Southeast Asian Bulletin of Mathematics (2022) 46: 749–770

Southeast Asian
Bulletin of
Mathematics

Relativistic Dissipatons in Integrable Nonlinear Majorana Type Spinor Model

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Received 26 January 2022
Accepted 4 February 2022

Communicated by Ngai-Ching Wong

Dedicated to the memory of Professor Ky Fan (1914–2010)

AMS Mathematics Subject Classification(2020): 35C08, 37K40, 35Q51

Abstract. By method of moving frame, the relativistic integrable nonlinear model for real, Majorana type spinor fields in 1+1 dimensions is introduced and gauge equivalence of this model with Papanicolau spin model on one sheet hyperboloid is established. In terms of the so called double numbers, the model is represented also as hyperbolic complex relativistic model, in the form similar to the massive Thirring model. By using Hirota's bilinear method, the one dissipaton solution of this model is constructed. We calculate first integrals of motion for this dissipaton and show that it represents a relativistic particle with highly nonlinear mass. Analyzing resonance conditions for scattering of two relativistic dissipatons, we find a solution describing resonant property of the dissipatons.

Keywords: Thirring model; JT gravity; Dissipative soliton; Hirota method; Relativistic particle; Double numbers.

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1. Introduction

The exponentially decaying and finite energy solutions of nonlinear partial differential equations are known as solitons, and they become indispensable part of integrable systems like KdV and NLS equations, possesing elastic collision property of soliton interaction. Another type of solutions exists in dissipative nonlinear equations of reaction-dissusion type, which can grow or decay exponentially. The finite energy solutions of this kind are called the "dissipatons" [32], as solutions of integrable system of reaction-diffusion equations, which was introduced for description of low dimensional gravity model of constant curvature, the Jackiw-Teitelboim (JT) gravity [24]. This system admits arbitrary N-dissipaton solutions, showing the resonant property under collision of dissipatons, by creating long time living resonances. Dissipaton resonances were related with black hole solutions of JT gravity and main characteristic of black holes as existence of the event horizon, intrinsically connected with resonant property of dissipatons [25]. Reformulation of the reaction-diffusion model as the nonlinear Schrödinger equation with de Broglie-Bohm quantum potential term was proposed 20 years ago in our paper [33], as a result of our joint work between 1997-2000 in Academia Sinica, Taipei, Taiwan. The equation was coined as the Resonant Nonlinear Schrödinger equation (RNLS), since envelope solitons of this equation interact by creating resonant soliton states [34]. Then, different physical and mathematical aspects of the model were studied intensively. The RNLS equation as descriptive in cold plasma physics was proposed in [19, 20], and applied to kinetic of soliton gas in [4, 8], being subject of experiments in [38]. Since RNLS appeared in both, the gravity theory and in plasma physics, it inspired also research on studying the analog gravity with black hole type configurations in plasma physics [49]. Another application is related with cappilary models of Korteweg types [39, 42, 45]. From mathematical point of view, wide classes of solutions were derived for RNLS and its different modifications to higher dimensions and variable coefficients, see for example [48, 22], for symmetry analysis, loop algebraic structure and integrability see [1] - [5]. In addition, the mapping of the RNLS hierarchy, the second and the third flow to KP-II equation [35], established link between dissipaton and envelope soliton resonances with planar solitons of KP-II, creating the web type structure in shallow water [12, 3]. Several modifications of RNLS models, as the derivative RNLS [34, 16, 18], modified RNLS [17], generic RNLS [36] and related generalized equations [37, 21, 41, 27, 28], were studied. However, all these developments are related with non-relativistic dissipatons and envelope solitons, so it is not clear if there exist relativistic nonlinear equations with dissipaton solutions and resonant property of their mutual interaction.

The goal of the present paper is to show that there exist such model and it admits relativistic dissipaton solution with resonant character of interaction. The model is derived from σ model in constant external field on the one sheet hyperboloid $SO(2,1)/O(1,1)$ and represents relativistic, real-valued spinor fields, satisfying 1+1 dimensional Dirac type equation with Thirring type nonlinearity.

The real solutions of Dirac type equation are known as Majorana spinors or fermions [9], and these solutions appeared recently in condensed matter physics for modeling topological superconductor systems [47]. Another reason to study Majorana fermions is connected with problem of neutrino mass, which requires, instead of the Weyl massless equation to use as descriptive the massive Majorana equation. The model we propose here is integrable relativistic equation with nonlinearity of the Thirring type. The Thirring model is one of the best known relativistic nonlinear systems, for which many properties like integrability [26], solvability [15], inverse scattering transform [14, 10, 11] and bilinear representation were studied. The proposed model is also integrable system with Lax pair and bilinear representation. We show that in terms of hyperbolic complex numbers it can be even rewritten in form very similar to the Thirring model. But in contrast to Thirring model it admits dissipaton solutions with relativistic dispersion and leading to resonant character of dissipaton interactions.

The paper is organized as follows. In Section 2 we briefly review a relation between JT gravity model and flat connection BF gauge theory on the one sheet hyperboloid. The gauge fixing conditions in the theory are determined by non-linear σ models on $SO(2,1)/O(1,1)$ space and solutions of the models in tangent space give the Riemannian metric tensor of JT gravity. Several models, including non-relativistic RNLS and relativistic ones, but of non-local type are briefly discussed in Section 3. In Section 4 we introduce nonlinear σ model in this space with constant external field, which is the non-compact version of the model introduced by Papanicolau [31]. In tangent space this gives us the real spinor relativistic nonlinear model and corresponding zero curvature representation. In Section 5 the Hamiltonian structure and first integrals of motion are studied. In Section 6 we reformulate our model as hyperbolic comlex or the so called double number Thirring model. The bilinear form for the model and one dissipaton solution are constructed in Section 7. By calculating first three integrals of motion in Section 8 we show that our dissipaton represents relativistic particle type object with highly nonlinear mass. In Section 9, the analysis of resonance conditions for fusion and fission of scattering dissipatons with relativistic dispersion shows that dissipatons in the model can interact in the resonant way. In Section 10 we present our conclusions.

2. SL(2,R) Gauge Group and Dissipative Equations

2.1. Gauge Theory of Jackiw-Teitelboim Gravity

A nontrivial gravity model in 1+1 dimensions introduced by Jackiw and Teitelboim is described by the action with Lagrange density

$$L = \sqrt{-g}\eta(R - \Lambda),$$

where η is an additional gravitational variable called a world scalar Lagrange multiplier field, R is the Riemann scalar and Λ is a cosmological constant. This model can be reformulated as the BF topological gauge theory with the three-parameter $SO(2,1)$ de Sitter or anti-de Sitter groups,

$$S = \int_\Sigma Tr(\Phi F) = \Phi^a(de^a + \epsilon^a_b \omega e^b) + \Phi(d\omega + \frac{\Lambda}{4} e^a \epsilon_{ab} e^b),$$

where ϵ_{ab} and ϵ^a_b are Levi-Civitta symbols, Φ are zero-form Lagrange multipliers, e^a_μ is the Zweibein and ω is the spin connection. Variation of these fields produces field equations for the curvature two-form

$$F = dA + A^2 = (de^a + \epsilon^a_b \omega e^b)P_a + (d\omega + \frac{\Lambda}{4} e^a \epsilon_{ab} e^b)J = 0,$$

giving the torsionless and curvature conditions

$$de^a + \epsilon^a_b \omega e^b = 0, \quad d\omega + \frac{\Lambda}{4} e^a \epsilon_{ab} e^b = 0,$$

equivalent to the Jackiw-Teitelboim model. The Riemann metric tensor $g_{\mu\nu}$ can be recovered from Zweibein fields according to the relation

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = -\frac{4}{\Lambda}(q^+_\mu q^-_\nu + q^+_\nu q^-_\mu),$$

where $\eta_{ab} = diag(1,-1)$ is the flat tangent space metric, the spin connection and Zweibeins are

$$\begin{aligned} V_\mu &= 2\omega_\mu, \\ q^\pm_\mu &\equiv u_\mu \pm w_\mu = \frac{1}{2}\sqrt{-\frac{\Lambda}{2}}(e^0_\mu \pm e^1_\mu) \equiv \frac{1}{2}\sqrt{-\frac{\Lambda}{2}} e^\pm_\mu. \end{aligned}$$

These relations give us possibility of gravitational interpretation for our models, by considering different integrable nonlinear σ models on the one sheet hyperboloid $SL(2,R)/O(1,1)$. Reformulated in the tangent space, they represent the gauge fixing conditions for the BF topological gauge theory. The resulting equations with global $O(1,1)$ gauge symmetry group represent nonlinear dissipative equations in real variables.

2.2. Moving Frame for Poincare Gauge Group in 1+1 dimension

Here we briefly review the gauge theoretical treatment of noncompact $SO(2,1)$ σ models with Abelian $O(1,1)$ subgroup [32, 25]. These models are relevant to the 1+1 dimensional Jackiw-Teitelboim model, where the subgroup plays the role of Lorentz transformation in the tangent plane. On the other hand, they lead to the dissipative nonlinear systems, like the reaction-diffusion system. The relation between these two, at first sight looking different fields is instructive.

We consider the group $SL(2,R)$ with element g, generated by τ_i $(i = 1,2,3)$, satisfying

$$\tau_i \tau_j = h_{ij} + i c_{ijk} \tau_k,$$

where h_{ij} and c_{ijk} are the Killing metric and structure constants of $SL(2,R)$. Explicit realization in terms of Pauli matrices σ_i is $\tau_1 = -i\sigma_1, \tau_2 = \sigma_2, \tau_3 = -i\sigma_3$. We define an orthonormal trihedral set of unit vectors $\mathbf{n}_i$ in the adjoint representation of $SL(2,R)$,

$$(\mathbf{n}_i, \tau) = \mathbf{n}_i^k \tau_k = h_{kl} \mathbf{n}_i^k \tau^l = g \tau_i g^{-1}.$$

The inner and cross products between three-vectors are defined as

$$\begin{aligned} (\mathbf{n}_i, \mathbf{n}_j) &= h_{ij}, \\ \mathbf{n}_i \wedge \mathbf{n}_j &= c_{ijk} \mathbf{n}_k, \end{aligned}$$

where $h_{ij} = diag(-1, 1, -1)$ and

$$c_{ijk} = \frac{1}{2} tr(\tau_i \tau_j \tau_k) h_{kk},$$

or explicitly in terms of the antisymmetric constant tensor ϵ_{ijk},

$$c_{ijk} = -\epsilon_{ijk} h_{kk}.$$

Let $\mathbf{n}_i = \mathbf{n}_i(x^0, x^1)$ be smooth vector fields that define at each space-time coordinates (x^0, x^1) a moving frame (orthonormal basis). By the right-invariant chiral current,

$$J_\mu = g^{-1} \partial_\mu g, \quad \mu = 0, 1, \tag{1}$$

the moving frame rotates according to the equation

$$\partial_\mu \mathbf{n}_i = (J_\mu^R)_{ik}^{(ad)} \mathbf{n}_k.$$

We decompose matrix J_μ to diagonal and off diagonal parts,

$$J_\mu = J_\mu^{(0)} + J_\mu^{(1)},$$

parametrized in the following form

$$\begin{aligned} J_\mu^{(0)} &= \frac{i}{4} \tau_3 V_\mu, \\ J_\mu^{(1)} &= i u_\mu \tau_1 - i w_\mu \tau_2 = \begin{pmatrix} 0 & u_\mu - w_\mu \\ u_\mu + w_\mu & 0 \end{pmatrix}. \end{aligned}$$

Vector $\mathbf{s} \equiv \mathbf{n}_3$ satisfies the constraint $\mathbf{s}^2 = (\mathbf{s}(x), \mathbf{s}(x)) = -s_1^2 + s_2^2 - s_3^2 = -1$ and belongs to the one sheet hyperboloid $S^{1,1} \sim SL(2,R)/O(1,1)$. The real fields V_μ, u_μ and w_μ are recovered by projections,

$$V_\mu = 2(\mathbf{n}_2, \partial_\mu \mathbf{n}_1), \quad w_\mu = \frac{1}{2}(\mathbf{s}, \partial_\mu \mathbf{n}_1), \quad u_\mu = \frac{1}{2}(\mathbf{s}, \partial_\mu \mathbf{n}_2).$$

In the light-cone basis,

$$\mathbf{n}_+ = \mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{n}_- = \mathbf{n}_1 - \mathbf{n}_2,$$

satisfying following relations

$$
\begin{aligned}
(\mathbf{n}_+, \mathbf{n}_+) &= 0 = (\mathbf{n}_-, \mathbf{n}_-), \quad (\mathbf{n}_+, \mathbf{n}_-) = -2, \\
\mathbf{n}_+ \wedge \mathbf{s} &= +\mathbf{n}_+, \quad \mathbf{n}_- \wedge \mathbf{s} = -\mathbf{n}_-, \quad \mathbf{n}_- \wedge \mathbf{n}_+ = 2\mathbf{s},
\end{aligned}
$$

we define the real fields

$$
q_\mu^+ = u_\mu + w_\mu = +\frac{1}{2}(\mathbf{s}, \partial_\mu \mathbf{n}_+), \quad q_\mu^- = u_\mu - w_\mu = -\frac{1}{2}(\mathbf{s}, \partial_\mu \mathbf{n}_-).
$$

In terms of these variables the moving frame equations become

$$
\begin{aligned}
D_\mu^- \mathbf{n}_+ &= -2q_\mu^+ \mathbf{s}, && (2)\\
D_\mu^+ \mathbf{n}_- &= +2q_\mu^- \mathbf{s}, && (3)\\
\partial_\mu \mathbf{s} &= q_\mu^+ \mathbf{n}_- - q_\mu^- \mathbf{n}_+, && (4)
\end{aligned}
$$

where $D_\mu^\pm \equiv \partial_\mu \pm (1/2)V_\mu$ is the covariant derivative. This form is explicitly invariant under the local $O(1,1)$ gauge transformations,

$$
\mathbf{s} \to \mathbf{s}, \quad \mathbf{n}_+ \to e^{+\alpha}\mathbf{n}_+, \quad \mathbf{n}_- \to e^{-\alpha}\mathbf{n}_-,
$$

which are just the Lorentz boost rotations in the tangent to the vector $\mathbf{s}$ plane. Finally, consistency conditions of system (2), (3), (4), are equations for fields V_μ and q_μ,

$$
\begin{aligned}
D_\mu^- q_\nu^+ &= D_\nu^- q_\mu^+, && (5)\\
D_\mu^+ q_\nu^- &= D_\nu^+ q_\mu^-, && (6)\\
\partial_\mu V_\nu - \partial_\nu V_\mu &= 4(q_\mu^+ q_\nu^- - q_\nu^+ q_\mu^-), && (7)
\end{aligned}
$$

representing the zero-curvature conditions for current (1), parametrized now as

$$
J_\mu = \frac{i}{4} V_\mu \tau_3 + \begin{pmatrix} 0 & q_\mu^- \\ q_\mu^+ & 0 \end{pmatrix}. \tag{8}
$$

3. Resonant NLS Equation

For the Heisenberg model on one sheet hyperboloid $\mathbf{s} \in SO(2,1)/O(1,1)$,

$$
\partial_0 \mathbf{s} = \mathbf{s} \wedge \partial_1^2 \mathbf{s}
$$

the above described method produces integrable system of reaction-diffusion equations

$$
\mp\partial_0 q^\pm + \partial_1^2 q^\pm - 2q^+ q^- q^\pm = 0 \tag{9}
$$

for pair of real functions [32]. It can be transformed to the so-called Resonant NLS equation [33], which includes the de Broigle-Bohm quantum potential

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 2\frac{|\psi|_{xx}}{|\psi|}\psi.$$

This equation admits envelope solitons with resonant interaction and besides application for JT gravity model [32, 24, 25], it was derived also in several physical models such as cold plasma physics [19, 20] and capillary models [39, 42, 45]. Existence of regular dissipaton solutions for system (9) is crucial for resonant soliton interactions in RNLS.

3.1. Relativistic Models

Several relativistic models can be derived from the topological magnetic fluid model, proposed in [23], which can be bilinearized in arbitrary number of dimensions. For noncompact spin $\mathbf{s} \in SO(2,1)/O(1,1)$, the similar model can be formulated as the system of Landau-Lifshitz equations in moving frame with velocity v^μ, and relation between vorticity of the flow and topological spin density, correspondingly,

$$\partial_0 \mathbf{s} + v^\mu \partial_\mu \mathbf{s} = \mathbf{s} \wedge \partial^\mu \partial_\mu \mathbf{s}, \tag{10}$$

$$\partial_\mu v_\nu - \partial_\nu v_\mu = 2\mathbf{s} \cdot (\partial_\mu \mathbf{s} \wedge \partial_\nu \mathbf{s}). \tag{11}$$

The Heisenberg model on one sheet hyperboloid is particular reduction of this system in 1+1 dimensions, with vanishing velocity field $v^\mu = 0$. Due to resonant character of soliton interactions in that spin model (see [33]) and corresponding reaction-diffusion equations and RNLS equations, having several applications to non-relativistic physical systems, it is interesting problem to construct the relativistic invariant systems, admitting dissipaton solutions with resonant scattering properties.

Example 3.1. Self-dual σ model

For self-dual σ model

$$\partial_0 \mathbf{s} = \mathbf{s} \wedge \partial_1 \mathbf{s}$$

the corresponding equations in tangent space [25] are given by non-linear and non-local relativistic system of equations for real (Majorana type) fields

$$-\partial_- q_+^+ + q_+^+ \int^x q_+^+ q_-^- dx' = 0,$$

$$\partial_+ q_-^- + q_-^- \int^x q_+^+ q_-^- dx' = 0.$$

This system can be solved by substitution

$$q_+^+ = e^{R+S}, \;\; q_-^- = e^{R-S},$$

leading to the Liouville equation

$$\partial_0^2 R - \partial_1^2 R = e^{2R}. \tag{12}$$

The general solution of Liouville equation is given in terms of arbitrary real functions $A(s)$ and $B(s)$ of one variable s,

$$R(x^0, x^1) = \frac{1}{2} \ln \frac{A'(x^0 + x^1) B'(x^0 - x^1)}{(A(x^0 + x^1) + B(x^0 - x^1))^2}.$$

Then, for any solution of this equation, function S can be obtained by integration of linear system

$$\partial_0 S = \partial_1 R + \int^x e^{2R} dx', \quad \partial_1 S = \partial_0 R.$$

Compatibility condition for the last one is just the Liouville equation (12).

Example 3.2. Nonlinear σ model

Another relativistic model is associated with nonlinear σ model

$$\partial_+ \partial_- \mathbf{s} - (\partial_+ \mathbf{s} \cdot \partial_- \mathbf{s}) \mathbf{s} = 0.$$

The tangent space representation of this equation leads to a more general nonlinear relativistic model [25],

$$\begin{aligned} -\partial_- q_+^+ + q_+^+ \int^x \left(q_+^+ q_-^- - \frac{U_+ U_-}{q_+^+ q_-^-} \right) dx' &= 0, \\ \partial_+ q_-^- + q_-^- \int^x \left(q_+^+ q_-^- - \frac{U_+ U_-}{q_+^+ q_-^-} \right) dx' &= 0, \\ \partial_+ U_- = 0, \ \ \partial_- U_+ &= 0. \end{aligned}$$

We notice that both models, considered in these two examples, are non-local. However, in the next section we construct the model with local interaction term of four fermions type.

3.2. Majorana-Thirring Type Model

More general type of nonlinear σ model corresponds to time independent Landau - Lifshitz equation (10) in moving frame

$$v^\mu \partial_\mu \mathbf{s} = \mathbf{s} \wedge \partial^\mu \partial_\mu \mathbf{s},$$

with constant vector $\mathbf{v} = (v^0, v^1)$ and psudo-Euclidean metric $diag(1, -1)$. This corresponds to non-compact one sheet hyperbolic version of model [31]. In terms of the light cone variables $v^+ = \frac{1}{2}(v^0 + v^1)$, $v^- = \frac{1}{2}(v^0 - v^1)$ we have equation

$$v^+ \partial_+ \mathbf{s} + v^- \partial_- \mathbf{s} = \mathbf{s} \wedge \partial_+ \partial_- \mathbf{s},$$

where $\partial_\pm = \partial_0 \pm \partial_1$, producing following constraints

$$D_+^- q_-^+ = v^+ q_+^+ + v^- q_-^+, \tag{13}$$
$$D_+^+ q_-^- = -v^+ q_+^- - v^- q_-^-. \tag{14}$$

The system (5), (6), (7), with these gauge constraints completely characterizes the model. Indeed, combining them together we find four equations

$$D_-^- q_+^+ = v^+ q_+^+ + v^- q_-^+, \tag{15}$$
$$D_+^+ q_-^- = -v^- q_-^- - v^+ q_+^-, \tag{16}$$
$$D_+^- q_-^+ = v^- q_-^+ + v^+ q_+^+, \tag{17}$$
$$D_-^+ q_+^- = -v^+ q_+^- - v^- q_-^-, \tag{18}$$

which provide the conservation law

$$\partial_-(v^+ q_+^+ q_+^-) + \partial_+(v^- q_-^+ q_-^-) = 0.$$

By using this equation and the one in (7) we get the flatness condition

$$\partial_- A_+ - \partial_+ A_- = 0,$$

for Abelian vector potential

$$A_+ = V_+ - \frac{2}{v^-} q_+^+ q_+^-, \;\; A_- = V_- - \frac{2}{v^+} q_-^+ q_-^-. \tag{19}$$

In terms of the covariant derivatives, $\mathcal{D}^\pm = \partial \pm 1/2A$, Eqs. (15)-(18) become

$$\begin{aligned}
\mathcal{D}_-^- q_+^+ - v^+ q_+^+ - v^- q_-^+ - \frac{1}{v^+} q_+^+ q_-^- q_-^+ &= 0,\\
\mathcal{D}_+^+ q_-^- + v^- q_-^- + v^+ q_+^- + \frac{1}{v^-} q_+^+ q_-^- q_+^- &= 0,\\
\mathcal{D}_+^- q_-^+ - v^- q_-^+ - v^+ q_+^+ - \frac{1}{v^-} q_-^+ q_+^- q_+^+ &= 0,\\
\mathcal{D}_-^+ q_+^- + v^+ q_+^- + v^- q_-^- + \frac{1}{v^+} q+_- q_+^- q_-^- &= 0.
\end{aligned}$$

By choosing constant value potentials

$$A_+ = -2v^-, \;\; A_- = -2v^+,$$

rescaling

$$q_+^+ = \sqrt{v^-} Q_+^+, \;\; q_+^- = \frac{1}{\sqrt{v^+}} Q_+^-, \;\; q_-^+ = \sqrt{v^+} Q_-^+, \;\; q_-^- = \frac{1}{\sqrt{v^-}} Q_-^-,$$

and restricting velocity

$$v^+ v^- = 1, \tag{20}$$

so that

$$q^{\pm}_{+} = \sqrt{v^{-}}Q^{\pm}_{+},\ q^{\pm}_{-} = \frac{1}{\sqrt{v^{-}}}Q^{\pm}_{-}, \tag{21}$$

we obtain the system of nonlinear equations for the real valued analog of Thirring model

$$\partial_{-}Q^{+}_{+} - Q^{+}_{-} - Q^{+}_{+}Q^{-}_{-}Q^{+}_{-} = 0, \tag{22}$$

$$\partial_{+}Q^{-}_{-} + Q^{-}_{+} + Q^{+}_{+}Q^{-}_{-}Q^{-}_{+} = 0, \tag{23}$$

$$\partial_{+}Q^{+}_{-} - Q^{+}_{+} - Q^{+}_{-}Q^{-}_{+}Q^{+}_{+} = 0, \tag{24}$$

$$\partial_{-}Q^{-}_{+} + Q^{-}_{-} + Q^{+}_{-}Q^{-}_{+}Q^{-}_{-} = 0. \tag{25}$$

The above procedure allows us to derive also the linear problem corresponding to this model. The current (8) in the light cone variables, after redefining the Abelian gauge potentials by (19) and using (20) gives the pair

$$J_{\pm} = \frac{1}{2}\left(-v^{\mp} + \frac{1}{v^{\mp}}q^{+}_{\pm}q^{-}_{\pm}\right)\sigma_3 + \begin{pmatrix} 0 & q^{-}_{\pm} \\ q^{+}_{\pm} & 0 \end{pmatrix}.$$

Then, in terms of the rescaled fields (21) we have the Lax pair (in zero-curvature condition form) for our model

$$J_{+} = \frac{1}{2}\left(-\lambda^2 + Q^{+}_{+}Q^{-}_{+}\right)\sigma_3 + \lambda\begin{pmatrix} 0 & Q^{-}_{+} \\ Q^{+}_{+} & 0 \end{pmatrix},$$

$$J_{-} = \frac{1}{2}\left(-\frac{1}{\lambda^2} + Q^{+}_{-}Q^{-}_{-}\right)\sigma_3 + \frac{1}{\lambda}\begin{pmatrix} 0 & Q^{-}_{-} \\ Q^{+}_{-} & 0 \end{pmatrix},$$

where the spectral parameter λ is $v^{-} \equiv \lambda^2$.

4. Hamiltonian Structure

It is convenient to change notations

$$Q^{+}_{+} = p^{+},\ Q^{-}_{+} = p^{-},\ Q^{+}_{-} = q^{+},\ Q^{-}_{-} = q^{-}$$

and represent system (22) - (25) in the form

$$-\partial_{-}p^{+} + q^{+} + q^{+}q^{-}p^{+} = 0, \tag{26}$$

$$\partial_{-}p^{-} + q^{-} + q^{+}q^{-}p^{-} = 0, \tag{27}$$

$$-\partial_{+}q^{+} + p^{+} + p^{+}p^{-}q^{+} = 0, \tag{28}$$

$$\partial_{+}q^{-} + p^{-} + p^{+}p^{-}q^{-} = 0. \tag{29}$$

This system is Lagrangian, with density

$$L = -p^{+}\partial_0 p^{-} - q^{+}\partial_0 q^{-} + p^{+}\partial_1 p^{-} - q^{+}\partial_1 q^{-} - p^{+}q^{-} - q^{+}p^{-} - p^{+}p^{-}q^{+}q^{-},$$

and it is the Hamiltonian system with Hamiltonian functional

$$H = \int_{-\infty}^{\infty} (-p^+\partial_1 p^- + q^+\partial_1 q^- + p^+q^- + q^+p^- + p^+p^-q^+q^-)dx^1. \qquad (30)$$

The corresponding Poisson brackets

$$\{A, B\} = \int_{-\infty}^{\infty} \left(\frac{\partial A}{\partial p^+}\frac{\partial B}{\partial p^-} - \frac{\partial A}{\partial p^-}\frac{\partial B}{\partial p^+} + \frac{\partial A}{\partial q^+}\frac{\partial B}{\partial q^-} - \frac{\partial A}{\partial q^-}\frac{\partial B}{\partial q^+} \right) dx^1$$

for canonical variables

$$\{p^+(x^0, x^1), p^-(x^0, x'^1)\} = \delta(x^1 - x'^1),$$
$$\{q^+(x^0, x^1), q^-(x^0, x'^1)\} = \delta(x^1 - x'^1),$$

give Hamiltonian evolution equations

$$\dot{p}^{\pm} = \{p^{\pm}, H\} = \pm\frac{\partial H}{\partial p^{\mp}}, \quad \dot{q}^{\pm} = \{q^{\pm}, H\} = \pm\frac{\partial H}{\partial q^{\mp}}.$$

Besides Hamiltonian (30), there exists another integral of motion

$$M = \int_{-\infty}^{\infty} (p^+p^- + q^+q^-)\, dx^1, \qquad (31)$$

which plays role of the mass. In addition, one more conserved quantity, the momentum integral is

$$P = \int_{-\infty}^{\infty} (p^+\partial_1 p^- + q^+\partial_1 q^-)\, dx^1. \qquad (32)$$

These three intergrals are the first ones of an infinite set of integrals of motion, which can be calculated from the linear problem.

4.1. Dynamical System

For homogeneous configurations $\partial_1 = 0$, we get the four dimensional dynamical system

$$\begin{aligned}
\dot{X}_1 &= X_3 + X_1X_3X_4, \\
\dot{X}_2 &= -X_4 - X_2X_3X_4, \\
\dot{X}_3 &= X_1 + X_3X_1X_2, \\
\dot{X}_4 &= -X_2 - X_4X_1X_2,
\end{aligned}$$

with the first integral

$$I = X_1X_2 + X_3X_4 = const, \qquad (33)$$

where

$$X_1 \equiv Q_+^+,\ X_2 \equiv Q_+^-,\ X_3 \equiv Q_-^+,\ X_4 \equiv Q_-^-.$$

The system is Hamiltonian, with the canonical pairs

$$\{X_1, X_2\} = 1,\ \{X_3, X_4\} = 1,$$

and the Hamiltonian function

$$H = X_2X_3 + X_1X_4 + X_1X_2X_3X_4.$$

The Hamiltonian provides the second integral of the motion, and as easy to check the integrals I and H are in involution, $\{I, H\} = 0$. Integral (33) generates the scaling transformation

$$\delta X_i = \{X_i, I\}\alpha = \alpha X_i, (i = 1, 3),\ \ \delta X_j = \{X_j, I\}\alpha = -\alpha X_j, (j = 2, 4),$$

or after integration

$$X_1' = e^{\alpha} X_1,\ X_2' = e^{-\alpha} X_2,\ X_3' = e^{\alpha} X_3,\ X_4' = e^{-\alpha} X_4.$$

5. Hyperbolic Complex Thirring Form

Here, by introducing the hyperbolic complex variables or the "double numbers" [50], we represent our main system (26) - (29) in form of the hyperbolic complex Thirring type model. By introducing four real functions

$$q^{\pm} = u_1 \pm v_1,\quad p^{\pm} = u_2 \pm v_2,$$

the system can be rewritten as

$$-\partial_+ v_1 + u_2 + (u_2^2 - v_2^2)u_1 = 0, \tag{34}$$
$$-\partial_+ u_1 + v_2 + (u_2^2 - v_2^2)v_1 = 0, \tag{35}$$
$$-\partial_- v_2 + u_1 + (u_1^2 - v_1^2)u_2 = 0, \tag{36}$$
$$-\partial_- u_2 + v_1 + (u_1^2 - v_1^2)v_2 = 0. \tag{37}$$

Now we combine these functions as the hyperbolic complex valued functions (or double number valued functions)

$$\chi_1 = u_1 + jv_1,\quad \chi_2 = u_2 + jv_2,$$

and corresponding conjugate functions

$$\bar{\chi}_1 = u_1 - jv_1,\quad \bar{\chi}_2 = u_2 - jv_2,$$

so that

$$\bar{\chi}_1\chi_1 = |\chi_1|^2 = u_1^2 - v_1^2, \quad \bar{\chi}_2\chi_2 = |\chi_2|^2 = u_2^2 - v_2^2,$$

where hyperbolic imaginary unit j satisfies

$$j^2 = 1, \quad \bar{j} = -j.$$

In matrix representation this unit can be defined as $j = \sigma_1$. In terms of these functions, our model takes the form

$$\begin{aligned} -j\partial_+\chi_1 + \chi_2 + |\chi_2|^2\chi_1 &= 0, \\ -j\partial_-\chi_2 + \chi_1 + |\chi_1|^2\chi_2 &= 0, \end{aligned}$$

of the hyperbolic complex Thirring model. This representation is remarkable since the equation formally looks similar to the usual Thirring model for complex functions ψ_1, ψ_2 and hyperbolic imaginary unit j replaced by usual complex unit $i = \sqrt{-1}$.

6. Bilinear Form and Dissipaton Solution

The bilinear form for system (26)-(29) can be derived in terms of six real functions, $g^\pm, h^\pm, f^\pm$, such that

$$p^\pm = \frac{g^\pm}{f^\mp} = \frac{g^\pm f^\pm}{f^\pm f^\mp}, \quad q^\pm = \frac{h^\pm}{f^\pm} = \frac{h^\pm f^\mp}{f^\mp f^\pm}.$$

By representing the system in following form

$$\begin{aligned} \mp\frac{D_-(g^\pm \cdot f^\pm)}{f^\pm f^\mp} \mp \frac{g^\pm}{f^\pm}\frac{D_-(f^\pm \cdot f^\mp)}{(f^\mp)^2} + \frac{h^\pm f^\mp}{f^\pm f^\mp} + \frac{h^+h^-}{f^+f^-}\frac{g^\pm}{f^\mp} &= 0, \\ \mp\frac{D_+(h^\pm \cdot f^\mp)}{f^\mp f^\pm} \mp \frac{h^\pm}{f^\mp}\frac{D_+(f^\mp \cdot f^\pm)}{(f^\pm)^2} + \frac{g^\pm f^\pm}{f^\mp f^\pm} + \frac{g^+g^-}{f^-f^+}\frac{h^\pm}{f^\pm} &= 0, \end{aligned}$$

it can be split to bilinear system of equations

$$\begin{aligned} \mp D_-(g^\pm \cdot f^\pm) + h^\pm f^\mp &= 0, \\ \mp D_+(h^\pm \cdot f^\mp) + g^\pm f^\pm &= 0, \\ D_+(f^+ \cdot f^-) + g^+g^- &= 0, \\ -D_-(f^+ \cdot f^-) + h^+h^- &= 0. \end{aligned}$$

Let $x^0 \equiv T$, $x^1 \equiv X$ be time and space coordinates in laboratory coordinate systems, and

$$x = \frac{1}{2}(X + T), \quad t = \frac{1}{2}(X - T),$$

are the light-cone coordinates (the characteristics), so that $X = x+t$, $T = x-t$, and

$$\partial_- = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1} = \frac{\partial}{\partial T} - \frac{\partial}{\partial X} = -\frac{\partial}{\partial t},$$
$$\partial_+ = \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} = \frac{\partial}{\partial T} + \frac{\partial}{\partial X} = \frac{\partial}{\partial x}.$$

By rewriting Hirota derivatives in light-cone coordinates, $D_- = -D_t$, $D_+ = D_x$ the bilinear system becomes

$$\begin{aligned} \pm D_t(g^\pm \cdot f^\pm) + h^\pm f^\mp &= 0, \\ \mp D_x(h^\pm \cdot f^\mp) + g^\pm f^\pm &= 0, \\ D_x(f^+ \cdot f^-) + g^+ g^- &= 0, \\ D_t(f^+ \cdot f^-) + h^+ h^- &= 0, \end{aligned}$$

so that

$$q^+ q^-(x,t) = -\left(\ln \frac{f^+}{f^-}\right)_t, \quad p^+ p^-(x,t) = -\left(\ln \frac{f^+}{f^-}\right)_x.$$

By Hirota expansion

$$\begin{aligned} g^\pm(x,t) &= \epsilon g_1^\pm(x,t) + \epsilon^3 g_3^\pm(x,t) + ... \\ h^\pm(x,t) &= \epsilon h_1^\pm(x,t) + \epsilon^3 h_3^\pm(x,t) + ... \\ f^\pm(x,t) &= 1 + \epsilon^2 f_2^\pm(x,t) + ..., \end{aligned}$$

we find exact solution in the form

$$\begin{aligned} g_1^\pm &= e^{\eta_1^\pm}, \quad h_1^\pm = a_1^\pm e^{\eta_1^\pm}, \quad f_2^\pm = b_2^\pm e^{\eta_1^+ + \eta_1^-}, \\ b_2^+ &= \frac{(a_1^+)^2 a_1^-}{(a_1^+ - a_1^-)^2}, \quad b_2^- = \frac{a_1^+ (a_1^-)^2}{(a_1^+ - a_1^-)^2}, \\ \eta_1^\pm &= k_1^\pm x + \omega_1^\pm t + \eta_{1_0}^\pm, \quad \omega_1^\pm = \mp a_1^\pm, \quad k_1^\pm = \pm \frac{1}{a_1^\pm}, \end{aligned}$$

parametrized by real constants $a_1^\pm$, $\eta_{1_0}^\pm$, so that

$$\eta_1^\pm = \pm\left(\frac{1}{a_1^\pm} x - a_1^\pm t\right) + \eta_{1_0}^\pm.$$

This gives dissipative one-soliton solution, which we call the dissipaton, in the following form

$$p^\pm(x,t) = \frac{g^\pm}{f^\mp} = \frac{e^{\eta^\pm}}{1 + b_2^\mp e^{\eta_1^+ + \eta_1^-}},$$
$$q^\pm(x,t) = \frac{h^\pm}{f^\pm} = \frac{a^{\pm_1} e^{\eta^\pm}}{1 + b_2^\pm e^{\eta_1^+ + \eta_1^-}}.$$

Components of this solution are decaying and growing exponentially

$$p^{\pm}(x,t) = \frac{e^{\pm\frac{\eta_1^+ - \eta_1^-}{2} - \frac{\alpha_{\mp}}{2}}}{2\cosh\frac{\eta_1^+ + \eta_1^- + \alpha_{\mp}}{2}},$$

$$q^{\pm}(x,t) = \frac{a_1^{\pm} e^{\pm\frac{\eta_1^+ - \eta_1^-}{2} - \frac{\alpha_{\pm}}{2}}}{2\cosh\frac{\eta_1^+ + \eta_1^- + \alpha_{\pm}}{2}},$$

where $\alpha_{\pm} = \frac{1}{2}\ln b_2^{\pm}$. But the mutual products are in perfect soliton form

$$p^+p^- = \frac{(a_1^+ - a_1^-)^2}{a_1^+ a_1^-} \frac{1}{2\sqrt{a_1^+ a_1^-}\cosh\left(\eta_1^+ + \eta_1^- + \frac{\alpha_+ + \alpha_-}{2}\right) + a_1^+ + a_1^-},$$

$$q^+q^- = (a_1^+ - a_1^-)^2 \frac{1}{2\sqrt{a_1^+ a_1^-}\cosh\left(\eta_1^+ + \eta_1^- + \frac{\alpha_+ + \alpha_-}{2}\right) + a_1^+ + a_1^-}.$$

To have non-singular solution we choose real parameters $a_1^+ > 0$, $a_1^- > 0$ and as follows $b_2^{\pm} > 0$. This implies $k_1^+ > 0$ and $k_1^- < 0$. By introducing parametrization

$$a_1^+ = \lambda_1 + \mu_1, \quad a_1^- = \lambda_1 - \mu_1,$$

so that $a_1^+ a_1^- = \lambda_1^2 - \mu_1^2$, $a_1^+ + a_1^- = 2\lambda_1$, $a_1^+ - a_1^- = 2\mu_1$ we have

$$p^+p^- = \frac{1}{\lambda_1^2 - \mu_1^2} \frac{2\mu_1^2}{\sqrt{\lambda_1^2 - \mu_1^2}\cosh\left(\eta_1^+ + \eta_1^- + \frac{\alpha_+ + \alpha_-}{2}\right) + \lambda_1},$$

$$q^+q^- = \frac{2\mu_1^2}{\sqrt{\lambda_1^2 - \mu_1^2}\cosh\left(\eta_1^+ + \eta_1^- + \frac{\alpha_+ + \alpha_-}{2}\right) + \lambda_1}.$$

The traveling wave factor in these expressions can be rewritten in the laboratory coordinates (X, T) with relativistic Lorentz contraction factor

$$\eta_1^+ + \eta_1^- + \frac{\alpha_+ + \alpha_-}{2} = -2k\frac{X - X_0 - vT}{\sqrt{1 - v^2}},$$

where velocity of the dissipaton is defined as

$$v \equiv \frac{a_1^+ a_1^- - 1}{a_1^+ a_1^- + 1} = \frac{\lambda_1^2 - \mu_1^2 - 1}{\lambda_1^2 - \mu_1^2 + 1}$$

and it is restricted by the speed of light $c = 1$: $|v| < 1$. The initial position is fixed by

$$\frac{2k}{\sqrt{1 - v^2}} X_0 \equiv \eta_{1_0}^+ + \eta_{1_0}^- + \ln\frac{1}{4k^2}\sqrt{\frac{1 + v}{1 - v}}$$

and

$$k \equiv \mu_1\sqrt{\frac{1 - v}{1 + v}}.$$

In terms of parameters v, k and X_0, dissipaton densities become just

$$p^+p^- = \sqrt{\frac{1-v}{1+v}}\frac{2k^2}{\cosh\left(2k\frac{X-X_0-vT}{\sqrt{1-v^2}}\right)+\sqrt{k^2+1}}, \tag{38}$$

$$q^+q^- = \sqrt{\frac{1+v}{1-v}}\frac{2k^2}{\cosh\left(2k\frac{X-X_0-vT}{\sqrt{1-v^2}}\right)+\sqrt{k^2+1}}. \tag{39}$$

By expressing

$$\alpha_\pm = \ln\left(\frac{\sqrt{k^2+1}\pm k}{4k^2}\sqrt{\frac{1+v}{1-v}}\right),$$

$$k\frac{X_{0\pm}}{\sqrt{1-v^2}} = \frac{\eta^+_{1_0}+\eta^-_{1_0}}{2}+\frac{1}{2}\alpha_\pm,$$

and denoting $\eta^+_{1_0}-\eta^-_{1_0}\equiv\nu_{1_0}$, we finally get dissipaton solution in the form

$$p^\pm = \left(\frac{1-v}{1+v}\right)^{\frac{1}{4}}\frac{k\sqrt{\sqrt{k^2+1}\pm k}}{\cosh k\frac{X-X_{0\pm}-vT}{\sqrt{1-v^2}}}e^{\pm\left[\frac{\sqrt{k^2+1}}{\sqrt{1-v^2}}(T-vX)+\nu_{1_0}\right]}, \tag{40}$$

$$q^\pm = \left(\frac{1+v}{1-v}\right)^{\frac{1}{4}}\frac{k\sqrt{\sqrt{k^2+1}\pm k}}{\cosh k\frac{X-X_{0\pm}-vT}{\sqrt{1-v^2}}}e^{\pm\left[\frac{\sqrt{k^2+1}}{\sqrt{1-v^2}}(T-vX)+\nu_{1_0}\right]}. \tag{41}$$

It is noticed that initial positions are related by the mean value formula

$$X_0 = \frac{1}{2}(X_{0_+}+X_{0_-}).$$

7. Relativistic Dissipaton

For physical interpretation of one dissipaton solution (40), (41), we have to calculate the mass (31), momentum (32) and energy integrals (30). By substituting densities (38), (39) to (31), after integration we get the mass integral as function of k only,

$$M = 2\ln\frac{\sqrt{k^2+1}+|k|}{\sqrt{k^2+1}-|k|}.$$

The momentum integral (32) for one dissipaton solution takes the form

$$P = \frac{4kv}{\sqrt{1-v^2}}.$$

To calculate the energy, we first rewrite (30) in the form

$$\begin{aligned} H &= \int_{-\infty}^{\infty} [\frac{1}{2}(-p^+\partial_1 p^- + p^-\partial_1 p^+ + q^+\partial_1 q^- - q^-\partial_1 q^+) \\ &+p^+q^- + q^+p^- + p^+p^-q^+q^-]\, dX \\ &= \int_{-\infty}^{\infty} [\frac{1}{2}p^+p^-\partial_1 \left(\ln \frac{g^+}{g^-}\right) - \frac{1}{2}q^+q^-\partial_1 \left(\ln \frac{h^+}{h^-}\right) \\ &+ + p^+q^- + p^-q^+ - \frac{1}{4}(p^+p^- - q^+q^-)^2]\, dX. \end{aligned}$$

Then, for one dissipaton solution it gives

$$E = \frac{4k}{\sqrt{1-v^2}}.$$

Denoting

$$m_0 \equiv 4k$$

as the rest mass, we find usual expressions for momentum and energy of relativisitic particle

$$P = \frac{m_0 v}{\sqrt{1-v^2}}, \quad E = \frac{m_0}{\sqrt{1-v^2}},$$

with speed of light $c = 1$. This shows that our one-dissipaton solution describes a finite energy relativistic particle with the rest mass m_0, corresponding to the rest frame, when $v = 0$. The dispersion relation for one-dissipaton is in the relativistic form

$$E^2 - P^2 = m_0^2$$

or

$$E = \pm\sqrt{m_0^2 + P^2}.$$

However, the rest mass m_0 is connected with first integral M by nonlinear formula

$$M = 2 \ln \frac{\sqrt{m_0^2+16} + m_0}{\sqrt{m_0^2+16} - m_0}$$

or

$$m_0 = 4 \sinh \frac{M}{4}.$$

It shows that in contrast with known non-relativistic dissipatons of RNLS [33], our dissipaton is a composite object with properties of relativistic particle.

8. Resonant Interaction

It is well known that non-relativistic dissipatons, related to RNLS and RDNLS equations show resonant properties under collisions [33, 34]. An interesting point is to see if such property is preserved also in the relativistic case. To explore this possibility, we consider a collision of two dissipatons with masses and velocities (m_1, v_1) and (m_2, v_2), which fuse into a single dissipaton (m, v).The conservation laws for this process imply following relations

$$M = M_1 + M_2, \quad P = P_1 + P_2, \quad E = E_1 + E_2.$$

Substituting to above formulas we have equations for resonant interaction of dissipatons

$$\frac{\sqrt{m_1^2+16}+m_1}{\sqrt{m_1^2+16}-m_1}\,\frac{\sqrt{m_2^2+16}+m_2}{\sqrt{m_2^2+16}-m_2} = \frac{\sqrt{m^2+16}+m}{\sqrt{m^2+16}-m}, \tag{42}$$

$$\frac{m_1 v_1}{\sqrt{1-v_1^2}} + \frac{m_2 v_2}{\sqrt{1-v_2^2}} = \frac{mv}{\sqrt{1-v^2}}, \tag{43}$$

$$\frac{m_1}{\sqrt{1-v_1^2}} + \frac{m_2}{\sqrt{1-v_2^2}} = \frac{m}{\sqrt{1-v^2}}. \tag{44}$$

If this system of algebraic equations admits nontrivial solution, then interaction of our relativistic dissipatons could have resonant character. We postpone the study of general solution for this system and will consider here only special case, namely collision of two equal mass dissipatons $m_1 = m_2$, with equal and opposite velocities $v_1 = -v_2$. It implies $P_1 = -P_2 \to P_1 + P_2 = 0 \to P = 0$. This process creates a dissipaton with mass m at the rest with $v = 0$. From (42) and (44) we have

$$\left(\frac{\sqrt{m_1^2+16}+m_1}{\sqrt{m_1^2+16}-m_1}\right)^2 = \frac{\sqrt{m^2+16}+m}{\sqrt{m^2+16}-m},$$

$$m = \frac{2m_1}{\sqrt{1-v_1^2}}.$$

Solution of this system is given by relation between velocity and mass of colliding dissipatons

$$v_1 = \frac{m_1}{\sqrt{m_1^2+16}},$$

so that $v_1 < 1$, and the mass of dissipaton at the rest is

$$m = \frac{1}{2} m_1 \sqrt{m_1^2+16}.$$

It shows that similar to non-relativistic case, the relativistic dissipatons admit resonant interaction. Calculations of two dissipaton solution and study of their mutual resonant interaction for specific choice of parameters would be done in forthcoming publication.

9. Conclusions

The hidden solitonic-type integrability aspects of general relativity, notably, in historic terms, with regard to Ernst-type systems and their admitted Bäcklund transformations are well-documented. In recent work, integrable connections between relativistic gasdynamics and the Heisenberg spin equation have been established in [46]. It is recalled also that Heisenberg spin connections, in spatial hydrodynamics were originally discovered and the Heisenberg spin connection was subsequently elaborated upon in [40]. The Heisenberg spin model on one sheet hyperboloid leads to RNLS [33] with resonant interaction of non-relativistic envelope solitons.

In present paper we have derived new integrable nonlinear relativistic real valued spinor model with four fermionic interaction in Thirring type form. As was shown, the model is gauge equivalent to non-compact version of Papanicolau model on one sheet hyperboloid and it provides a specific gauge constraint in JT gravity. By introducing bilinear form of the nonlinear system we calculated one dissipaton solution and corresponding integrals of motion as mass, momentum and energy. The obtained dispersion relation shows that dissipaton represents relativistic particle with highly nonlinear mass term. By analyzing resonans conditions for dissipaton scattering we found nontrivial solution with resonant properties. Description of this resonant scattering requires calculation of two dissipaton solution in a specific range of parameters. Moreover, it is interesting to calculate one soliton solution of the spin model, corresponding to dissipaton solution and the metric tensor in JT gravity on existence of relativistic black holes. This work is in progress now.

One more aspect of resonant soliton equations is linked with connection to Ermakov-Painleve symmetries and Whitham-Kaup-Broer system of equations. It would be of research interest to investigate potential hybrid Ermakov- Painleve II symmetry reduction of multi-component 2+1- dimensional resonant NLS systems linked to Whitham-Kaup-Broer systems. The Ermakov-Painleve II integrable reduction of basic two-component such resonant systems has recently been established in [43, 44]. The relativistic aspect of these type symmetries and corresponding equations in framework of the present paper is intriguing question for future research.

Acknowledgement. This is a good chance to contribute a paper in this special issue of SEABM in honor of Ky Fan. Professor Ky Fan has been the Director of Institute of Mathematics, Academia Sinica, Taipei in the period of 1978–1984. He emphasized the importance of basic research and the need for the young research assistants to be exposed to more mathematics. One of the authors J.L entered this Institute in 1983 when Prof. Ky Fan was the Director.

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