Southeast Asian Bulletin of Mathematics (2022) 46: 783–800

Southeast Asian
Bulletin of
Mathematics

The Set of Automorphisms of Pell Forms and Pell Equations

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Received 27 October 2021
Accepted 9 January 2022

Communicated by Ngai-Ching Wong

Dedicated to the memory of Professor Ky Fan (1914–2010)

AMS Mathematics Subject Classification(2020): 11E18, 11E25, 11D09, 11D45

Abstract. Let $d = k^2 + 4$ for some integer $k \geq 2$. In this work, we first determined the set of automorphisms of the Pell form $F_\Delta(x, y) = x^2 - dy^2$ of discriminant $\Delta = 4d$. Later, we deduced the set of all integer solutions of the Pell equations $F_\Delta(x, y) = \pm 1$ and $F_\Delta(x, y) = \pm k^2$.

Keywords: Quadratic form; Pell form; Automorphism; Pell equation.

1. Introduction

A real binary quadratic form F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. F is an integral form if and only if $a, b, c \in \mathbb{Z}$; F is primitive if and only if $\gcd(a, b, c) = 1$; F is indefinite if $\Delta > 0$ and F is positive definite if and only if $a, c > 0$ and $\Delta < 0$.

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Let $\mathrm{GL}(2,\mathbb{Z})$ be the multiplicative group of 2×2 matrices $g=\begin{bmatrix} r & s \\ t & u\end{bmatrix}$ such that $r,s,\ t,u\in\mathbb{Z}$ and $\det(g)=\pm1$. Gauss defined the group action of $\mathrm{GL}(2,\mathbb{Z})$ on the set of forms as

$$gF(x,y)=F(rx+ty,sx+uy) \tag{1}$$

for some $g=\begin{bmatrix} r & s \\ t & u\end{bmatrix}\in\mathrm{GL}(2,\mathbb{Z})$. If there exists a $g\in\mathrm{GL}(2,\mathbb{Z})$ such that $gF=G$, then F and G are called equivalent. If $\det(g)=1$, then F and G are called properly equivalent and if $\det(g)=-1$, then F and G are called improperly equivalent. An element $g\in\mathrm{GL}(2,\mathbb{Z})$ is called an automorphism of F if $gF=F$. If $\det(g)=1$, then g is called a proper automorphism of F and if $\det(g)=-1$, then g is called an improper automorphism of F. The set of proper automorphisms of F is denoted by $Aut^+(F)$ and the set of improper automorphisms of F is denoted by $Aut^-(F)$. We also set $Aut^*(F)=\{g\in\mathrm{GL}(2,\mathbb{Z}):gF=-F,\ \det(g)=-1\}$ (for further details see [3, 4, 5]).

2. Automorphisms of Pell Forms

In [13], the first author derived some new results on the proper cycles of indefinite forms and their right neighbors. In [14], the first author considered the cycles of indefinite quadratic forms and cycles of ideals, in [15], the first author considered the indefinite quadratic forms and Pell equations involving quadratic ideals and in [16], the first author derived some new results on base points, bases and positive definite forms.

In the present paper, we consider the set of automorphisms of Pell forms. Recall that a Pell form is the form

$$F_\Delta(x,y)=\begin{cases} x^2-\frac{\Delta}{4}y^2 & \text{if } \Delta\equiv 0(\operatorname{mod}4)\\ x^2+xy-\frac{\Delta-1}{4}y^2 & \text{if } \Delta\equiv 1(\operatorname{mod}4)\end{cases} \tag{2}$$

for a non-zero discriminant Δ. So the Pell equation is the equation $F_\Delta(x,y)=\pm1$. $F_\Delta(x,y)=1$ is called the positive Pell equation and $F_\Delta(x,y)=-1$ is called the negative Pell equation. Let $\mathrm{Pell}(\Delta)=\{(x,y)\in\mathbb{Z}^2:F_\Delta(x,y)=1\}$ and $\mathrm{Pell}^{\pm}(\Delta)=\{(x,y)\in\mathbb{Z}^2:F_\Delta(x,y)=\pm1\}$. Then for any $(x,y)\in\mathrm{Pell}^{\pm}(\Delta)$, we set

$$g_F(x,y)=\begin{cases}\begin{bmatrix} x-\frac{b}{2}y & ay\\ -cy & x-\frac{b}{2}y\end{bmatrix} & \text{if } \Delta\equiv 0(\operatorname{mod}4)\\[2ex] \begin{bmatrix} x+\frac{1-b}{2}y & ay\\ -cy & x+\frac{1+b}{2}y\end{bmatrix} & \text{if } \Delta\equiv 1(\operatorname{mod}4).\end{cases} \tag{3}$$

Then $\det(g_F(x,y))=F_\Delta(x,y)$, $g_F:\mathrm{Pell}^{\pm}(\Delta)\to\mathrm{GL}(2,\mathbb{Z})$ is a group homomorphism and $g_F(x,y)$ is a proper automorphism of F for all $(x,y)\in\mathrm{Pell}(\Delta)$. If F is primitive, then $g_F:\mathrm{Pell}^{\pm}(\Delta)\to Aut^+(F)$ is a group isomorphism.

Now let $d = k^2 + 4$ for some integer $k \geq 1$ and let $\Delta = 4d$. Then from (2), we get the Pell form

$$F_\Delta(x, y) = x^2 - dy^2. \tag{4}$$

For the set of automorphisms of (4), we can give the following theorem.

Theorem 2.1. *Let F_Δ be the Pell form defined in (4). Then we have the following statements:*

(i) *If $k \geq 3$ is odd, then*

$$Aut^+(F_\Delta) = \{\pm(g_F^+)^t : t \in \mathbb{Z}\},\ Aut^-(F_\Delta) = \{\pm g_F^-(g_F^+)^t : t \in \mathbb{Z}\} \text{ and}$$
$$Aut^*(F_\Delta) = \{\pm(g_F^*)^{2t-1} : t \in \mathbb{Z}\},$$

where

$$g_F^+ = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^5+4k^3+3k}{2} \\ \frac{k^7+8k^5+19k^3+12k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix},$$

$$g_F^- = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^5+4k^3+3k}{2} \\ -\frac{k^7+8k^5+19k^3+12k}{2} & -\frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix},$$

$$g_F^* = \begin{bmatrix} \frac{k^3+3k}{2} & \frac{k^2+1}{2} \\ \frac{k^4+5k^2+4}{2} & \frac{k^3+3k}{2} \end{bmatrix}.$$

(ii) *If $k \geq 2$ is even, then*

$$Aut^+(F_\Delta) = \{\pm(g_F^+)^t : t \in \mathbb{Z}\},\ Aut^-(F_\Delta) = \{\pm g_F^-(g_F^+)^t : t \in \mathbb{Z}\} \text{ and}$$
$$Aut^*(F_\Delta) = \{\},$$

where

$$g_F^+ = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k}{2} \\ \frac{k^3+4k}{2} & \frac{k^2+2}{2} \end{bmatrix} \text{ and } \quad g_F^- = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k}{2} \\ -\frac{k^3+4k}{2} & -\frac{k^2+2}{2} \end{bmatrix}.$$

Proof. It is known that (see [7, Corollary 5.7]), if $d > 0$ is not a perfect square and $\sqrt{d}$ has continued fraction expansion $[a_0, \overline{a_1, a_2, \cdots, a_l}]$ of period length l, then the fundamental solution of $x^2 - dy^2 = 1$ is given by $(x_1, y_1) = (A_{l-1}, B_{l-1})$ if l is even or (A_{2l-1}, B_{2l-1}) if l is odd. Moreover if l is odd, then the fundamental solution of $x^2 - dy^2 = -1$ is given by $(x_1, y_1) = (A_{l-1}, B_{l-1})$, where $A_{-2} = 0, A_{-1} = 1$, $A_k = a_k A_{k-1} + A_{k-2}$ and $B_{-2} = 1, B_{-1} = 0, B_k = a_k B_{k-1} + B_{k-2}$.

(i) Let $k \geq 3$ be an odd integer. Then it is easily seen that the continued fraction expansion of $\sqrt{d}$ is

$$\sqrt{k^2+4} = k + (\sqrt{k^2+4} - k) = k + \cfrac{1}{\frac{k-1}{2} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\frac{k-1}{2} + \cfrac{1}{2k + (\sqrt{k^2+4} - k)}}}}}.$$

So $\sqrt{d} = [k; \overline{\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k}]$ with period length 5 and hence the fundamental solution of $F_\Delta(x, y) = 1$ is $(x_1, y_1) = (\frac{k^6+6k^4+9k^2+2}{2}, \frac{k^5+4k^3+3k}{2})$. Thus from (3), we deduce that

$$g_F^+ = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^5+4k^3+3k}{2} \\ \frac{k^7+8k^5+19k^3+12k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}$$

is a proper automorphism of F_Δ. Since $\det(g_F^- g_F^+) = -1$ and $g_F^- g_F^+ F_\Delta = F_\Delta$, $g_F^- g_F^+$ is an improper automorphism of F_Δ, that is, $g_F^- g_F^+ \in Aut^-(F_\Delta)$ for

$$g_F^- = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^5+4k^3+3k}{2} \\ -\frac{k^7+8k^5+19k^3+12k}{2} & -\frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}.$$

For any $t \in \mathbb{Z}$, $g_F^-(g_F^+)^t$ is also an improper automorphisms of F_Δ. Since the fundamental solution of $F_\Delta(x, y) = -1$ is $(x_1, y_1) = (\frac{k^3+3k}{2}, \frac{k^2+1}{2})$, we get

$$g_F^* = \begin{bmatrix} \frac{k^3+3k}{2} & \frac{k^2+1}{2} \\ \frac{k^4+5k^2+4}{2} & \frac{k^3+3k}{2} \end{bmatrix}$$

with $\det(g_F^*) = -1$ and $g_F^* F_\Delta = -F_\Delta$. So $g_F^* \in Aut^*(F_\Delta)$. Since $(g_F^*)^2 = g_F^+$, even powers of g_F^* are the proper automorphisms of F_Δ. Therefore $Aut^*(F_\Delta) = \{\pm(g_F^*)^{2t-1} : t \in \mathbb{Z}\}$.

(i) Let $k \geq 2$ be an even integer. Then $\sqrt{d} = [k; \overline{\frac{k}{2}, 2k}]$. So the fundamental solution of $F_\Delta(x, y) = 1$ is $(x_1, y_1) = (\frac{k^2+2}{2}, \frac{k}{2})$. Thus

$$g_F^+ = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k}{2} \\ \frac{k^3+4k}{2} & \frac{k^2+2}{2} \end{bmatrix}$$

is a proper automorphism of F_Δ. For

$$g_F^- = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k}{2} \\ -\frac{k^3+4k}{2} & -\frac{k^2+2}{2} \end{bmatrix},$$

we get $\det(g_F^- g_F^+) = -1$ and since $g_F^- g_F^+ F_\Delta = F_\Delta$, $g_F^- g_F^+$ is an improper automorphism of F_Δ, that is, $g_F^- g_F^+ \in Aut^-(F_\Delta)$. Since the period length is 2 which is an even number, $F_\Delta(x, y) = -1$ has no integer solutions. Therefore there is no a matrix g_F^* with $\det(g_F^*) = -1$ such that $g_F^* F_\Delta = -F_\Delta$. Consequently $Aut^*(F_\Delta) = \{\}$. ∎

3. The Pell Equation $F_\Delta(x, y) = \pm 1$

Let F_Δ be the Pell form defined in (4). In this section, we consider the set of all (positive) integer solutions of the Pell equation (see [1, 6, 7])

$$F_\Delta(x, y) = \pm 1$$

in two cases: $k \geq 3$ is odd or $k \geq 2$ is even.

3.1. $k \geq 3$ Is Odd

Theorem 3.1. *Let $k \geq 3$ be odd. Then we have the following statements:*

(i) *For the positive Pell equation $F_\Delta(x,y) = 1$, we have*

(a) *the fundamental solution is $(x_1, y_1) = (\frac{k^6+6k^4+9k^2+2}{2}, \frac{k^5+4k^3+3k}{2})$.*

(b) *the set of all integer solutions is $\Omega = \{(x_n, y_n)\}$, where*

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $n \geq 1$ and

$$M = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}. \tag{5}$$

(c) *the integer solutions (x_n, y_n) satisfy the recurrence relations*

$$x_n = (k^6 + 6k^4 + 9k^2 + 1)(x_{n-1} + x_{n-2}) - x_{n-3}$$
$$y_n = (k^6 + 6k^4 + 9k^2 + 1)(y_{n-1} + y_{n-2}) - y_{n-3}$$

for $n \geq 4$.

(d) *the n^{th} integer solution (x_n, y_n) can be given by the aid of continued fraction expansion, namely,*

$$\frac{x_n}{y_n} = \begin{cases} \left[3; \underbrace{1,1,1,1,6}_{2n-1 \text{ times}}, 1,1,2\right] & \text{for } k = 3 \\ \left[k; \underbrace{\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k}_{2n-1 \text{ times}}, \frac{k-1}{2}, 1, 1, \frac{k-1}{2}\right] & \text{for } k \geq 5 \end{cases}$$

for $n \geq 1$.

(ii) *For the negative Pell equation $F_\Delta(x,y) = -1$, we have*

(a) *the fundamental solution is $(x_1, y_1) = (\frac{k^3+3k}{2}, \frac{k^2+1}{2})$.*

(b) *the set of all integer solutions is $\Omega = \{(x_{2n-1}, y_{2n-1})\}$, where*

$$\begin{bmatrix} x_{2n-1} \\ y_{2n-1} \end{bmatrix} = M^{2n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $n \geq 1$ and

$$M = \begin{bmatrix} \frac{k^3+3k}{2} & \frac{k^4+5k^2+4}{2} \\ \frac{k^2+1}{2} & \frac{k^3+3k}{2} \end{bmatrix}.$$

(c) *the integer solutions* (x_{2n-1}, y_{2n-1}) *satisfy the recurrence relations*

$$x_{2n-1} = (k^6 + 6k^4 + 9k^2 + 1)(x_{2n-3} + x_{2n-5}) - x_{2n-7}$$
$$y_{2n-1} = (k^6 + 6k^4 + 9k^2 + 1)(y_{2n-3} + y_{2n-5}) - y_{2n-7}$$

for $n \geq 4$.

(d) *the* $(2n-1)^{st}$ *integer solution* (x_{2n-1}, y_{2n-1}) *can be given by the aid of continued fraction expansion, namely,*

$$\frac{x_{2n-1}}{y_{2n-1}} = \begin{cases} \left[3; \underbrace{1,1,1,1,6}_{2n-2\ times}, 1, 1, 2\right] & \text{for } k = 3 \\ \left[k; \underbrace{\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k}_{2n-2\ times}, \frac{k-1}{2}, 1, 1, \frac{k-1}{2}\right] & \text{for } k \geq 5 \end{cases}$$

for $n \geq 1$.

Proof. (i)(a) It can be easily seen that $(x_1, y_1) = (\frac{k^6+6k^4+9k^2+2}{2}, \frac{k^5+4k^3+3k}{2})$ is the fundamental solution by (1) of Theorem 2.1.

(i)(b) We prove it by induction. Let $n = 1$. Then $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} \\ \frac{k^5+4k^3+3k}{2} \end{bmatrix}$ which is true. Assume that it is satisfied for $n-1$, that is,

$$\begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^7+8k^5+19k^3+12k}{2} \\ \frac{k^5+4k^3+3k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} (\frac{k^6+6k^4+9k^2+2}{2})x_{n-1} + (\frac{k^7+8k^5+19k^3+12k}{2})y_{n-1} \\ (\frac{k^5+4k^3+3k}{2})x_{n-1} + (\frac{k^6+6k^4+9k^2+2}{2})y_{n-1} \end{bmatrix}. \end{aligned}$$

So

$$x_n = (\frac{k^6+6k^4+9k^2+2}{2})x_{n-1} + (\frac{k^7+8k^5+19k^3+12k}{2})y_{n-1}$$

and

$$y_n = (\frac{k^5+4k^3+3k}{2})x_{n-1} + (\frac{k^6+6k^4+9k^2+2}{2})y_{n-1}.$$

Thus we conclude that

$$\begin{aligned}x_n^2 - dy_n^2 &= \left[(\frac{k^6+6k^4+9k^2+2}{2})x_{n-1} + (\frac{k^7+8k^5+19k^3+12k}{2})y_{n-1}\right]^2 \\ &\quad - (k^2+4)\left[(\frac{k^5+4k^3+3k}{2})x_{n-1} + (\frac{k^6+6k^4+9k^2+2}{2})y_{n-1}\right]^2 \\ &= x_{n-1}^2 - (k^2+4)y_{n-1}^2 \\ &= 1.\end{aligned}$$

So it is true for every $n \geq 1$.

(i)(c) For $x_1 = \frac{k^6+6k^4+9k^2+2}{2}$ and $y_1 = \frac{k^5+4k^3+3k}{2}$, we set $\alpha = x_1 + y_1\sqrt{d}$ and $\beta = x_1 - y_1\sqrt{d}$. Then it is known that $x_n = \frac{\alpha^n+\beta^n}{2}$ and $y_n = \frac{\alpha^n-\beta^n}{2\sqrt{d}}$. Hence we deduce that

$$\begin{aligned}&(k^6+6k^4+9k^2+1)(x_{n-1}+x_{n-2}) - x_{n-3} \\ =&(k^6+6k^4+9k^2+1)(\frac{\alpha^{n-1}+\beta^{n-1}}{2} + \frac{\alpha^{n-2}+\beta^{n-2}}{2}) - \frac{\alpha^{n-3}+\beta^{n-3}}{2} \\ =&\frac{\alpha^n}{2}\left[\frac{(k^6+6k^4+9k^2+1)\alpha^2 + (k^6+6k^4+9k^2+1)\alpha - 1}{\alpha^3}\right] \\ &+ \frac{\beta^n}{2}\left[\frac{(k^6+6k^4+9k^2+1)\beta^2 + (k^6+6k^4+9k^2+1)\beta - 1}{\beta^3}\right] \\ =&\frac{\alpha^n+\beta^n}{2} \\ =&x_n\end{aligned}$$

since $(k^6+6k^4+9k^2+1)\alpha^2 + (k^6+6k^4+9k^2+1)\alpha - 1 = \alpha^3$ and $(k^6+6k^4+9k^2+1)\beta^2 + (k^6+6k^4+9k^2+1)\beta - 1 = \beta^3$. Similarly it can be shown that $y_n = (k^6+6k^4+9k^2+1)(y_{n-1}+y_{n-2}) - y_{n-3}$ for $n \geq 4$. The other cases can be proved similarly. ■

In order to determine the set of all integer solutions (x_n, y_n) of $F_\Delta(x,y) = \pm 1$, we need the n^{th} power of M defined in (5) which is given below.

Theorem 3.2. *The* n^{th} *power of* M *defined in* (5) *is* $M^n = \begin{bmatrix} R & S \\ T & U \end{bmatrix}$, *where*

$$R = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i = U, \quad S = \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^{i+1},$$

$$T = \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i$$

for even $n \geq 2$ *or*

$$R = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i = U, \quad S = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^{i+1},$$

$$T = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i$$

for odd $n \geq 1$, *where* $x_1 = \frac{k^6+6k^4+9k^2+2}{2}$ *and* $y_1 = \frac{k^5+4k^3+3k}{2}$.

Proof. It can be proved by induction on n. ■

From Theorems 3.1 and 3.2, we deduce that

Theorem 3.3. *Let* $k \geq 3$ *be odd. Then the following statements hold:*

(i) *For* $x_1 = \frac{k^6+6k^4+9k^2+2}{2}$ *and* $y_1 = \frac{k^5+4k^3+3k}{2}$, *the set of all integer solutions of* $F_\Delta(x,y) = 1$ *is* $\Omega = \{(x_n, y_n)\}$, *where*

$$(x_n, y_n) = \left(\sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i, \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i \right)$$

for even $n \geq 2$ *or*

$$(x_n, y_n) = \left(\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i, \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i \right)$$

for odd $n \geq 1$.

(ii) *For* $x_1 = \frac{k^3+3k}{2}$ *and* $y_1 = \frac{k^2+1}{2}$, *the set of all integer solutions of* $F_\Delta(x,y) = -1$ *is* $\Omega = \{(x_{2n-1}, y_{2n-1})\}$, *where*

$$x_{2n-1} = \sum_{i=0}^{\frac{2n-1}{2}} \binom{2n-1}{2i} x_1^{2n-1-2i} y_1^{2i} d^i \quad \text{and}$$

$$y_{2n-1} = \sum_{i=0}^{\frac{2n-1}{2}} \binom{2n-1}{2i+1} x_1^{2n-2-2i} y_1^{2i+1} d^i$$

for $n \geq 1$.

3.2. $k \geq 2$ Is Even

Theorem 3.4. *Let* $k \geq 2$ *be even. Then we have the following statements:*

(i) *For the positive Pell equation* $F_\Delta(x,y) = 1$, *we have*

(a) *the fundamental solution is* $(x_1, y_1) = (\frac{k^2+2}{2}, \frac{k}{2})$.

(b) *the set of all integer solutions is* $\Omega = \{(x_n, y_n)\}$, *where*

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $n \geq 1$ *and*

$$M = \begin{bmatrix} \frac{k^2+2}{2} & \frac{k^3+4k}{2} \\ \\ \frac{k}{2} & \frac{k^2+2}{2} \end{bmatrix}. \tag{6}$$

(c) *the integer solutions* (x_n, y_n) *satisfy the recurrence relations*

$$\begin{aligned} x_n &= (k^2+1)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n &= (k^2+1)(y_{n-1} + y_{n-2}) - y_{n-3} \end{aligned}$$

for $n \geq 4$.

(d) *the* n^{th} *integer solution* (x_n, y_n) *can be given by the aid of continued fraction expansion, namely,*

$$\frac{x_n}{y_n} = \begin{cases} \left[2; \underbrace{1, 4}_{n-2\ times}, 1, 5\right] & \textit{for } k = 2 \textit{ and } n \geq 2 \\ \\ \left[k; \underbrace{\frac{k}{2}, 2k}_{n-1\ times}, \frac{k}{2}\right] & \textit{for } k \geq 4 \textit{ and } n \geq 1. \end{cases}$$

(ii) *The negative Pell equation* $F_\Delta(x,y) = -1$ *has no integer solutions.*

Proof. It can be proved as in the same way that Theorem 3.1 was proved. ∎

The n^{th} power of M defined in (6) is given below.

Theorem 3.5. *The n^{th} power of M defined in (6) is* $M^n = \begin{bmatrix} R & S \\ T & U \end{bmatrix}$, *where*

$$R = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i = U, \quad S = \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^{i+1},$$
$$T = \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i$$

for even $n \geq 2$ or

$$R = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i = U, \quad S = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^{i+1},$$
$$T = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i$$

for odd $n \geq 1$, where $x_1 = \frac{k^2+2}{2}$ and $y_1 = \frac{k}{2}$.

Proof. It can be proved by induction on n. ■

From Theorems 3.4 and 3.5, we can give the following theorem.

Theorem 3.6. *Let $x_1 = \frac{k^2+2}{2}$ and $y_1 = \frac{k}{2}$. Then the set of all integer solutions of $F_\Delta(x, y) = 1$ is $\Omega = \{(x_n, y_n)\}$, where*

$$(x_n, y_n) = \left(\sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i, \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i \right)$$

for even $n \geq 2$ or

$$(x_n, y_n) = \left(\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} x_1^{n-2i} y_1^{2i} d^i, \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} x_1^{n-1-2i} y_1^{2i+1} d^i \right)$$

for odd $n \geq 1$.

Remark 3.7. Here one may wonder why we only consider the case $\Delta = 4d$. In fact, when we consider the case $\Delta = 1 + 4d$, we see that there is no a general formula, indeed, for the fundamental solutions of the positive Pell equation $F_\Delta(x, y) = x^2 + xy - dy^2 = 1$, we have $(x_1, y_1) = (22, 7)$ is the fundamental solution for $k = 3$, $(x_1, y_1) = (5, 1)$ is the fundamental solution for $k = 5, (x_1, y_1) = (34, 5)$ is the fundamental solution for $k = 7, (x_1, y_1) = (131, 15)$

is the fundamental solution for $k = 9, (x_1, y_1) = (38, 3)$ is the fundamental solution for $k = 13, (x_1, y_1) = (571, 39)$ is the fundamental solution for $k = 15$ and $(x_1, y_1) = (133, 8)$ is the fundamental solution for $k = 17$.

4. The Pell Equation $F_\Delta(x, y) = \pm k^2$

In this section we consider the set of all (positive) integer solutions of

$$F_\Delta(x, y) = \pm k^2. \tag{7}$$

Now let Δ be a non-square discriminant. The Δ-order O_Δ is defined to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 (\text{mod } 4)$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 (\text{mod } 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ.

Let $F = (a, b, c)$ be an indefinite integral quadratic form of discriminant $\Delta = b^2 - 4ac$. Then we can rewrite $F(x, y) = ((xa + y\frac{b+\sqrt{\Delta}}{2})(xa + y\ \frac{b-\sqrt{\Delta}}{2}))/a$. So the module M_F of F is $M_F = \{xa + y\frac{b+\sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta})$. Therefore we get $(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \quad y'] = \begin{cases} [x \quad y] \begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 (\text{mod } 4), \\ \\ [x \quad y] \begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 (\text{mod } 4). \end{cases} \tag{8}$$

Let m be any integer and let Ω denote the set of all integer solutions of $F(x, y) = m$, that is, $\Omega = \{(x, y) : F(x, y) = m\}$. Then there is a bijection $\Psi : \Omega \to \{\gamma \in M_F : N(\gamma) = am\}$. The action of $O_{\Delta,1}^u = \{\alpha \in O_\Delta^u : N(\alpha) = 1\}$ on the set Ω is most interesting when Δ is a positive non-square since $O_{\Delta,1}^u$ is infinite. Therefore the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O_{\Delta,1}^u$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ that is grater than 1 and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$ or ε_Δ^2 if $N(\varepsilon_\Delta) = -1$. Then every $O_{\Delta,1}^u$ orbit of integral solutions of $F(x, y) = m$ contains a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \le y \le U$, where $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}} (1 - \frac{1}{\tau_\Delta})$ if $am > 0$ or $U = \left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}} (1 + \frac{1}{\tau_\Delta})$ if $am < 0$. So for finding a set of representatives of the $O_{\Delta,1}^u$ orbits of integral solutions of $F(x, y) = m$, we must find for each integer y_0 in the range $0 \le y_0 \le U$, whether $\Delta y_0^2 + 4am$ is a perfect square or not since $ax_0^2 + bx_0y_0 + cy_0^2 = m \Leftrightarrow \Delta y_0^2 + 4am = (2ax_0 + by_0)^2$. If $\Delta y_0^2 + 4am$ is a perfect square, then $x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a}$. So there is a set of representatives $\text{Rep} = \{[x_0 \ \ y_0]\}$. Thus for the matrix M derived from (8), the set of all integer solutions of $F(x, y) = m$ is $\Omega = \{\pm(x, y) : [x \quad y] = [x_0 \quad y_0]M^n, n \in \mathbb{Z}\}$. If $\Delta y_0^2 + 4am$ is not a perfect square, then there are no integer solutions.

4.1. $k \geq 3$ Is Odd

Theorem 4.1. *Let $k \geq 3$ be odd. Then we have the following statements:*

(i) *For the positive Pell equation $F_\Delta(x,y) = k^2$,*

(a) *If $k \geq 3$ is not a perfect square and $\#Rep = 4$, then the set of representatives is $Rep = \{[\pm x_0^* \quad 0], [\pm x_1^* \quad y_1^*]\}$, where*

$$x_0^* = k, x_1^* = \frac{k^4 - 2k^3 + 5k^2 - 6k + 4}{2} \quad and \qquad (9)$$
$$y_1^* = \frac{k^3 - 2k^2 + 3k - 2}{2}$$

and the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n})\}$, where

$$(x_{3n+1}, y_{3n+1}) = (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0,$$
$$(x_{3n-1}, y_{3n-1}) = (x_1^* R - y_1^* S, x_1^* T - y_1^* U) \text{ for } n \geq 1,$$
$$(x_{3n}, y_{3n}) = (x_0^* R, x_0^* T) \text{ for } n \geq 1.$$

(b) *If $k \geq 9$ is a perfect square, say $k = t^2$ for some integer $t \geq 1$ and $\#Rep = 6$, then the set of representatives is*

$$Rep = \{[\pm x_0^* \quad 0], [\pm x_1^{**} \quad y_1^{**}], [\pm x_1^* \quad y_1^*]\},$$

where x_0^*, x_1^*, y_1^* is defined in (9), $x_1^{**} = \frac{t^5 - t^3 + 2t}{2}, y_1^{**} = \frac{t^3 - t}{2}$ and the set of all integer solutions is $\Omega = \{(x_{5n+1}, y_{5n+1}), (x_{5n+2}, y_{5n+2}), (x_{5n-2}, y_{5n-2}), (x_{5n-1}, y_{5n-1}), (x_{5n}, y_{5n})\}$, where

$$(x_{5n+1}, y_{5n+1}) = (x_1^{**} R + y_1^{**} S, x_1^{**} T + y_1^{**} U) \text{ for } n \geq 0,$$
$$(x_{5n+2}, y_{5n+2}) = (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0,$$
$$(x_{5n-2}, y_{5n-2}) = (x_1^* R - y_1^* S, x_1^* T - y_1^* U) \text{ for } n \geq 1,$$
$$(x_{5n-1}, y_{5n-1}) = (x_1^{**} R - y_1^{**} S, x_1^{**} T - y_1^{**} U) \text{ for } n \geq 1,$$
$$(x_{5n}, y_{5n}) = (x_0^* R, x_0^* T) \text{ for } n \geq 1.$$

(ii) *For the negative Pell equation $F_\Delta(x,y) = -k^2$,*

(a) *If $k \geq 3$ is not a perfect square and $\#Rep = 4$, then the set of representatives is $Rep = \{[\pm x_0^* \quad 1], [\pm x_1^* \quad y_1^*]\}$, where*

$$x_0^* = 2, x_1^* = \frac{k^4 + 3k^2}{2}, y_1^* = \frac{k^3 + k}{2}, \qquad (10)$$

and the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}), (x_{3n}, y_{3n})\}$, where

$$(x_{3n+1}, y_{3n+1}) = (x_0^* R + S, x_0^* T + U) \text{ for } n \geq 0,$$
$$(x_{3n+2}, y_{3n+2}) = (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0,$$
$$(x_{3n}, y_{3n}) = (-x_0^* R + S, -x_0^* T + U) \text{ for } n \geq 1.$$

(b) *If $k \geq 9$ is a perfect square, say $k = t^2$ for some integer $t \geq 1$ and $\#Rep = 6$, then the set of representatives is*

$$Rep = \{[\pm x_0^* \quad 1], [\pm x_1^{**} \quad y_1^{**}], [\pm x_1^* \quad y_1^*]\},$$

where x_0^*, x_1^*, y_1^* is defined in (10), $x_1^{**} = \frac{t^5+t^3+2t}{2}, y_1^{**} = \frac{t^3+t}{2}$ and the set of all integer solutions is $\Omega = \{(x_{5n+1}, y_{5n+1}), (x_{5n+2}, y_{5n+2}), (x_{5n+3}, y_{5n+3}), (x_{5n-1}, y_{5n-1}), (x_{5n}, y_{5n})\}$, where

$$\begin{aligned}
(x_{5n+1}, y_{5n+1}) &= (x_0^* R + S, x_0^* T + U) \text{ for } n \geq 0\\
(x_{5n+2}, y_{5n+2}) &= (x_1^{**} R + y_1^{**} S, x_1^{**} T + y_1^{**} U) \text{ for } n \geq 0\\
(x_{5n+3}, y_{5n+3}) &= (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0\\
(x_{5n-1}, y_{5n-1}) &= (-x_1^{**} R + y_1^{**} S, -x_1^{**} T + y_1^{**} U) \text{ for } n \geq 1\\
(x_{5n}, y_{5n}) &= (-x_0^* R + S, -x_0^* T + U) \text{ for } n \geq 1.
\end{aligned}$$

In all cases R, S, T, U is defined in Theorem 3.2.

Proof. (i)(a) For the positive Pell equation $F_\Delta(x, y) = k^2$, we have $F = (1, 0, -d)$ of discriminant $\Delta = 4d$. Since the fundamental solution is $(x_1, y_1) = ((k^6+6k^4+9k^2+2)/2, (k^5+4k^3+3k)/2)$, we get $\tau_\Delta = \frac{k^6+6k^4+9k^2+2+(k^5+4k^3+3k)\sqrt{k^2+4}}{2}$. In this case, the set of representatives is Rep $= \{[\pm x_0^* \quad 0], [\pm x_1^* \quad y_1^*]\}$, where

$$x_0^* = k, x_1^* = \frac{k^4 - 2k^3 + 5k^2 - 6k + 4}{2} \quad \text{and} \quad y_1^* = \frac{k^3 - 2k^2 + 3k - 2}{2}.$$

Here $[x_0^* \quad 0]H^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$, $[x_1^* \quad y_1^*]H^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$ and $[x_1^* \quad -y_1^*]H^n$ generates all integer solutions (x_{3n-1}, y_{3n-1}) for $n \geq 1$, where

$$H = \begin{bmatrix} \frac{k^6+6k^4+9k^2+2}{2} & \frac{k^5+4k^3+3k}{2} \\ \frac{k^7+8k^5+19k^3+12k}{2} & \frac{k^6+6k^4+9k^2+2}{2} \end{bmatrix}$$

which is the transpose of M defined in (5). Thus the set of all integer solutions of $F_\Delta(x, y) = k^2$ is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n})\}$, where

$$\begin{aligned}
(x_{3n+1}, y_{3n+1}) &= (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0\\
(x_{3n-1}, y_{3n-1}) &= (x_1^* R - y_1^* S, x_1^* T - y_1^* U) \text{ for } n \geq 1\\
(x_{3n}, y_{3n}) &= (x_0^* R, x_0^* T) \text{ for } n \geq 1.
\end{aligned}$$

Indeed, we only prove $(x_{3n+1}, y_{3n+1}) = (x_1^* R + y_1^* S, x_1^* T + y_1^* U)$ for $n \geq 0$. Let $n \geq 0$ be even. For $n = 0$, we have $R = 1, S = 0, T = 0$ and $U = 1$. So $(x_1, y_1) = (x_1^*, y_1^*)$ and hence

$$x_1^2 - dy_1^2 = (\frac{k^4 - 2k^3 + 5k^2 - 6k + 4}{2})^2 - (k^2 + 4)(\frac{k^3 - 2k^2 + 3k - 2}{2})^2 = 1.$$

So it is true for $n = 0$. Assume that it is satisfied for $n - 2$, that is, $x_{3n-5} = x_1^* R + y_1^* S$ and $y_{3n-5} = x_1^* T + y_1^* U$, where

$$R = \sum_{i=0}^{\frac{n-2}{2}} \binom{n-2}{2i} x_1^{n-2-2i} y_1^{2i} d^i = U, \quad S = \sum_{i=0}^{\frac{n-4}{2}} \binom{n-2}{2i+1} x_1^{n-3-2i} y_1^{2i+1} d^{i+1},$$
$$T = \sum_{i=0}^{\frac{n-4}{2}} \binom{n-2}{2i+1} x_1^{n-3-2i} y_1^{2i+1} d^i.$$

Then $x_{3n+1} = x_{3n-5}(R^2 + TS) + y_{3n-5}(SR + US)$ and $y_{3n+1} = x_{3n-5}(RT + TU) + y_{3n-5}(ST + U^2)$ and clearly,

$$\begin{aligned} x_{3n+1}^2 - dy_{3n+1}^2 &= [x_{3n-5}(R^2 + TS) + y_{3n-5}(SR + US)]^2 \\ &\quad - d[x_{3n-5}(RT + TU) + y_{3n-5}(ST + U^2)]^2 \\ &= x_{3n-5}^2[(R^2 + TS)^2 - d(RT + TU)^2] \\ &\quad + 2x_{3n-5}y_{3n-5}[(R^2 + TS)(SR + US) \\ &\quad - d(RT + TU)(ST + U^2)] \\ &\quad + y_{3n-5}^2[(SR + US)^2 - d(ST + U^2)^2] \\ &= x_{3n-5}^2 - dy_{3n-5}^2 \\ &= k^2 \end{aligned}$$

since $(R^2+TS)^2 - d(RT+TU)^2 = 1$, $(R^2+TS)(SR+US) - d(RT+TU)(ST+U^2) = 0$ and $(SR + US)^2 - d(ST + U^2)^2 = -d$. The other cases can be proved similarly. ■

4.2. $k \geq 2$ Is Even

Theorem 4.2. *Let $k \geq 2$ be even. Then we have the following statements:*

(i) *For the positive Pell equation $F_\Delta(x, y) = k^2$,*

(a) *If $k = 2$, then the set of representatives is $Rep = \{[\pm 2 \;\; 0]\}$ and the set of all integer solutions is $\Omega = \{(x_n, y_n)\}$, where $x_n = 2C_n$ and $y_n = 2B_n$ for $n \geq 1$ (Here B_n is the n^{th} balancing number and C_n is the n^{th} Lucas-balancing number).*

(b) *If $k = 4$, then the set of representatives is $Rep = \{[\pm 4 \;\; 0], [\pm 6 \;\; 1]\}$ and the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n})\}$, where*

$$\begin{aligned} (x_{3n+1}, y_{3n+1}) &= (6R + S, 6T + U) \text{ for } n \geq 0 \\ (x_{3n-1}, y_{3n-1}) &= (6R - S, 6T - U) \text{ for } n \geq 1 \\ (x_{3n}, y_{3n}) &= (4R, 4T) \text{ for } n \geq 1. \end{aligned}$$

(c) *If* $k \geq 6$ *is not a perfect square and* $\#Rep = 4$*, then the set of representatives is* $Rep = \{[\pm x_0^* \quad 0],\ [\pm x_1^* \quad y_1^*]\}$*, where*

$$x_0^* = k, x_1^* = \frac{k^2 - 2k + 4}{2}, y_1^* = \frac{k-2}{2}, \tag{11}$$

and the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n})\}$*, where*

$$\begin{aligned}(x_{3n+1}, y_{3n+1}) &= (x_1^*R + y_1^*S, x_1^*T + y_1^*U) \text{ for } n \geq 0\\ (x_{3n-1}, y_{3n-1}) &= (x_1^*R - y_1^*S, x_1^*T - y_1^*U) \text{ for } n \geq 1\\ (x_{3n}, y_{3n}) &= (x_0^*R, x_0^*T) \text{ for } n \geq 1.\end{aligned}$$

(d) *If* $k \geq 16$ *is a perfect square, say* $k = t^2$ *for some integer* $t \geq 1$ *and* $\#Rep = 6$*, then the set of representatives is*

$$Rep = \{[\pm x_0^* \quad 0], [\pm x_1^{**} \quad y_1^{**}], [\pm x_1^* \quad y_1^*]\},$$

where x_0^*, x_1^*, y_1^* *is defined in* (11)*,* $x_1^{**} = \frac{t^3+2t}{2}, y_1^{**} = \frac{t}{2}$*, and the set of all integer solutions is* $\Omega = \{(x_{5n+1}, y_{5n+1}), (x_{5n+2}, y_{5n+2}), (x_{5n-2}, y_{5n-2}), (x_{5n-1}, y_{5n-1}), (x_{5n}, y_{5n})\}$*, where*

$$\begin{aligned}(x_{5n+1}, y_{5n+1}) &= (x_1^{**}R + y_1^{**}S, x_1^{**}T + y_1^{**}U) \text{ for } n \geq 0\\ (x_{5n+2}, y_{5n+2}) &= (x_1^*R + y_1^*S, x_1^*T + y_1^*U) \text{ for } n \geq 0\\ (x_{5n-2}, y_{5n-2}) &= (x_1^*R - y_1^*S, x_1^*T - y_1^*U) \text{ for } n \geq 1\\ (x_{5n-1}, y_{5n-1}) &= (x_1^{**}R - y_1^{**}S, x_1^{**}T - y_1^{**}U) \text{ for } n \geq 1\\ (x_{5n}, y_{5n}) &= (x_0^*R, x_0^*T) \text{ for } n \geq 1.\end{aligned}$$

(ii) *For the negative Pell equation* $F_\Delta(x, y) = -k^2$*,*

(a) *If* $k = 2$*, then the set of representatives is* $Rep = \{[\pm 2 \quad 1]\}$*, and the set of all integer solutions is* $\Omega = \{(x_n, y_n)\}$*, where* $x_n = 2c_n$ *and* $y_n = P_{2n-1}$ *for* $n \geq 1$ *(Here* c_n *is the* n^{th} *Lucas-cobalancing number and* P_n *is the* n^{th} *Pell number).*

(b) *If* $k = 4$*, then the set of representatives is* $Rep = \{[\pm 2 \quad 1], [\pm 8 \quad 2]\}$*, and the set of all integer solutions is* $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}), (x_{3n}, y_{3n})\}$*, where*

$$\begin{aligned}(x_{3n+1}, y_{3n+1}) &= (2R + S, 2T + U) \text{ for } n \geq 0\\ (x_{3n+2}, y_{3n+2}) &= (8R + 2S, 8T + 2U) \text{ for } n \geq 0\\ (x_{3n}, y_{3n}) &= (-2R + S, -2T + U) \text{ for } n \geq 1.\end{aligned}$$

(c) *If* $k \geq 6$ *is not a perfect square and* $\#Rep = 4$*, then the set of representatives is* $Rep = \{[\pm x_0^* \quad 1], [\pm x_1^* \quad y_1^*]\}$*, where*

$$x_0^* = 2, x_1^* = \frac{k^2}{2}, y_1^* = \frac{k}{2}, \tag{12}$$

and the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}), (x_{3n}, y_{3n})\}$, *where*

$$\begin{aligned}(x_{3n+1}, y_{3n+1}) &= (x_0^* R + S, x_0^* T + U) \text{ for } n \geq 0\\ (x_{3n+2}, y_{3n+2}) &= (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0\\ (x_{3n}, y_{3n}) &= (-x_0^* R + S, -x_0^* T + U) \text{ for } n \geq 1.\end{aligned}$$

(d) *If* $k \geq 16$ *is a perfect square, say* $k = t^2$ *for some integer* $t \geq 1$ *and* $\#Rep = 6$, *then the set of representatives is*

$$Rep = \{[\pm x_0^* \quad 1], [\pm x_1^{**} \quad y_1^{**}], [\pm x_1^* \quad y_1^*]\},$$

where x_0^*, x_1^*, y_1^* *is defined in* (12), $x_1^{**} = \frac{t^3-2t}{2}, y_1^{**} = \frac{t}{2}$ *and the set of all integer solutions is* $\Omega = \{(x_{5n+1}, y_{5n+1}), (x_{5n+2}, y_{5n+2}), (x_{5n+3}, y_{5n+3}), (x_{5n-1}, y_{5n-1}), (x_{5n}, y_{5n})\}$, *where*

$$\begin{aligned}(x_{5n+1}, y_{5n+1}) &= (x_0^* R + S, x_0^* T + U) \text{ for } n \geq 0\\ (x_{5n+2}, y_{5n+2}) &= (x_1^{**} R + y_1^{**} S, x_1^{**} T + y_1^{**} U) \text{ for } n \geq 0\\ (x_{5n+3}, y_{5n+3}) &= (x_1^* R + y_1^* S, x_1^* T + y_1^* U) \text{ for } n \geq 0\\ (x_{5n-1}, y_{5n-1}) &= (-x_1^{**} R + y_1^{**} S, -x_1^{**} T + y_1^{**} U) \text{ for } n \geq 1\\ (x_{5n}, y_{5n}) &= (-x_0^* R + S, -x_0^* T + U) \text{ for } n \geq 1.\end{aligned}$$

In all cases R, S, T, U *is defined in Theorem 3.5.*

Proof. (i)(a) Let $k = 2$. The the set of representatives is Rep $= \{[\pm 2 \quad 0]\}$ for the Pell equation $x^2 - 8y^2 = 4$. Here $[2 \quad 0]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$ and $M = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$.

Behera and Panda [2] introduced balancing numbers $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{13}$$

for some positive integer r which is called balancer. If n is a balancing number with balancer r, then from (13) one has

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}. \tag{14}$$

Let B_n denote the n^{th} balancing number. Then from (14), we note that B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square. Thus $\sqrt{8B_n^2 + 1}$ is an integer which is called n^{th} Lucas-balancing number and is denoted by C_n, that is, $C_n = \sqrt{8B_n^2 + 1}$ (for further details see also [8, 9, 10, 12]). It can be easily seen that the n^{th} power of $M = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$ is $M^n = \begin{bmatrix} C_n & B_n \\ 8B_n & C_n \end{bmatrix}$. Thus the set of all integer solutions is $\Omega = \{(x_n, y_n)\}$, where $x_n = 2C_n$ and $y_n = 2B_n$ for $n \geq 1$.

(i)(b) If $k = 4$, then the set of representatives is Rep $= \{[\pm 4 \quad 0], [\pm 6 \quad 1]\}$ and $M = \begin{bmatrix} 9 & 2 \\ 40 & 9 \end{bmatrix}$. In this case $[4 \quad 0]M^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$, $[6 \quad 1]M^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$ and $[6 \quad -1]M^n$ generates all integer solutions (x_{3n-1}, y_{3n-1}) for $n \geq 1$. So the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n-1}, y_{3n-1}), (x_{3n}, y_{3n})\}$, where $(x_{3n+1}, y_{3n+1}) = (6R+S, 6T+U)$ for $n \geq 0$, $(x_{3n-1}, y_{3n-1}) = (6R-S, 6T-U)$ for $n \geq 1$ and $(x_{3n}, y_{3n}) = (4R, 4T)$ for $n \geq 1$. The other two cases can be proved similarly.

(ii)(a) Let $k = 2$. The the set of representatives is Rep $= \{[\pm 2 \quad 1]\}$ and $M = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$. Here $[-2 \ \ 1]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$.

Panda and Ray [11] defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \tag{15}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (15), we get

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \tag{16}$$

Let b_n denote the n^{th} cobalancing number. Then from (16), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus $\sqrt{8b_n^2 + 8b_n + 1}$ is an integer which is called n^{th} Lucas-cobalancing number and is denoted by c_n, that is, $c_n = \sqrt{8b_n^2 + 8b_n + 1}$. Recall that Pell numbers are the numbers given by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Since

$$[-2 \ \ 1] \begin{bmatrix} C_n & B_n \\ 8B_n & C_n \end{bmatrix} = [-2C_n + 8B_n \quad -2B_n + C_n]$$

and since $-C_n + 4B_n = c_n$ and $-2B_n + C_n = P_{2n-1}$, the set of all integer solutions is $\Omega = \{(x_n, y_n)\}$, where $x_n = 2c_n$ and $y_n = P_{2n-1}$ for $n \geq 1$.

(ii)(b) Let $k = 4$. Then the set of representatives is Rep $= \{[\pm 2 \quad 1], [\pm 8 \quad 2]\}$ and $M = \begin{bmatrix} 9 & 2 \\ 40 & 9 \end{bmatrix}$. Here $[2 \quad 1]M^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$, $[-2 \quad 1]M^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$ and $[8 \quad 2]M^n$ generates all integer solutions (x_{3n+2}, y_{3n+2}) for $n \geq 0$ ($[-8 \quad 2]M^n$ generates all integer solutions (x_{3n-1}, y_{3n-1}) for $n \geq 1$). Thus the set of all integer solutions is $\Omega = \{(x_{3n+1}, y_{3n+1}), (x_{3n+2}, y_{3n+2}), (x_{3n}, y_{3n})\}$, where $(x_{3n+1}, y_{3n+1}) = (2R+S, 2T+U)$ for $n \geq 0$, $(x_{3n+2}, y_{3n+2}) = (8R+2S, 8T+2U)$ for $n \geq 0$ and $(x_{3n}, y_{3n}) = (-2R+S, -2T+U)$ for $n \geq 1$. The others are similar. ∎

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