Southeast Asian Bulletin of Mathematics (2022) 46: 771–782

Southeast Asian
Bulletin of
Mathematics

On Some K-g-Fusion Frames in Finite and Infinite Hilbert Spaces

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Received 11 April 2020
Accepted 30 September 2020

Communicated by Ngai-Ching Wong

Dedicated to the memory of Professor Ky Fan (1914–2010)

AMS Mathematics Subject Classification(2020): 42C15, 46C99, 41A58

Abstract. In this note, we aim to present a method for reconstruction of generalized fusion frames for operators and error operator with its upper bound. Also, the approximation operator for these frames will be introduced and we study some results about them specially, a new identity about the norm of these frames is presented.

Keywords: Parseval frame; K-frame; K-g-fusion frame; Erasures.

1. Introduction and Preliminaries

Frames have a significant role in both pure and applied mathematics, so that these are a fundamental research area in mathematics, computer science and quantum information and several new applications have been developed, e.g. besides traditional application as signal processing, image processing, data compression and sampling theory. After introducing Frames by Duffin and Scheaffer [12], there have been many generalizations such as c-frame [16], g-frame [25], fusion frame [7, 20], K-frame [15], controlled frames [4] and also the combination of each two of them, lead to c-fusion frames [13], c-g-frames [18], g-fusion frames [23] and etc.

Frames for operators or K-frames have been introduced by Găvruţa in [15] to study the nature of atomic systems for a separable Hilbert space with respect to a bounded linear operator K and presented some another kind of these frames e.g. [2, 3, 21, 22, 26]. It is a well-known fact that K-frames are more general than the classical frames and due to higher generality of K-frames, many properties of frames may not hold for K-frames like the frame operator in all kind of K-frames is not invertible. Recently, Sadri et al. presented K-g-fusion frames (and g-fusion frames) in [23, 24].

Robustness of Parseval fusion frames under erasure have been employed by Bodmann et al. in [5] for optimal transmission of quantum states and packet encoding. After, Kutyniok et al. in [19] were able to present fusion frames which are optimally resilient against noise and erasure for random signals and further, Casazza and Kutyniok in [8] have studied this topic and they presented sufficient conditions on the robustness of a fusion frame with respect to erasures of subspaces. In this paper, we focus on the study of those topics on K-g-fusion frames and we will show some new results about these frames.

Throughout this paper, H and H_j are separable Hilbert spaces for each $j \in \mathbb{J}$ where $\mathbb{J}$ is a subset of $\mathbb{Z}$ and $\mathcal{B}(H_1, H_2)$ is the collection of all the bounded linear operators of H_1 into H_2. If $H_1 = H_2 = H$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$. Also, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $K \in \mathcal{B}(H)$.

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following senses.

Lemma 1.1. [11] *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

In this part, we review notations of K-frames and K-g-fusion frames from [15, 24]. We notice that the operators in frames and K-frames are similar.

Definition 1.2. [15] *Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H and $K \in \mathcal{B}(H)$. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame for H if there exist $0 < A \le B < \infty$ such that for each $f \in H$,*

$$A\|K^* f\|^2 \le \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \le B\|f\|^2. \tag{1}$$

The constants A and B are called K-frame bounds. If the right hand of (1) holds, we say that $\{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound B. The set $\{f_j\}_{j\in\mathbb{J}}$ is called a Parseval K-frame if

$$\sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 = \|K^* f\|^2.$$

If $\{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence, then the synthesis and the analysis operators of frames are defined by

$$\begin{aligned} T :&\ell^2(\mathbb{N}) \to H, & T^* &: H \to \ell^2(\mathbb{N}), \\ T\{c_j\}_{j\in\mathbb{J}} &= \sum_{j\in\mathbb{J}} c_j f_j, & T^* f &= \{\langle f, f_j\rangle\}_{j\in\mathbb{J}}. \end{aligned}$$

Now, the frame operator is defined by $S = TT^*$.

Definition 1.3. [3] *Let $K \in \mathcal{B}(H)$. A sequence $\{\Lambda_j \in \mathcal{B}(H, H_j)\}_{j\in\mathbb{J}}$ is called a K-g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|K^* f\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leq B\|f\|^2. \tag{2}$$

Definition 1.4. [24] *Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e. $v_j > 0$, $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$ and $K \in \mathcal{B}(H)$. We say $\Lambda := (W_j, \Lambda_j, v_j)_j$ is a K-g- fusion frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|K^* f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{3}$$

Throughout this paper, Λ will be a triple $(W_j, \Lambda_j, v_j)_j$ with $j \in \mathbb{J}$ unless otherwise noted. If the right hand of (3) holds, we say that Λ is a g-fusion Bessel sequence with the bound B. If $A = B$, then we say Λ is a tight K-g-fusion frame and we say Λ is a Parseval K-g-fusion frame whenever $A = B = 1$ and we get

$$\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \|K^* f\|^2.$$

If Λ is a g-fusion Bessel sequence, then the synthesis and the analysis operators of the g-fusion frames are defined by ([24, 23])

$$\begin{gathered} T_\Lambda : \mathscr{H}_2 \longrightarrow H, \\ T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j, \\ T_\Lambda^* : H \longrightarrow \mathscr{H}_2, \\ T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}}, \end{gathered}$$

where $\mathscr{H}_2 = \big\{\{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\big\}$, with the inner product defined by $\langle\{f_j\}, \{g_j\}\rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j\rangle$. It is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations. Thus, the g-fusion frame operator is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f$$

and so, we can get

$$\langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\pi_{W_j} f\|^2, \tag{4}$$

for all $f \in H$. The following shows an interesting property between T_Λ and T_Θ^* for two g-fusion Bessel sequences.

Theorem 1.5. *Let $|\mathbb{J}| < \infty$, also $\Lambda = (W_j, \Lambda_j, v_j)_j$ and $\Theta = (W_j, \Theta_j, w_j)_j$ be two g-fusion Bessel sequences for H, where $\Lambda_j, \Theta_j \in \mathcal{B}(H, H_j)$. Let $\phi := T_\Lambda T_\Theta^*$. Then ϕ is a trace class operator.*

Proof. Suppose that $\phi = u|\phi|$ is a polar decomposition of the operator ϕ, where $u \in \mathcal{B}(H)$ is a partial isometry, therefore $|\phi| = u^* T_\Lambda T_\Theta^*$. Assume that $\{e_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for H. Then

$$\begin{aligned}
tr(|\phi|) &= \sum_{j\in\mathbb{J}} \langle |\phi| e_j, e_j\rangle = \sum_{j\in\mathbb{J}} \langle T_\Theta^* e_j, T_\Lambda^* u e_j\rangle \\
&= \sum_{j\in\mathbb{J}} \big\langle \{w_k\Theta_k\pi_{W_k} e_j\}_{k\in\mathbb{J}}, \{v_k\Lambda_k\pi_{W_k} u e_j\}_{k\in\mathbb{J}}\big\rangle \\
&= \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{J}} \langle w_k\Theta_k\pi_{W_k} e_j, v_k\Lambda_k\pi_{W_k} u e_j\rangle \\
&\le \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{J}} \|w_k\Theta_k\pi_{W_k} e_j\| . \|v_k\Lambda_k\pi_{W_k} u e_j\| \\
&\le \sum_{j\in\mathbb{J}} \Big(\sum_{k\in\mathbb{J}} \|w_k\Theta_k\pi_{W_k} e_j\|^2\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{J}} \|v_k\Lambda_k\pi_{W_k} u e_j\|^2\Big)^{\frac{1}{2}} \\
&\le \sum_{j\in\mathbb{J}} \sqrt{B_\Lambda B_\Theta}\, \|u e_j\| \\
&= \sqrt{B_\Lambda B_\Theta}\ \|u\| |\mathbb{J}| < \infty.
\end{aligned}$$

∎

In next Theorem, we show a relation between K-frames with K-g-fusion frames which is a generalized of Theorem 3.2 in [7].

Theorem 1.6. *For each $j \in \mathbb{J}$ let $\Lambda_j \in \mathcal{B}(H, H_j)$ and $v_j > 0$. Let $\{f_{ij}\}_{i\in\mathbb{I}_j}$ be a K-frame for H_j with bounds A_j and B_j. Define a sequence of subspaces $W_j = \overline{\text{span}}\{\Lambda_j^* f_{ij}\}_{i\in\mathbb{I}_j}$ for each $j \in \mathbb{J}$ and suppose that*

$$0 < A := \inf_{j\in\mathbb{J}} A_j \le B := \sup_{j\in\mathbb{J}} B_j < \infty.$$

The following assertions are equivalent:

(i) $\{v_j\Lambda_j^* f_{ij}\}_{j\in\mathbb{J},i\in\mathbb{I}_j}$ *is a* K*-frame for* H.

(ii) $\Lambda_j(W_j)$ *are closed subspaces of* H_j *for every* $j\in\mathbb{J}$ *and* $\{e_{ij}\}_{j\in\mathbb{J},i\in\mathbb{I}_j}$ *are orthonormal bases for them such that* $\{v_j\pi_{W_j}\Lambda_j^* e_{ij}\}_{j\in\mathbb{J},i\in\mathbb{I}_j}$ *is a* K*-frame for* H.

(iii) $\Lambda=(W_j,\Lambda_j,v_j)_{j\in\mathbb{J}}$ *is a* K*-g-fusion frame for* H.

Proof. First, we prove that (i) and (iii) are equivalent. Suppose that $\{v_j\Lambda_j^* f_{ij}\}_{j\in\mathbb{J},i\in\mathbb{I}_j}$ is a K-frame for H with frame bounds C and D. For each $f\in H$, we have

$$\begin{aligned}
A\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 &\le \sum_{j\in\mathbb{J}} A_j v_j^2\|\Lambda_j\pi_{W_j}f\|^2\\
&\le \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle v_j\Lambda_j\pi_{W_j}f, f_{ij}\rangle|^2\\
&= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle \pi_{W_j}f, v_j\Lambda_j^* f_{ij}\rangle|^2\\
&= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle f, v_j\Lambda_j^* f_{ij}\rangle|^2\\
&\le D\|f\|^2.
\end{aligned}$$

This means that Λ is a g-fusion Bessel sequence for H with a bound $\frac{D}{A}$. With same method, we can show that $\frac{C}{B}$ is a lower K-frame bound for Λ. For the opposite case, assume that Λ is a K-g-fusion frame with bounds C and D. For each $f\in H$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle f, v_j\Lambda_j^* f_{ij}\rangle|^2 &= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j}|\langle \pi_{W_j}f, v_j\Lambda_j^* f_{ij}\rangle|^2\\
&= \sum_{j\in\mathbb{J}}\sum_{i\in\mathbb{I}_j} v_j^2|\langle \Lambda_j\pi_{W_j}f, f_{ij}\rangle|^2\\
&\ge \sum_{j\in\mathbb{J}} A_j v_j^2\|\Lambda_j\pi_{W_j}f\|^2\\
&\ge AC\|K^*f\|^2,
\end{aligned}$$

and it is easy to check that BD is a lower frame bound.

Now, according to the following:

$$v_j^2\|\Lambda_j\pi_{W_j}f\|^2 = v_j^2\Big\|\sum_{i\in\mathbb{I}_j}\langle\Lambda_j\pi_{W_j}f, e_{ij}\rangle e_{ij}\Big\|^2 = \sum_{i\in\mathbb{I}_j}|\langle f, v_j\pi_{W_j}\Lambda_j^* e_{ij}\rangle|^2,$$

we aim that (ii) and (iii) are equivalent. ∎

2. Main Results

Suppose that $\{W_j\}_{j\in\mathbb{J}}$ and $\{Z_j\}_{j\in\mathbb{J}}$ are two closed subspaces of H and $\{v_j\}_{j\in\mathbb{J}}$ is a set of weights. Also, Λ_j and Θ_j are bounded operators in $\mathcal{B}(H, H_j)$. We define *the approximation operator* with respect to Λ and $\Theta := (Z_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ as follows:

$$\Psi : H \longrightarrow H,$$
$$\Psi f = \sum_{j\in\mathbb{J}} v_j \pi_{Z_j} \Theta_j^*(v_j \Lambda_j \pi_{W_j} f).$$

The following can be found in the text of Banach spaces:

Lemma 2.1. *Let $(X, \|.\|)$ be a Banach space and $U : X \to X$ be a bounded operator such that $\|I - U\| < 1$. Then U is invertible and*

$$U^{-1} = \sum_{k=0}^{n} (I - U)^k.$$

Moreover

$$\|U^{-1}\| \leq \frac{1}{1 - \|I - U\|}.$$

Theorem 2.2. *Let $C_1, C_2 > 0$ and $0 \leq \gamma < 1$ be real numbers such that for each $f \in H$ and $\{f_j\}_{\mathrm{j}\in\mathbb{J}} \in \mathscr{H}_2$ the following assertions holds:*

(i) $\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq C_1 \|f\|^2$;

(ii) $\|\sum_{\mathrm{j}\in\mathbb{J}} v_j \pi_{Z_j} \Theta_j^* f_j\|^2 \leq C_2 \|\{f_j\}\|_2^2$;

(iii) $\|f - \Psi f\|^2 \leq \gamma \|f\|^2$.

Then Λ is a K-g-fusion frame for H with bounds $C_2^{-1}(1-\gamma)^2$ and C_1. Also, Θ is a K-g-fusion frame for H with bounds $C_1^{-1}(1-\gamma)^2$ and C_2.

Proof. Assume that $f \in H$, with items (i) and (ii) we get

$$\|\Psi f\|^2 \leq C_2 \|\{v_j \Lambda_j \pi_{W_j} f\}\|_2^2 = C_2 \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq C_2 C_1 \|f\|^2.$$

Hence, Ψ is a bounded operator. By Lemma 2.1, Ψ is invertible and $\|\Psi^{-1}\| \leq (1-\gamma)^{-1}$. Thus,

$$\begin{aligned}
\|K\|^{-2} \|K^* f\|^2 &\leq \|f\|^2 \\
&= \|\Psi^{-1} \Psi f\|^2 \\
&\leq (1-\gamma)^{-2} \|\Psi f\|^2 \\
&\leq C_2 (1-\gamma)^{-2} \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq C_2 C_1 (1-\gamma)^{-2} \|f\|^2.
\end{aligned}$$

So, we conclude that

$$C_2^{-1}(1-\gamma)^2\|K\|^{-2}\|K^*f\|^2 \le \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 \le C_1\|f\|^2,$$

and the first part is proved. Next, we verify two inequalities which are dual to (i) and (ii) for Θ. Let $f\in H$. Then we have

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^2 &= \big(\langle\sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^*\Theta_j\pi_{Z_j}f,f\rangle\big)^2\\ &\le \|\sum_{j\in\mathbb{J}} v_j\pi_{Z_j}\Theta_j^*\Theta_j\pi_{Z_j}f\|^2\|f\|^2\\ &\le C_2\|f\|^2\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2.\end{aligned}$$

Therefore,

$$\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{Z_j}f\|^2 \le C_2\|f\|^2.$$

For second inequality, for $\{f_j\}_{j\in\mathbb{J}}\in\mathscr{H}_2$, we can write

$$\begin{aligned}\|\sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*f_j\|^2 &= \Big(\sup_{\|f\|=1}\big|\big\langle\sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*f_j,f\big\rangle\big|\Big)^2\\ &\le \Big(\sup_{\|f\|=1}\big|\sum_{j\in\mathbb{J}}\big\langle f_j, v_j\Lambda_j\pi_{W_j}f\big\rangle\big|\Big)^2\\ &\le \|\{f_j\}\|_2^2\Big(\sup_{\|f\|=1}\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)\\ &\le C_1\|\{f_j\}\|_2^2.\end{aligned}$$

Now by similar argument and applying an approximation operator of the form

$$\Psi^*f = \sum_{j\in\mathbb{J}} v_j\pi_{W_j}\Lambda_j^*(v_j\Theta_j\pi_{Z_j}f),$$

we can establish Θ has required properties. ∎

The next result is a generalization of Theorem 3.2 from [8] for K-g-fusion frames.

Theorem 2.3. *Let K be closed range and Λ be a K-g-fusion frame for H with bounds A and B and $\mathbb{I}\subset\mathbb{J}$. Then the following statements hold:*

(i) *If $\{\Lambda_j\}_{j\in\mathbb{I}}$ is a K-g-frame for H with the lower frame bound B, also $\bigcap_{j\in\mathbb{I}} W_j\subseteq\mathcal{R}(K)$ and $v_j>\|K^\dagger\|$ for each $j\in\mathbb{I}$, then*

$$\bigcap_{j\in\mathbb{I}} W_j = \{0\}.$$

(ii) *If* $\{\Lambda_j\}_{j\in\mathbb{I}}$ *is a tight* K*-g-frame for* H *with the lower frame bound* B *and* $\|K^\dagger\| \leq 1$, *also* $\bigcap_{j\in\mathbb{I}} W_j \subseteq \mathcal{R}(K)$ *and we have* $v_j = 1$ *for each* $j \in \mathbb{I}$, *then*

$$\bigcap_{j\in\mathbb{I}} W_j \perp \operatorname{span}\{W_j\}_{j\in\mathbb{J}\setminus\mathbb{I}}.$$

(iii) *If* $C := \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\|^2 < A$, *then* $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}\setminus\mathbb{I}}$ *is a* K*-g-fusion frame for* $\mathcal{R}(K)$ *with bounds* $A - C\|K^\dagger\|$ *and* B.

Proof. (i). For every $f \in \bigcap_{j\in\mathbb{I}} W_j$ and $j \in \mathbb{I}$ we have $\pi_{W_j} f = f$. So,

$$\begin{aligned}
B\|f\|^2 &\leq B\|K^\dagger\|^2\|K^*f\|^2 \\
&< \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j f\|^2 \\
&= \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq B\|f\|^2.
\end{aligned}$$

Thus, $f = 0$.

(ii). For each $f \in \bigcap_{j\in\mathbb{I}} W_j$, we have

$$\begin{aligned}
B\|K^*f\|^2 &= \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\
&\leq B\|f\|^2 \\
&\leq B\|K^\dagger\|^2\|K^*f\|^2 \\
&\leq B\|K^*f\|^2.
\end{aligned}$$

Therefore, $\sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 = 0$ and it shows that $f \perp \operatorname{span}\{W_j\}_{j\in\mathbb{J}\setminus\mathbb{I}}$.

(iii) The upper bound is evident. For the lower bound, if $f \in \mathcal{R}(K)$ we get

$$\begin{aligned}
\sum_{j\in\mathbb{J}\setminus\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 - \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j \pi_{W_j} f\|^2 \\
&\geq A\|K^*f\|^2 - \sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\|^2\|f\|^2 \\
&\geq (A - C\|K^\dagger\|^2)\|K^*f\|^2.
\end{aligned}$$

∎

When the set $\mathbb{I}$ which is introduced in Theorem 2.3 is singleton, then we can get the following result.

Corollary 2.4. *Let K be closed range and Λ be a K-g-fusion frame for H with bounds A and B. If there exists $j_0 \in \mathbb{J}$ such that $v_{j_0}^2 \|\Lambda_{j_0}\|^2 < A$, then $(W_j, \Lambda_j, v_j)_{j \neq j_0}$ is a K-g-fusion frame for $\mathcal{R}(K)$ with bounds $A - v_{j_0}^2 \|\Lambda_{j_0}\|^2 \|K^\dagger\|^2$ and B.*

The following corallary is a generalized of Corollary 3.4 from [8].

Corollary 2.5. *Let K be closed range and Λ be a tight K-g-fusion frame for H with bound A and $j_0 \in \mathbb{J}$. Then the following assertions are equivalent:*

(i) $v_{j_0}^2 \|\Lambda_{j_0} \pi_{W_{j_0}}\|^2 < A$.

(ii) $(W_j, \Lambda_j, v_j)_{j \neq j_0}$ *is a K-g-fusion frame for $\mathcal{R}(K)$.*

Proof. The proof of (i)⇒(ii) is clear from Corollary 2.4. For the opposite, assume that C is a lower frame bound of $(W_j, \Lambda_j, v_j)_{j \neq j_0}$. For each $0 \neq f \in \mathcal{R}(K)$ we have

$$\begin{aligned} C\|K^* f\|^2 &\leq \sum_{j \neq j_0} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \\ &= \sum_{j \in \mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 - v_{j_0}^2 \|\Lambda_{j_0} \pi_{W_{j_0}} f\|^2 \\ &= (A\|f\|^2 - v_{j_0}^2 \|\Lambda_{j_0} \pi_{W_{j_0}} f\|^2). \end{aligned}$$

Hence,

$$0 < C \frac{\|K^* f\|^2}{\|f\|^2} \leq A - v_{j_0}^2 \frac{\|\Lambda_{j_0} \pi_{W_{j_0}} f\|^2}{\|f\|^2}.$$

Therefore $A - v_{j_0}^2 \|\Lambda_{j_0} \pi_{W_{j_0}}\|^2 > 0$. ∎

In next result, we provide a new K-g-fusion frame for the space H with by deleting a number of members of a Parseval frame for H_j.

Theorem 2.6. *Let Λ be a K-g-fusion frame for H with bounds A and B. For each $j \in \mathbb{J}$, let $\{f_{ij}\}_{i \in \mathbb{I}_j} \in \Lambda_j(W_j)$ be a Parseval frame for H_j which $\{f_{ij}\}_{i \in \mathbb{I}_j \setminus \mathbb{L}_j}$ is a frame for H_j with the lower frame bound C_j for each finite subset $\mathbb{L}_j \subset \mathbb{I}_j$ and all $j \in \mathbb{J}$. If $\widetilde{W}_j := \overline{\operatorname{span}}\{\Lambda_j^* f_{ij}\}_{i \in \mathbb{I}_j \setminus \mathbb{L}_j}$, then $(\widetilde{W}_j, \Lambda_j, v_j)_{j \in \mathbb{J}}$ is a K-g-fusion frame for H with bounds $(\min_{j \in \mathbb{J}} C_j)A$ and B.*

Proof. It is clear that $\widetilde{W}_j$ are closed subspaces of H for each $j \in \mathbb{J}$ and B is a upper frame bound of $(\widetilde{W}_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$. For each $f \in H$ we have

$$\begin{aligned}\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{\widetilde{W}_j} f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j} |\langle \Lambda_j \pi_{\widetilde{W}_j} f, f_{ij}\rangle|^2 \\ &\geq \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j\setminus\mathbb{L}_j} |\langle \pi_{\widetilde{W}_j} f, \Lambda_j^* f_{ij}\rangle|^2 \\ &= \sum_{j\in\mathbb{J}} v_j^2 \sum_{i\in\mathbb{I}_j\setminus\mathbb{L}_j} |\langle \Lambda_j \pi_{W_j} f, f_{ij}\rangle|^2 \\ &\geq \sum_{j\in\mathbb{J}} v_j^2 C_j \|\Lambda_j \pi_{W_j} f\|^2 \\ &\geq (\min_{j\in\mathbb{J}} C_j) \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \\ &\geq (\min_{j\in\mathbb{J}} C_j) A \|K^* f\|^2.\end{aligned}$$

■

3. Error Operator for Parseval K-g-Fusion Frames

Now, we aim to study the approximation Ψ in finite case similar to the view presented in [8]. Suppose that $\mathbb{J} = \{1, 2, \cdots, m\}$ is finite and Λ is a Parseval K-g-fusion frame for H where $\dim H < \infty$. For every $j_0 \in \mathbb{J}$, we consider the following operator:

$$\begin{aligned}&D_{j_0} : \mathscr{H}_2 \longrightarrow \mathscr{H}_2, \\ &D_{j_0}\{f_j\}_{j\in\mathbb{J}} = \delta_{j,j_0} f_{j_0}.\end{aligned}$$

We define the associated 1-erasure reconstruction error $\mathcal{E}_1(\Lambda)$ to be

$$\mathcal{E}_1(\Lambda) = \max_{j\in\mathbb{J}} \|T_\Lambda D_j T_\Lambda^*\|.$$

Since

$$\|T_\Lambda D_j T_\Lambda^*\| = \sup_{\|f\|=1} \|T_\Lambda D_j T_\Lambda^* f\| = v_j^2 \sup_{\|f\|=1} \|\pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\| \leq v_j^2 \|\Lambda_j\|^2,$$

therefore,

$$\mathcal{E}_1(\Lambda) = \max_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\|^2.$$

Theorem 3.1. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for H and $\{\lambda_j\}_{j=1}^n$ the eigenvalues for the frame operator S where $n = \dim H$. Then*

$$\sum_{j\in\mathbb{J}} \|f_j\|^2 = n\|K\|^2.$$

Proof. First, it is clear that the equality

$$\sum_{j=1}^{n} \lambda_j = \sum_{j\in\mathbb{J}} \|f_j\|^2 \tag{5}$$

which is presented in [11] for ordinary frames, is also true for K-frames. Suppose that $f \in H$. Since $\langle Sf, f\rangle = \|K^*f\|^2$, so $S = KK^*$. Assume that λ_j is an eigenvalue of S for arbitrary j. Therefore, $\|K^*f\|^2 = \lambda_j\|f\|^2$ and we conclude that $\|K^*\|^2 = \lambda_j$ for each $j = 1, 2, \cdots, n$. By (5), this completes the proof. ■

Theorem 3.2. *Let $\Lambda_j(W_j)$ be closed subspaces, $\mathbb{J} = \{1, 2, \cdots, m\}$ and Λ be a Parseval g-fusion frame for H where $\dim H = n$ and also $|H_j| < \infty$ for each $j \in \mathbb{J}$. Then the following conditions are equivalent:*

(i) *Λ satisfies $\mathcal{E}_1(\Lambda) = \min_{j\in\mathbb{J}} \mathcal{E}_1(\widetilde{W}_j, \Lambda_j, \widetilde{v}_j)_{j\in\mathbb{J}}$, where $(\widetilde{W}_j, \Lambda_j, \widetilde{v}_j)_{j\in\mathbb{J}}$ is a Parseval K-g-fusion frame for H with $\dim \widetilde{W}_j = \dim W_j$ for each $j \in \mathbb{J}$.*

(ii) *For each $j \in \mathbb{J}$ we have*

$$v_j^2\|\Lambda_j\|^2 = \frac{n\|K\|^2}{m.\dim W_j}.$$

Proof. Assume that $\{e_{ij}\}_{i\in\mathbb{I}_j}$ is a orthonormal basis for $\Lambda_j(W_j)$ for each $j \in \mathbb{J}$. Via Theorem 1.6, the sequence $\{v_j\pi_{W_j}\Lambda_j^* e_{ij}\}_{j=1,i=1}^{m,\ \dim\Lambda_j(W_j)}$ is a Parseval K-frame for H. By Theorem 3.1, we can get

$$n\|K\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{\dim\Lambda_j(W_j)} v_j^2\|\pi_{W_j}\Lambda_j^* e_{ij}\|^2 \le \sum_{j=1}^{m} \dim\Lambda_j(W_j)v_j^2\|\Lambda_j\|^2.$$

So, there exists j such that

$$n\|K\|^2 \le m.\dim\Lambda_j(W_j)v_j^2\|\Lambda_j\|^2.$$

Since the dimensions as well as the number of subspaces are fixed, we conclude that $\mathcal{E}_1(\Lambda)$ is minimal if and only if $n\|K\|^2 = m.\dim\Lambda_j(W_j)v_j^2\|\Lambda_j\|^2$ for all $j \in \mathbb{J}$. ■

Acknowledgement. We gratefully thank the referee for carefully reading the paper and for the suggestions that greatly improved the presentation of the paper.

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