1, U_2, U_3 such that $A = U_1(\operatorname{diag} s(A))U_2^*$ and $B = U_3(\operatorname{diag} s(B))U_2^*$. In such a case one has $AB^* = U_1(\operatorname{diag}(s(A) \circ s(B)))U_3^*$, where $\circ$ is the Hadamard product on $\mathbb{F}^n$. In consequence, $s_1(AB^*) = s_1(A)s_1(B)$. This is an equality case of inequality (1) (cf. [1, 2]).

In the present paper, our aim is to derive some refinements of the basic inequality (1) by using the mentioned equality cases of (1). Some analogous subadditivity problems have been investigated recently, too (see e.g., [5, 9, 8, 7]). They are related to Dunkl-Williams and Maligranda's inequalities and angular distance of vectors (see e.g., [6, 10, 11]). In the next sections we initiate preparing a framework for corresponding studies on submultiplicative case.

2. Inequalities for the Largest Singular Value

For given $X, Y \in \mathbb{M}_n$, we write $Y \preceq X$ if there exists $W \in \mathbb{M}_n$ such that

$$X = YW \quad \text{and} \quad s_1(X) = s_1(Y)s_1(W). \tag{2}$$

Observe that (2) gives

$$s_1(YW) = s_1(Y)s_1(W), \tag{3}$$

which is an equality case of inequality (1).

Example 2.1. Let $A \geq 0$ be an $n \times n$ (hermitian) positive semidefinite matrix. Then

$$A^{1/2} \preceq A.$$

In fact, for $X = A$, $Y = W = A^{1/2}$ we have $X = YW$ and $A = U(\operatorname{diag}\lambda(A))U^*$ and $A^{1/2} = U(\operatorname{diag}\lambda(A^{1/2}))U^*$ for some $n \times n$ unitary matrix U. In consequence,

$$s_1(X) = s_1(A) = \lambda_1(A) = \left(\lambda_1(A^{1/2})\right)^2 = \left(s_1(A^{1/2})\right)^2 = s_1(Y)s_1(W),$$

as wanted.

In the sequel we shall study properties of the relation $\preceq$ on $\mathbb{M}_n$ in connection with the problem of refining the basic inequality (1). To this end we shall utilize statements of type (3).

Theorem 2.2. *The relation $\preceq$ is transitive and reflexive on $\mathbb{M}_n$, and hence $\preceq$ is a preorder on $\mathbb{M}_n$.*

Proof. Assume that $X, Y, Z \in \mathbb{M}_n$ are such that $Z \preceq Y$ and $Y \preceq X$. Then there exist $U, W \in \mathbb{M}_n$ such that

$$Y = ZU \quad \text{and} \quad X = YW \tag{4}$$

and

$$s_1(Y) = s_1(Z)s_1(U) \quad \text{and} \quad s_1(X) = s_1(Y)s_1(W) \tag{5}$$

(see (2)). By denoting $V = UW$ we get from (4) that

$$X = YW = ZUW = ZV. \tag{6}$$

By (5) we can write

$$s_1(X) = s_1(Y)s_1(W) = s_1(Z)s_1(U)s_1(W). \tag{7}$$

Simultaneosly, we have

$$s_1(V) = s_1(UW) \leq s_1(U)s_1(W)$$

by (1). Hence,

$$s_1(Z)s_1(V) \leq s_1(Z)s_1(U)s_1(W),$$

which together with (7) leads to

$$s_1(Z)s_1(V) \leq s_1(X). \tag{8}$$

However, due to (6) we obtain

$$s_1(X) = s_1(ZV) \le s_1(Z)s_1(V). \tag{9}$$

So, by combining (8) and (9) we establish

$$s_1(X) = s_1(Z)s_1(V).$$

Additionally, in light of (6) we see that $X = ZV$. Therefore, by (2), we infer that $Z \preceq X$. Thus we have proved that $\preceq$ is transitive on $\mathbb{M}_n$.

Finally, it is clear that $s_1(I_n) = 1$. It now follows that

$$X = XI_n \quad \text{and} \quad s_1(X) = s_1(X)s_1(I_n) \quad \text{for } X \in \mathbb{M}_n.$$

So, we get $X \preceq X$ for $X \in \mathbb{M}_n$. That is, $\preceq$ is reflexive on $\mathbb{M}_n$, as wanted. In summary, $\preceq$ is transitive and reflexive. So, it is a preorder on $\mathbb{M}_n$. ■

With the help of the relation $\preceq$, we now present a refinement of inequality (1).

Theorem 2.3. *If $X, Y \in \mathbb{M}_n$ and $Z \in \mathbb{M}_n \setminus \{0\}$ are such that $Z \preceq Y$, then*

$$s_1(XY) \le s_1(XZ)s_1(Z)^{-1}s_1(Y) \le s_1(X)s_1(Y). \tag{10}$$

Proof. As $Z \preceq Y$, we see that

$$Y = ZW \quad \text{and} \quad s_1(Y) = s_1(Z)s_1(W)$$

for some $W \in \mathbb{M}_n$. Therefore, we get

$$s_1(W) = s_1(Z)^{-1}s_1(Y).$$

In consequence, by (1),

$$\begin{aligned} s_1(XY) &= s_1(XZW) \le s_1(XZ)s_1(W) = s_1(XZ)s_1(Z)^{-1}s_1(Y) \\ &\le s_1(X)s_1(Z)s_1(Z)^{-1}s_1(Y) = s_1(X)s_1(Y). \end{aligned}$$

Thus (10) is proven, as claimed. ■

In the example below we show that (10) in Theorem 2.3 can be a strict inequality.

Example 2.4. We choose $n = 2$ and

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad Y = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \quad Z = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then

$$s_1(X) = 3, \quad s_1(Y) = 6, \quad s_1(Z) = 2, \quad s_1(W) = 3.$$

Moreover, $Y = ZW$ and $s_1(Y) = s_1(Z)s_1(W)$, which gives $Z \preceq Y$. In addition,

$$XY = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad XZ = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$s_1(XY) = 6, \quad s_1(XZ) = 3.$$

Therefore we have the strict inequalities

$$s_1(XY) < s_1(XZ)s_1(Z)^{-1}s_1(Y) < s_1(X)s_1(Y). \tag{11}$$

This shows that (10) is really an improvement of the basic inequality (1).

Remark 2.5. A preliminary version of Theorem 2.3 is as follows.

If $X, Y \in \mathbb{M}_n$ and $Z, W \in \mathbb{M}_n$ are such that $Y = ZW$, then

$$s_1(XY) \leq s_1(XZ)s_1(W) \leq s_1(X)s_1(Y). \tag{12}$$

Here we do not assume that $s_1(Y) = s_1(Z)s_1(W)$. However, with this assumption, the statements (10) and (12) are the same (for $Z \neq 0$).

The initial inequality $s_1(XY) \leq s_1(X)s_1(Y)$ is of the form (12) for the decomposition $Y = I_nY$. Because $s_1(I_n) = 1$ and $s_1(Y) = s_1(I_n)s_1(Y)$, so one has $I_n \preceq Y$, and, consequently, the last scalar inequality is of the form (10).

Example 2.6. Let $A, B \geq 0$ be two commuting positive semidefinite matrices in $\mathbb{M}_n$ such that $A = U(\operatorname{diag}\lambda(A))U^*$ and $B = U(\operatorname{diag}\lambda(B))U^*$ for some $n \times n$ unitary matrix U. So, A, B are simultaneously diagonalizable.

We obtain

$$\begin{aligned}
s_1(AB) &= s_1(U(\operatorname{diag}\lambda(A))U^*U(\operatorname{diag}(\lambda(B)))U^*) \\
&= s_1(U((\operatorname{diag}\lambda(A))(\operatorname{diag}\lambda(B)))U^*) \\
&= s_1(U(\operatorname{diag}(\lambda(A) \circ \lambda(B)))U^*) \\
&= s_1(U(\operatorname{diag}(s(A) \circ s(B)))U^*) \\
&= s_1(\operatorname{diag}(s(A) \circ s(B))) \\
&= s_1(A)s_1(B).
\end{aligned}$$

Therefore $A \preceq AB$ (see (2)). So, according to Theorem 2.3, we conclude that

$$s_1(XAB) \leq s_1(XA)s_1(B) \leq s_1(X)s_1(AB). \tag{13}$$

In particular, for any $X \in \mathbb{M}_n$ and positive semidefinite A, applying (13) to $A^{1/2}$ and $A^{1/2}$ in place of A and B, respectively, we get

$$s_1(XA) \leq s_1(XA^{1/2})s_1(A^{1/2}) \leq s_1(X)s_1(A).$$

For given $X, Y \in \mathbb{M}_n$, we define the following functional

$$\varphi(Z) = s_1(XZ)s_1(Z)^{-1}s_1(Y) \quad \text{for } Z \in \mathbb{M}_n \setminus \{0\},\, Z \preceq Y. \tag{14}$$

The last theorem shows a role of the values of functional (14) as refining numbers in (1). We now demonstrate monotonicity of the functional.

Theorem 2.7. *If $Y \in \mathbb{M}_n$ and $Z_1, Z_2 \in \mathbb{M}_n \setminus \{0\}$ are such that $Z_2 \preceq Z_1$ and $Z_1 \preceq Y$, then for any $X \in \mathbb{M}_n$,*

$$s_1(XZ_1)s_1(Z_1)^{-1}s_1(Y) \le s_1(XZ_2)s_1(Z_2)^{-1}s_1(Y). \tag{15}$$

Proof. Because $Z_2 \preceq Z_1$ and $Z_1 \preceq Y$, we get

$$Z_1 = Z_2U \quad \text{and} \quad Y = Z_1W \tag{16}$$

and

$$s_1(Z_1) = s_1(Z_2)s_1(U) \quad \text{and} \quad s_1(Y) = s_1(Z_1)s_1(W)$$

for some $U, W \in \mathbb{M}_n$. Hence,

$$s_1(U) = s_1(Z_2)^{-1}s_1(Z_1) \quad \text{and} \quad s_1(W) = s_1(Z_1)^{-1}s_1(Y). \tag{17}$$

From (16) we deduce that

$$s_1(XZ_1) = s_1(XZ_2U) \le s_1(XZ_2)s_1(U),$$

and further

$$s_1(XZ_1)s_1(W) \le s_1(XZ_2)s_1(U)s_1(W). \tag{18}$$

According to (17) and (18) we obtain

$$\begin{aligned} s_1(XZ_1)s_1(Z_1)^{-1}s_1(Y) &\le s_1(XZ_2)s_1(Z_2)^{-1}s_1(Z_1)s_1(Z_1)^{-1}s_1(Y) \\ &= s_1(XZ_2)s_1(Z_2)^{-1}s_1(Y), \end{aligned}$$

which gives (15), as required. ∎

Remark 2.8. Under the hypotheses of Theorem 2.7, it holds that if $Z_2 \preceq Z_1 \preceq Y$ then

$$\begin{aligned} s_1(XY) &\le s_1(XZ_1)s_1(Z_1)^{-1}s_1(Y) \\ &\le s_1(XZ_2)s_1(Z_2)^{-1}s_1(Y) \le s_1(X)s_1(Y). \end{aligned} \tag{19}$$

To see this, combine Theorems 2.2, 2.3 and 2.7.

Because $s_1(I_n) = 1$, one has $I_n \preceq Z_2 \preceq Z_1 \preceq Y$. Then (19) reads as

$$\varphi(Y) \le \varphi(Z_1) \le \varphi(Z_2) \le \varphi(I_n),$$

where φ is given by (14). This is the announced property of the monotonicity of the functional φ.

Example 2.9. For $i = 1, 2$, we assume $A_i, B_i \in \mathbb{M}_n$ are two matrices simultaneously diagonalized via singular values, that is, there exist two unitaries U_{i1}, U_{i2} of size $n \times n$ such that $A_i = U_{i1}(\operatorname{diag} s(A_i))U_{i2}^*$ and $B_i = U_{i1}(\operatorname{diag} s(B_i))U_{i2}^*$. Assume $A_1 B_1^* = A_2 B_2^*$. It is not hard to show that $s_1(A_i B_i^*) = s_1(A_i) s_1(B_i^*)$.

By taking $Y = A_i B_i^*$, $Z_i = A_i$ and $W_i = B_i^*$, we see that $Y = Z_i W_i$ and $s_1(Y) = s_1(Z_i)s_1(W_i)$. Therefore $Z_i \preceq Y$ (see (2)).

According to Theorem 2.7, for any $X \in \mathbb{M}_n$ we conclude that if $Z_2 \preceq Z_1$ then (15) holds.

3. Identities and Inequalities

A map $(\cdot)^- : \mathbb{M}_n \to \mathbb{M}_n$ is called a *generalized inverse* on $\mathbb{M}_n$ if

$$AA^-A = A \quad \text{for all } A \in \mathbb{M}_n \tag{20}$$

(see [12]).

For example, if

$$A = U_1(\operatorname{diag}(s_1(A), \ldots, s_n(A)))U_2^*$$

is the Singular Value Decomposition of A with unitary U_1 and U_2 and singular values $s_1(A) \geq \ldots \geq s_n(A) \geq 0$, then one can put

$$A^- = U_2(\operatorname{diag}(\sigma_1(A), \ldots, \sigma_n(A)))U_1^*,$$

where $\sigma_i(A) = \frac{1}{s_i(A)}$ if $s_i(A) \neq 0$, and $\sigma_i(A) = 0$ if $s_i(A) = 0$.

We introduce a functional Φ by

$$\Phi(A, C, B) = \frac{s_1(B^-C)s_1(C^-A)}{s_1(B^-A)} \tag{21}$$

for all $A, B, C \in \mathbb{M}_n$ such that $B^-A \neq 0$.

Lemma 3.1. *Let $X, Y, Z, W \in \mathbb{M}_n$ with $Z^-X \neq 0$, $W^-X \neq 0$, $Z^-Y \neq 0$. Then*

$$\Phi(X, W, Z)\ \Phi(X, Y, W) = \Phi(X, Y, Z)\ \Phi(Y, W, Z). \tag{22}$$

Proof. By (21) we get

$$\begin{aligned}\Phi(X, W, Z)\ \Phi(X, Y, W) &= \frac{s_1(Z^-W)s_1(W^-X)}{s_1(Z^-X)}\ \frac{s_1(W^-Y)s_1(Y^-X)}{s_1(W^-X)} \\ &= \frac{s_1(Z^-W)s_1(W^-Y)s_1(Y^-X)}{s_1(Z^-X)}.\end{aligned}$$

Similarly, we have

$$\Phi(X,Y,Z)\,\Phi(Y,W,Z) = \frac{s_1(Z^-Y)s_1(Y^-X)}{s_1(Z^-X)}\,\frac{s_1(Z^-W)s_1(W^-Y)}{s_1(Z^-Y)} = \frac{s_1(Y^-X)s_1(Z^-W)s_1(W^-Y)}{s_1(Z^-X)}.$$

Therefore the result follows. ■

It is also not hard to check that

$$\Phi(Z,X,W)\,\Phi(X,Y,W) = \Phi(Z,X,Y)\,\Phi(Z,Y,W), \tag{23}$$

$$\Phi(X,W,Y)\,\Phi(X,Z,W) = \Phi(Z,W,Y)\,\Phi(X,Z,Y), \tag{24}$$

$$\Phi(Y,X,W)\,\Phi(X,Z,W) = \Phi(Y,Z,W)\,\Phi(Y,X,Z). \tag{25}$$

Throughout, for a matrix $A \in \mathbb{M}_n$ the symbol $R(A)$ stands for the *range* of A defined by $R(A) = \{Ax : x \in \mathbb{F}^n\}$.

It follows from (1) that

$$s_1(B^-A) \le s_1(B^-C)s_1(C^-A) \quad \text{for } A,B,C \in \mathbb{M}_n \tag{26}$$

such that $R(A) \subset R(C)$, because we get $A = CV$ for some $V \in \mathbb{M}_n$, and next

$$(B^-C)(C^-A) = B^-(CC^-CV) = B^-(CV) = B^-A. \tag{27}$$

Therefore, in light of (21) and (26),

$$\Phi(A,C,B) \ge 1 \tag{28}$$

for $A,B,C \in \mathbb{M}_n$ such that $B^-A \ne 0$ and $R(A) \subset R(C)$.

In the special case when C is invertible (i.e., there exists C^{-1}), we have $C^- = C^{-1}$, and the condition $R(A) \subset R(C)$ is fulfilled automatically.

In the next theorem we present and prove four inequalities of type (1) expressed in the terms of functional Φ. In doing so, we utilize identity (22). Such an approach is very useful in deriving refinements of (1) (see the next section). Each of the forthcoming inequalities (29)-(32) formally involves four matrices, but in fact it is of the form $s_1(c) \le s_1(a)s_1(b)$ for some matrices $a,b,c \in \mathbb{M}_n$. The used inclusions ensure that $c = ab$ by (20) and (27), and, in consequence, (1) via (28) can be applied in the proof.

Theorem 3.2. *Let $X,Y,Z,W \in \mathbb{M}_n$ be matrices with $Z^-X \ne 0$, $W^-X \ne 0$, $Z^-Y \ne 0$. Then*

$$R(X) \subset R(Y) \quad \textit{implies} \quad \Phi(X,W,Z) \le \Phi(X,Y,Z)\,\Phi(Y,W,Z), \tag{29}$$

$$R(X) \subset R(W) \quad \textit{implies} \quad \Phi(X,Y,W) \le \Phi(X,Y,Z)\,\Phi(Y,W,Z), \tag{30}$$

$$R(Y) \subset R(W) \quad \textit{implies} \quad \Phi(X,Y,Z) \le \Phi(X,W,Z)\ \Phi(X,Y,W), \tag{31}$$

$$R(X) \subset R(Y) \quad \textit{implies} \quad \Phi(Y,W,Z) \le \Phi(X,W,Z)\ \Phi(X,Y,W). \tag{32}$$

Proof. If $R(X) \subset R(Y)$ then $1 \le \Phi(X,Y,W)$ by (28), whence

$$\Phi(X,W,Z) \le \Phi(X,W,Z)\ \Phi(X,Y,W). \tag{33}$$

By (22) in Lemma 3.1 we have

$$\Phi(X,W,Z)\ \Phi(X,Y,W) = \Phi(X,Y,Z)\ \Phi(Y,W,Z). \tag{34}$$

This and (33) give

$$\Phi(X,W,Z) \le \Phi(X,Y,Z)\ \Phi(Y,W,Z).$$

Thus statement (29) is proven.

The proofs of (30)-(32) are similar, and therefore omitted. ■

Remark 3.3. By making use of (23), one can obtain further inequalities of type (29)-(32), for instance

$$R(Z) \subset R(X) \quad \text{implies} \quad \Phi(Z,X,W) \le \Phi(Z,X,Y)\ \Phi(Z,Y,W), \tag{35}$$

$$R(X) \subset R(Y) \quad \text{implies} \quad \Phi(X,Y,W) \le \Phi(Z,X,Y)\ \Phi(Z,Y,W) \tag{36}$$

for $X,Y,Z,W \in \mathbb{M}_n$ with $W^-Z \neq 0$, $Y^-Z \neq 0$, $W^-X \neq 0$.

4. Refining Inequalities of Type $\Phi(A,C,B) \geq 1$

Let $(\cdot)^- : \mathbb{M}_n \to \mathbb{M}_n$ be a generalized inverse map on $\mathbb{M}_n$ (see (20)). For $Y, Z \in \mathbb{M}_n$, we introduce the following set

$$\begin{aligned}[Z,Y]_{s_1(\cdot)} = \{&W \in \mathbb{M}_n : W = ZV_1 \text{ and } Y = WV_2 \text{ for some } V_1, V_2 \in \mathbb{M}_n, \\ &\text{and } s_1(Z^-W)s_1(W^-Y) = s_1(Z^-Y)\}.\end{aligned}$$

Notice that by (20) we have: if $W \in [Z,Y]_{s_1(\cdot)}$ then

$$(Z^-W)(W^-Y) = Z^-(WW^-WV_2) = Z^-WV_2 = Z^-Y.$$

Example 4.1. Let $Z, V_1, V_2 \in \mathbb{M}_n$ be matrices with positive singular values such that

$$Z = U_1(\operatorname{diag} s(Z))U_2^*, \quad V_1 = U_2(\operatorname{diag} s(V_1))U_3^*, \quad V_2 = U_3(\operatorname{diag} s(V_2))U_4^*$$

for some unitaries U_1, U_2, U_3, U_4. Evidently, the matrices Z nad V_1^* as well as V_1 and V_2^* are semi-simultaneously diagonalizable via singular values.

We introduce $W = ZV_1$ and $Y = WV_2$. Then we have

$$\begin{aligned} W &= U_1(\operatorname{diag}(s(Z) \circ s(V_1)))U_3^*, \\ Y &= U_1(\operatorname{diag}(s(Z) \circ s(V_1) \circ s(V_2)))U_4^*, \\ V_1V_2 &= U_2(\operatorname{diag}(s(V_1) \circ s(V_2)))U_4^*. \end{aligned}$$

As a corollary, we obtain

$$s_1(Z^{-1}W)s_1(W^{-1}Y) = s_1(V_1)s_1(V_2)$$

and

$$s_1(Z^-Y) = s_1(Z^{-1}Y) = s_1(Z^{-1}WW^{-1}Y) = s_1(V_1V_2) = s_1(V_1)s_1(V_2).$$

Thus we get $s_1(Z^-W)s_1(W^-Y) = s_1(Z^-Y)$.

In conclusion, we obtain $W \in [Z, Y]_{s_1(\cdot)}$, as desired.

Theorem 4.2. *Let $X, Y, Z, W \in \mathbb{M}_n$ with $Z^-Y \neq 0$, $X^-Y \neq 0$, $X^-W \neq 0$. If $W \in [Z, Y]_{s_1(\cdot)}$ then the following refinement of the standard inequality $1 \leq \Phi(Y, Z, X)$ holds:*

$$1 \leq \max\{\Phi(W, Z, X), \Phi(Y, W, X)\} \leq \Phi(Y, Z, X). \tag{37}$$

Proof. It is not hard to check that

$$\Phi(Y, W, Z)\ \Phi(Y, Z, X) = \Phi(W, Z, X)\ \Phi(Y, W, X). \tag{38}$$

Suppose $W \in [Z, Y]_{s_1(\cdot)}$. Hence $s_1(Z^-W)s_1(W^-Y) = s_1(Z^-Y)$ with $R(W) \subset R(Z)$ and $R(Y) \subset R(W)$. So, $\Phi(Y, W, Z) = 1$ and $R(Y) \subset R(Z)$. Therefore $\Phi(Y, Z, X) \geq 1$ by (28). For this reason we get

$$\Phi(Y, Z, X) = \Phi(Y, W, Z)\ \Phi(Y, Z, X).$$

On account of (38), this condition is equivalent to

$$\Phi(Y, Z, X) = \Phi(W, Z, X)\ \Phi(Y, W, X).$$

It now follows from this that

$$\begin{aligned} \Phi(Y, Z, X) =& \Phi(W, Z, X)\ \Phi(Y, W, X) \\ &\geq \max\{\Phi(W, Z, X), \Phi(Y, W, X)\} \geq 1, \end{aligned} \tag{39}$$

because

$$\Phi(Y, W, X) \geq 1 \quad \text{and} \quad \Phi(W, Z, X) \geq 1$$

by (28), since $R(Y) \subset R(W)$ and $R(W) \subset R(Z)$ with $X^-Y \neq 0$ and $X^-W \neq 0$.

Finally, (39) yields (37), as wanted. ∎

Corollary 4.3. *Let* $X, Y, Z, W \in \mathbb{M}_n$ *with* $Z^-Y \neq 0$, $X^-Y \neq 0$, $X^-W \neq 0$. *If* $W \in [Z, Y]_{s_1(\cdot)}$ *then*

$$s_1(X^-Y) \leq s_1(X^-W)s_1(W^-Y) \leq s_1(X^-Z)s_1(Z^-Y). \tag{40}$$

Proof. Assume that $W \in [Z, Y]_{s_1(\cdot)}$. Then $R(W) \subset R(Z)$ and $R(Y) \subset R(W)$. Therefore $\Phi(Y, W, X) \geq 1$ by (28).

By employing (37) we find that

$$\Phi(Y, W, X) \leq \Phi(Y, Z, X).$$

Hence, by (21), we have

$$1 \leq \frac{s_1(X^-W)s_1(W^-Y)}{s_1(X^-Y)} \leq \frac{s_1(X^-Z)s_1(Z^-Y)}{s_1(X^-Y)}. \tag{41}$$

In other words, we obtain

$$s_1(X^-Y) \leq s_1(X^-W)s_1(W^-Y) \leq s_1(X^-Z)s_1(Z^-Y), \tag{42}$$

which gives (40), as claimed. ∎

Acknowledgement. The author wishes to thank anonymous referee for giving valuable comments.

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