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Some Generalized Growth Properties of Composite Entire Functions

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Abstract. In this paper we wish to prove some results relating to the growth rates of composition of two entire functions with their corresponding left and right factors on the basis of their generalized order (α, β) and generalized type (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords: Entire function; Growth; Composition; Generalized order (α, β); Generalized type (α, β); Generalized index-pair (α, β).

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are defined as $M_f(r) = \max_{|z|=r} |f(z)|$ and $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$ respectively.

We use the standard notations and definitions of the theory of entire functions which are available in [15] and [17], and therefore we do not explain those in details. For $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Now considering this, let us recall that Juneja et al. [7] defined the (p,q)-th order and (p,q)-th lower order of an entire function respectively, as follows:

Definition 1.1. [7] *The* (p,q)*-th order and* (p,q)*-th lower order of an entire function* f *are defined as:*

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p *and* q *always denote positive integers with* $p \geq q$.

The function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Extending the notion of (p,q)-th order, Shen et al. [11] introduced the new concept of $[p,q]$-φ order of an entire function where $p \geq q$. Later on, combining the definition of (p,q)-th order and $[p,q]$-φ order, Biswas [1] redefined the (p,q)-th order of an entire function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f. For details about generalized order one may see [9]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

In this connection we just recall the following definition where we will give a minor modification to the original definition (see e.g. [7]):

Definition 1.2. *An entire function* f *is said to have index-pair* (p,q) *if* $b < \rho^{(p,q)}(f) < +\infty$ *and* $\rho^{(p-1,q-1)}(f)$ *is not a nonzero finite number, where* $b = 1$ *if* $p = q$ *and* $b = 0$ *for otherwise. Moreover if* $0 < \rho^{(p,q)}(f) < +\infty$*, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \quad . \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < +\infty$*, one can easily verify that*

$$\begin{cases} \lambda^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \quad . \end{cases}$$

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0\subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0.

Considering this, the value

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)}\ (\alpha\in L,\beta\in L)$$

is called [12] generalized order (α,β) of an entire function f. For details about generalized order (α,β) one may see [12]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications, Biswas et al. [3] rewrite the definition of the generalized order (α,β) of entire function in the following way after giving a minor modification to the original definition (see, e.g. [12]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [4]:

Definition 1.3. [3] *The generalized order (α,β) and generalized lower order (α,β) of an entire function f are defined as:*

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}\ \text{and}\ \lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}.$$

Since $\mu_f(r)\leq M_f(r)\leq\frac{R}{R-r}\mu_f(R)$ for $0\leq r<R$ {cf. [13] }, so it is easy to see that

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\mu_f(r))}{\beta(r)}\ \text{and}\ \lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\mu_f(r))}{\beta(r)}.$$

The function f is said to be of regular generalized (α,β) growth when generalized order (α,β) and generalized lower order (α,β) of f are the same. Functions which are not of regular generalized (α,β) growth are said to be of irregular generalized (α,β) growth.

Definition 1.1 is a special case of Definition 1.3 for $\alpha(r)=\log^{[p]}r$ and $\beta(r)=\log^{[q]}r$.

In this connection we also introduce the following definition which will be needed in the sequel:

Definition 1.4. *An entire function f is said to have generalized index-pair (α,β) if $b<\rho_{(\alpha,\beta)}[f]<+\infty$ and $\rho_{(\exp\alpha,\exp\beta)}[f]$ is not a non-zero finite number, where*

$b=1$ if $\alpha=\beta$ and $b=0$ for otherwise. Moreover if $0<\rho_{(\alpha,\beta)}[f]<+\infty$, then for any $\gamma_1\in L$ and $\gamma_1(r)\neq r$

$$\begin{cases}\rho_{(\gamma_1(\alpha),\beta)}[f]=+\infty \quad \text{when } \gamma_1(\alpha)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho\beta(r))}{\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f],\\ \rho_{(\gamma_1(\alpha),\beta)}[f]=0 \quad \text{when } \gamma_1(\alpha)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\rho_1\beta(r))}{\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f],\\ \rho_{(\alpha,\gamma_1(\beta))}[f]=+\infty \quad \text{when } \gamma_1(\beta)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\rho\beta(r)}{\gamma_1(\beta(r))}=+\infty \text{ for any } \rho<\rho_{(\alpha,\beta)}[f],\\ \rho_{(\alpha,\gamma_1(\beta))}[f]=0 \quad \text{when } \gamma_1(\beta)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\rho_1\beta(r)}{\gamma_1(\beta(r))}=0 \text{ for any } \rho_1>\rho_{(\alpha,\beta)}[f],\\ \rho_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]=1 \quad \text{when } \gamma_1\in L^0.\end{cases}$$

Similarly for $0<\lambda_{(\alpha,\beta)}[f]<+\infty$, one can easily verify that

$$\begin{cases}\lambda_{(\gamma_1(\alpha),\beta)}[f]=+\infty \quad \text{when } \gamma_1(\alpha)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\lambda\beta(r))}{\beta(r))}=+\infty \text{ for any } \lambda<\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\gamma_1(\alpha),\beta)}[f]=0 \quad \text{when } \gamma_1(\alpha)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\gamma_1(\lambda_1\beta(r))}{\beta(r))}=0 \text{ for any } \lambda_1>\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\alpha,\gamma_1(\beta))}[f]=+\infty \quad \text{when } \gamma_1(\beta)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\lambda\beta(r)}{\gamma_1(\beta(r))}=+\infty \text{ for any } \lambda<\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\alpha,\gamma_1(\beta))}[f]=0 \quad \text{when } \gamma_1(\beta)\in L^0\\ \text{and } \lim\limits_{r\to+\infty}\frac{\lambda_1\beta(r)}{\gamma_1(\beta(r))}=0 \text{ for any } \lambda_1>\lambda_{(\alpha,\beta)}[f],\\ \lambda_{(\gamma_1(\alpha),\gamma_1(\beta))}[f]=1 \quad \text{when } \gamma_1\in L^0.\end{cases}$$

Now in order to refine the growth scale namely the generalized order (α,β), Biswas et al. [2] have introduced the definitions of another growth indicators, called generalized type (α,β) and generalized lower type (α,β) respectively of an entire function which are as follows:

Definition 1.5. [2] *The generalized type (α,β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α,β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α,β) $\left(0<\rho_{(\alpha,\beta)}[f]<+\infty\right)$ are defined as:*

$$\sigma_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\rho_{(\alpha,\beta)}[f]}},$$
$$\overline{\sigma}_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\rho_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0\leq\overline{\sigma}_{(\alpha,\beta)}[f]\leq\sigma_{(\alpha,\beta)}[f]\leq+\infty$.

Analogously, to determine the relative growth of two entire functions having same non zero finite generalized lower order (α, β), Biswas et al. [2] have introduced the definitions of generalized weak type (α,β) and generalized upper weak type (α,β) of an entire function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 1.6. [2] *The generalized upper weak type (α,β) denoted by $\tau_{(\alpha,\beta)}[f]$ and generalized weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α, β) $\left(0 < \lambda_{(\alpha,\beta)}[f] < +\infty\right)$ are defined as:*

$$\tau_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\lambda_{(\alpha,\beta)}[f]}},$$
$$\overline{\tau}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \tau_{(\alpha,\beta)}[f] \leq +\infty$.

Using the characteristic of entire functions many researchers have already contributed their works in the different directions of the present literature (see [6, 8, 10]). For any two entire functions f and g the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to +\infty$ and $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to +\infty$ are called the growth of f with respect to g in terms of their maximum modulus and the maximum term respectively. Actually the studies of the growths of composite entire functions in the light of their generalized order (α, β) and generalized type (α, β) after improving some results of [14] and [16] are the prime concern of this paper.

2. Main Results

First of all we present a lemma which will be needed in the sequel.

Lemma 2.1. [5] *Let f and g be any two entire functions with $g(0) = 0$. Also let B satisfy $0 < B < 1$ and $c(B) = \frac{(1-B)^2}{4B}$. Then for all sufficiently large values of r,*

$$M_f\left(c\left(B\right)M_g\left(Br\right)\right) \leq M_{f\circ g}\left(r\right) \leq M_f\left(M_g\left(r\right)\right).$$

In addition if $B = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}\left(r\right) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right).$$

Now we present the main results of this paper.

Theorem 2.2. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively. Then*

(i) *the generalized index-pair of $f\circ g$ is (α_1,β_2) when $\beta_1(r)=\alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$. Also*

 (a) $\lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_1,\beta_2)}[f\circ g] \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]>0$,

 (b) $\lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_1,\beta_2)}[f\circ g] \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g]>0$.

(ii) *the generalized index-pair of $f\circ g$ is $(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))$ when $\beta_1(\alpha_2^{-1}(r))\in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$. Also*

 (a) $\lambda_{(\alpha_1,\beta_1)}[f]\le\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]\leqslant\rho_{(\alpha_1,\beta_1)}[f]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]>0$.

 (b) $\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\rho_{(\alpha_1,\beta_1)}[f]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g]>0$.

(i) *the generalized index-pair of $f\circ g$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r))\in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$. Also*

 (a) $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]=\rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]>0$.

 (b) $\lambda_{(\alpha_2,\beta_2)}[g]\le\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]\leqslant\rho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g]>0$.

Proof. In view of the first part of Lemma 2.1, it follows for all sufficiently large values of r that

$$\alpha_1(M_{f\circ g}(r))\ge(1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big) \tag{1}$$

and also for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_{f\circ g}(r)\right)\ge(1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right))\beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big). \tag{2}$$

Similarly, in view of the second part of Lemma 2.1, we have for all sufficiently large values of r that

$$\alpha_1\left(M_{f\circ g}(r)\right)\leqslant\left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_1\left(M_g(r)\right). \tag{3}$$

Now the following two cases may arise:

Case I. Let $\beta_1(r)=\alpha_2(r)$.

Now we have from (3) for all sufficiently large values of r that

$$\alpha_1\left(M_{f\circ g}(r)\right)\leqslant\left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_2(r)}\leqslant\rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]. \tag{4}$$

Also from (1), we obtain for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_{f\circ g}(r)\right)\ge(1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\left(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_2(r)}\ge\lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]. \tag{5}$$

Moreover, we have from (2) for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_{f\circ g}(r)\right) \geq (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\left(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2(r)$$

$$i.e., \ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_2(r)} \geq \rho_{(\alpha_1,\beta_1)}[f]\lambda_{(\alpha_2,\beta_2)}[g]. \tag{6}$$

Therefore from (4) and (5), we get for $\lambda_{(\alpha_1,\beta_1)}[f]>0$ that

$$\lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]$$

$$i.e., \ \lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_1,\beta_2)}[f\circ g] \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]. \tag{7}$$

Likewise, from (4) and (6), we obtain for $\lambda_{(\alpha_2,\beta_2)}[g]>0$ that

$$\rho_{(\alpha_1,\beta_1)}[f]\lambda_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_2(r)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]$$

$$i.e., \ \rho_{(\alpha_1,\beta_1)}[f]\lambda_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_1,\beta_2)}[f\circ g] \leqslant \rho_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]. \tag{8}$$

Also from (7) and (8) one can easily verify that

$$\rho_{(\alpha_1(\gamma_1^{-1}),\beta_2)}[f\circ g]=\infty \quad \text{when } \alpha_1(\gamma_1^{-1})\in L^0 \text{ and } \lim_{r\to+\infty}\tfrac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)}=+\infty,$$

$$\rho_{(\alpha_1,\beta_2(\gamma_1^{-1}))}[f\circ g]=0 \quad \text{when } \beta_2\left(\gamma_1^{-1}\right)\in L^0 \text{ and } \lim_{r\to+\infty}\tfrac{\beta_2(\gamma_1^{-1}(r))}{\beta_2(r)}=+\infty,$$

$$\rho_{(\alpha_1(\gamma_1),\beta_2(\gamma_1))}[f\circ g]=1 \quad \text{when } \lim_{r\to+\infty}\tfrac{\alpha_1(\gamma_1(r))}{\alpha_1(r)}=0 \text{ and } \lim_{r\to+\infty}\tfrac{\beta_2(\gamma_1(r))}{\beta_2(r)}=0.$$

Therefore we obtain that the generalized index-pair of $f\circ g$ is (α_1,β_2) when $\beta_1(r)=\alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and thus the first part of the theorem is established.

Case II. Let $\beta_1(\alpha_2^{-1}(r))\in L^0$.

Now we obtain from (3) for all sufficiently large values of r that

$$\alpha_1\left(M_{f\circ g}(r)\right) \leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\alpha_2\left(M_g(r)\right)\right)\right)$$

$$i.e., \ \alpha_1\left(M_{f\circ g}(r)\right) \leqslant \left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2(r)\right)\right)$$

$$i.e., \ \alpha_1\left(M_{f\circ g}(r)\right) \leqslant (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)$$

$$i.e., \ \lim_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]. \tag{9}$$

Also from (1), we have for a sequence of values of r tending to infinity that

$$\alpha_1\left(M_{f\circ g}(r)\right) \geq (1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_1\Big(\alpha_2^{-1}\Big(\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2\Big(\frac{r}{2}\Big)\Big)\Big)$$

$$i.e.,\ \alpha_1\left(M_{f\circ g}(r)\right) \geq (1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \geq \lambda_{(\alpha_1,\beta_1)}[f]. \tag{10}$$

Further, we get from (2) for a sequence of values of r tending to infinity that

$$\begin{aligned}&\alpha_1\left(M_{f\circ g}(r)\right)\\ &\geq (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_1\Big(\alpha_2^{-1}\Big(\left(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2\Big(\frac{r}{2}\Big)\Big)\Big)\end{aligned}$$

$$i.e.,\ \alpha_1\left(M_{f\circ g}(r)\right) \geq (1+o(1))\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \geq \rho_{(\alpha_1,\beta_1)}[f]. \tag{11}$$

Therefore from (9) and (10), we get for $\lambda_{(\alpha_1,\beta_1)}[f]>0$ that

$$\begin{aligned}\lambda_{(\alpha_1,\beta_1)}[f] &\leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]\\ i.e.,\ \lambda_{(\alpha_1,\beta_1)}[f] &\leq \rho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}[f\circ g] \leqslant \rho_{(\alpha_1,\beta_1)}[f].\end{aligned} \tag{12}$$

Likewise, from (9) and (11) we get for $\lambda_{(\alpha_2,\beta_2)}[g]>0$ that

$$\rho_{(\alpha_1,\beta_1)}[f] \leq \limsup_{r\to+\infty}\frac{\alpha_1\left(M_{f\circ g}(r)\right)}{\beta_1\left(\alpha_2^{-1}\left(\beta_2(r)\right)\right)} \leqslant \rho_{(\alpha_1,\beta_1)}[f]$$

$$i.e.,\ \rho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]. \tag{13}$$

Further from (12) and (13) one can easily verify that

$$\begin{aligned}&\rho_{(\alpha_1(\gamma_1^{-1}),\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\infty\\ &\qquad\text{when}\ \ \alpha_1(\gamma_1^{-1})\in L^0\ \text{and}\ \lim_{r\to+\infty}\tfrac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)}=+\infty,\\ &\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}))))}[f\circ g]=0,\\ &\qquad\text{when}\ \ \beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1})))\in L^0\ \text{and}\ \lim_{r\to+\infty}\tfrac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))}=+\infty,\\ &\rho_{(\alpha_1(\gamma_1),\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1))))}[f\circ g]=1\\ &\qquad\text{when}\ \ \lim_{r\to+\infty}\tfrac{\alpha_1(\gamma_1(r))}{\alpha_1(r)}=0\ \text{and}\ \lim_{r\to+\infty}\tfrac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))}=0.\end{aligned}$$

Therefore we get that the generalized index-pair of $f\circ g$ is $\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)$ when $\beta_1\left(\alpha_2^{-1}(r)\right)\in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and thus the second part of the theorem follows.

Case III. Let $\alpha_2(\beta_1^{-1}(r))\in L^0$.

Then we obtain from (3) for all sufficiently large values of r that

$$\begin{aligned}
&\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \le \alpha_2\left(\beta_1^{-1}\left(\left(\rho_{(\alpha_1,\beta_1)}\left[f\right]+\varepsilon\right)\beta_1\left(M_g\left(r\right)\right)\right)\right)\\
i.e.,\ &\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \leqslant (1+o(1))\,\alpha_2\left(M_g\left(r\right)\right)\\
i.e.,\ &\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \leqslant (1+o(1))\left(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2\left(r\right)
\end{aligned}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right)}{\beta_2\left(r\right)} \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \tag{14}$$

Also from (1) we have for a sequence of values of r tending to infinity that

$$\begin{aligned}
&\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \ge (1+o(1))\,\beta_2\Big(M_g\Big(\frac{r}{2}\Big)\Big)\\
i.e.,\ &\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \ge (1+o(1))\left(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2\left(r\right)
\end{aligned}$$

$$\limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right)}{\beta_2\left(r\right)} \ge \rho_{(\alpha_2,\beta_2)}[g]. \tag{15}$$

Similarly, we get from (2) for a sequence of values of r tending to infinity that

$$\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right) \ge (1+o(1))\left(\lambda_{(\alpha_2,\beta_2)}\left[g\right]-\varepsilon\right)\beta_2\left(r\right)$$

$$\limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right)}{\beta_2\left(r\right)} \ge \lambda_{(\alpha_2,\beta_2)}\left[g\right]. \tag{16}$$

Therefore from (14) and (15), we obtain for $\lambda_{(\alpha_1,\beta_1)}[f]>0$ that

$$\rho_{(\alpha_2,\beta_2)}\left[g\right] \le \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right)}{\beta_2\left(r\right)} \leqslant \rho_{(\alpha_2,\beta_2)}\left[g\right]$$

$$i.e.,\ \rho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}\left[f\circ g\right] = \rho_{(\alpha_2,\beta_2)}\left[g\right]. \tag{17}$$

Similarly, from (14) and (16) we get for $\lambda_{(\alpha_2,\beta_2)}\left[g\right]>0$ that

$$\begin{aligned}
&\lambda_{(\alpha_2,\beta_2)}[g] \le \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(M_{f\circ g}\left(r\right)\right)\right)\right)}{\beta_2\left(r\right)} \leqslant \rho_{(\alpha_2,\beta_2)}[g]\\
i.e.,\ &\lambda_{(\alpha_2,\beta_2)}[g] \le \rho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}\left[f\circ g\right] \leqslant \rho_{(\alpha_2,\beta_2)}[g].
\end{aligned} \tag{18}$$

So from (17) and (18) one can easily verify that

$$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}))),\beta_2)}[f\circ g] = \infty,$$
$$\text{when }\ \alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}))) \in L^0 \text{ and } \lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))} = +\infty,$$
$$\rho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\left(\gamma_1^{-1}\right)\right)}[f\circ g] = 0,$$
$$\text{when }\ \beta_2\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim_{r\to+\infty}\frac{\beta_2\left(\gamma_1^{-1}(r)\right)}{\beta_2(r)} = +\infty$$
$$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1))),\beta_2(\gamma_1))}[f\circ g] = 1,$$
$$\text{when }\ \lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))} = 0 \text{ and } \lim_{r\to+\infty}\frac{\beta_2(\gamma_1(r))}{\beta_2(r)} = 0.$$

So we obtain that the generalized index-pair of $f\circ g$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the third part of the theorem is established. ∎

Theorem 2.3. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively.*

(i) *If $\beta_1(r) = \alpha_2(r)$, $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\begin{aligned}\lambda_{(\alpha_1,\beta_1)}[f]\lambda_{(\alpha_2,\beta_2)}[g] &\le \lambda_{(\alpha_1,\beta_2)}[f\circ g]\\ &\le \min\left\{\rho_{(\alpha_1,\beta_1)}[f]\lambda_{(\alpha_2,\beta_2)}[g], \lambda_{(\alpha_1,\beta_1)}[f]\rho_{(\alpha_2,\beta_2)}[g]\right\}.\end{aligned}$$

(ii) *If $\beta_1\left(\alpha_2^{-1}(r)\right) \in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] = \lambda_{(\alpha_1,\beta_1)}[f].$$

(iii) *If $\alpha_2\left(\beta_1^{-1}(r)\right) \in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[g].$$

In the line of Theorem 2.2 one can easily deduce the conclusion of Theorem 2.3 and so its proof is omitted.

Theorem 2.4. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively.*

(i) *If $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\begin{aligned}&\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\\ &\le \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\\ &\le \min\left\{\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\}\\ &\le \max\left\{\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\\ &\le \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.\end{aligned}$$

(ii) *If* $\beta_1(\alpha_2^{-1}(r)) \in L^0$ *and either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*, then*

$$\begin{aligned}
&\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]} \\
&\leq \liminf_{r \to +\infty} \frac{\alpha_1(M_{f \circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))} \\
&\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \\
&\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \\
&\leq \limsup_{r \to +\infty} \frac{\alpha_1(M_{f \circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))} \\
&\leq \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.
\end{aligned}$$

(iii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$ *and either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*, then*

$$\begin{aligned}
&\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]} \\
&\leq \liminf_{r \to +\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f \circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \\
&\leq \min\left\{\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \\
&\leq \max\left\{\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}, \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}\right\} \\
&\leq \limsup_{r \to +\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f \circ g}(r))))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \\
&\leq \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.
\end{aligned}$$

Proof. Let $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Then in view of Theorem 2.2, the generalized index-pair of $f \circ g$ is (α_1,β_2).

Now from the definition of $\rho_{(\alpha_1,\beta_1)}[f]$ and $\lambda_{(\alpha_1,\beta_2)}[f \circ g]$, we have for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\alpha_1(M_{f \circ g}(r)) \geqslant \left(\lambda_{(\alpha_1,\beta_2)}[f \circ g] - \varepsilon\right) \beta_2(r), \tag{19}$$

$$\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))) \leq \left(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon\right) \beta_2(r). \tag{20}$$

Now from (19) and (20), it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f \circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))} \geqslant \frac{\left(\lambda_{(\alpha_1,\beta_2)}[f \circ g] - \varepsilon\right) \beta_2(r)}{\left(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon\right) \beta_2(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geqslant\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{21}$$

Again we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r))\leq\left(\lambda_{(\alpha_1,\beta_2)}[f\circ g]+\varepsilon\right)\beta_2(r) \tag{22}$$

and for all sufficiently large positive numbers of r that

$$\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))\geqslant\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_2(r). \tag{23}$$

Combining (22) and (23), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\left(\lambda_{(\alpha_1,\beta_2)}[f\circ g]+\varepsilon\right)\beta_2(r)}{\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_2(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{24}$$

Also for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))\leq\left(\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_2(r). \tag{25}$$

Now from (19) and (25), we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\left(\lambda_{(\alpha_1,\beta_2)}[f\circ g]-\varepsilon\right)\beta_2(r)}{\left(\lambda_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_2(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{26}$$

Also we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(M_{f\circ g}(r))\leq\left(\rho_{(\alpha_1,\beta_2)}[f\circ g]+\varepsilon\right)\beta_2(r). \tag{27}$$

Now it follows from (23) and (27) for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\left(\rho_{(\alpha_1,\beta_2)}[f\circ g]+\varepsilon\right)\beta_2(r)}{\left(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_2(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{28}$$

Further from the definition of $\rho_{(\alpha_1,\beta_1)}[f]$, we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))\geqslant\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_2(r). \tag{29}$$

Now from (27) and (29), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\left(\rho_{(\alpha_1,\beta_2)}[f\circ g]+\varepsilon\right)\beta_2(r)}{\left(\rho_{(\alpha_1,\beta_1)}[f]-\varepsilon\right)\beta_2(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{30}$$

Again we obtain for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_{f\circ g}(r))\geqslant\left(\rho_{(\alpha_1,\beta_2)}[f\circ g]-\varepsilon\right)\beta_2(r). \tag{31}$$

So combining (20) and (31), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geqslant\frac{\left(\rho_{(\alpha_1,\beta_2)}[f\circ g]-\varepsilon\right)\beta_2(r)}{\left(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon\right)\beta_2(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))}\geqslant\frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{32}$$

Thus the first part of the theorem follows from (21),(24),(26), (28), (30) and (32).

Analogously, the second and third part of the theorem can be derived in a like manner. ■

The following theorem can be proved in the line of Theorem 2.4 and so its proof is omitted.

Theorem 2.5. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively.*

(i) *If* $\beta_1(r) = \alpha_2(r)$ *and either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*, then*

$$\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]} \leq \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(r))}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(r))} \leq \frac{\rho_{(\alpha_1,\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

(ii) *If* $\beta_1(\alpha_2^{-1}(r)) \in L^0$ *and either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*, then*

$$\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}$$
$$\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_{f\circ g}(r))}{\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))}$$
$$\leq \frac{\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

(iii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$ *and either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*, then*

$$\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}$$
$$\leq \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_2(M_g(r))}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}, \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\rho_{(\alpha_2,\beta_2)}[g]}\right\}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r))))}{\alpha_2(M_g(r))}$$
$$\leq \frac{\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 2.6. The same results of Theorems 2.4 and 2.5 in terms of maximum terms of entire functions can also be deduced with the help of Definition 1.3.

The proofs of the following four theorems can be carried out as of Theorem 2.4, therefore we omit the details.

Theorem 2.7. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively.*

(i) *If $\beta_1(r)=\alpha_2(r)$, either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$, $0<\overline{\sigma}_{(\alpha_1,\beta_2)}[f\circ g]\le\sigma_{(\alpha_1,\beta_2)}[f\circ g]<\infty$, $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]\le\sigma_{(\alpha_1,\beta_1)}[f]<\infty$ and $\rho_{(\alpha_1,\beta_2)}[f\circ g]=\rho_{(\alpha_1,\beta_1)}[f]$, then*

$$\begin{aligned}
&\frac{\overline{\sigma}_{(\alpha_1,\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\\
&\le\liminf_{r\to+\infty}\frac{\exp\left(\alpha_1(M_{f\circ g}(r))\right)}{\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))\right)}\\
&\le\min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_1,\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le\max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_1,\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le\limsup_{r\to+\infty}\frac{\exp\left(\alpha_1(M_{f\circ g}(r))\right)}{\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))\right)}\\
&\le\frac{\sigma_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.
\end{aligned}$$

(ii) *If $\beta_1(\alpha_2^{-1}(r))\in L^0$, either $\lambda_{(\alpha_1,\beta_1)}[f]>0$ or $\lambda_{(\alpha_2,\beta_2)}[g]>0$, $0<\overline{\sigma}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]\le\sigma_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]<\infty$, $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]\le\sigma_{(\alpha_1,\beta_1)}[f]<\infty$ and $\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\rho_{(\alpha_1,\beta_1)}[f]$, then*

$$\begin{aligned}
&\frac{\overline{\sigma}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\\
&\le\liminf_{r\to+\infty}\frac{\exp\left(\alpha_1(M_{f\circ g}(r))\right)}{\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\right)}\\
&\le\min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le\max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le\limsup_{r\to+\infty}\frac{\exp\left(\alpha_1(M_{f\circ g}(r))\right)}{\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\right)}\\
&\le\frac{\sigma_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.
\end{aligned}$$

(iii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, $0 < \overline{\sigma}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] \le \sigma_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] < \infty$, $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]$, *then*

$$\begin{aligned}
&\frac{\overline{\sigma}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\\
&\le \liminf_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\\
&\le \min\left\{\frac{\overline{\sigma}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le \max\left\{\frac{\overline{\sigma}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]},\frac{\sigma_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\sigma_{(\alpha_1,\beta_1)}[f]}\right\}\\
&\le \limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\\
&\le \frac{\sigma_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.
\end{aligned}$$

Remark 2.8. In Theorem 2.7 (i), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \le \sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f\circ g] = \rho_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (i) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$", "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" respectively.

In Theorem 2.7 (ii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \le \sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] = \rho_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (ii) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$", "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r))))))\right)$" respectively.

In Theorem 2.7 (iii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \le \sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] = \rho_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (iii) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$", "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" respectively.

Remark 2.9. In Theorem 2.7 (i), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f\circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] \le \tau_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f\circ g] = \lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (i) remains valid with

"$\tau_{(\alpha_1,\beta_1)}[f]$", "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))\right)$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" respectively.

In Theorem 2.7 (ii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \leq \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] \leq \tau_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7(ii) remains valid with "$\tau_{(\alpha_1,\beta_1)}[f]$", "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\right)$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\right)$" respectively.

In Theorem 2.7 (iii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \leq \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] \leq \tau_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7(iii) remains valid with "$\tau_{(\alpha_1,\beta_1)}[f]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" respectively.

Remark 2.10. In Theorem 2.7 (i), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \leq \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \leq \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_1,\beta_2)}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (i) remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" respectively.

In Theorem 2.7 (ii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \leq \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \leq \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7(ii) remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\left(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\right)$" respectively.

In Theorem 2.7 (iii), if we replace the conditions "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \leq \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" by "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \leq \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\rho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.7 (iii) remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" instead of "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" respectively.

Analogously one may formulate the following theorem without its proof.

Theorem 2.11. *Let f and g be any two entire functions with generalized index-pairs (α_1,β_1) and (α_2,β_2) respectively.*

(i) *If $\beta_1(r) = \alpha_2(r)$, either $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, $0 < \overline{\tau}_{(\alpha_1,\beta_2)}[f \circ g] \leq \tau_{(\alpha_1,\beta_2)}[f \circ g] < \infty$, $0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \leq \tau_{(\alpha_2,\beta_2)}[g] < \infty$ and $\lambda_{(\alpha_1,\beta_2)}[f \circ g]$*

$= \lambda_{(\alpha_2,\beta_2)}[g]$ *then*

$$\begin{aligned}
&\frac{\overline{\tau}_{(\alpha_1,\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\\
&\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(M_{f\circ g}(r)))}{\exp(\alpha_2(M_g(r)))}\\
&\le \min\left\{\frac{\overline{\tau}_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_1,\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\right\}\\
&\le \max\left\{\frac{\overline{\tau}_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_1,\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\right\}\\
&\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(M_{f\circ g}(r)))}{\exp(\alpha_2(M_g(r)))}\\
&\le \frac{\tau_{(\alpha_1,\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}.
\end{aligned}$$

(ii) *If* $\beta_1(\alpha_2^{-1}(r)) \in L^0$, *either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, $0 < \overline{\tau}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] \le \tau_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] < \infty$, $0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$ *then*

$$\begin{aligned}
&\frac{\overline{\tau}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\\
&\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(M_{f\circ g}(r)))}{\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))}\\
&\le \min\left\{\frac{\overline{\tau}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\right\}\\
&\le \max\left\{\frac{\overline{\tau}_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\right\}\\
&\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(M_{f\circ g}(r)))}{\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))}\\
&\le \frac{\tau_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}.
\end{aligned}$$

(iii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *either* $\lambda_{(\alpha_1,\beta_1)}[f] > 0$ *or* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, $0 < \overline{\tau}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] \le \tau_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] < \infty$, $0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$ *and* $\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$ *then*

$$\begin{aligned}
&\frac{\overline{\tau}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\\
&\le \liminf_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))))}{\exp(\alpha_2(M_g(r)))}
\end{aligned}$$

$$\le \min\Big\{\frac{\overline{\tau}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\Big\}$$
$$\le \max\Big\{\frac{\overline{\tau}_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]},\frac{\tau_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\tau_{(\alpha_2,\beta_2)}[g]}\Big\}$$
$$\le \limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_{f\circ g}(r)))))}{\exp(\alpha_2(M_g(r)))}$$
$$\le \frac{\tau_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]}{\overline{\tau}_{(\alpha_2,\beta_2)}[g]}.$$

Remark 2.12. In Theorem 2.11 (i), if we replace the conditions "$0<\overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$\lambda_{(\alpha_1,\beta_2)}[f\circ g]=\lambda_{(\alpha_2,\beta_2)}[g]$" by "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]\le \tau_{(\alpha_1,\beta_1)}[f]<\infty$" and "$\lambda_{(\alpha_1,\beta_2)}[f\circ g]=\lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (i) remains valid with "$\tau_{(\alpha_1,\beta_1)}[f]$", "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\big(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))\big)$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" respectively.

In Theorem 2.11 (ii), if we replace the conditions "$0<\overline{\tau}_{(\alpha_2,\beta_2)}[g]\le\tau_{(\alpha_2,\beta_2)}[g] <\infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\lambda_{(\alpha_2,\beta_2)}[g]$" by "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]\le\tau_{(\alpha_1,\beta_1)}[f] <\infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (ii) remains valid with "$\tau_{(\alpha_1,\beta_1)}[f]$", "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\big(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\big)$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))$" respectively.

In Theorem 2.11 (iii), if we replace the conditions "$0<\overline{\tau}_{(\alpha_2,\beta_2)}[g]\le\tau_{(\alpha_2,\beta_2)}[g] <\infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]=\lambda_{(\alpha_2,\beta_2)}[g]$" by "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]\le\tau_{(\alpha_1,\beta_1)}[f] <\infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f\circ g]=\lambda_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (iii) remains valid with "$\tau_{(\alpha_1,\beta_1)}[f]$", "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" respectively.

Remark 2.13. In Theorem 2.11 (i), if we replace the conditions "$0<\overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$\lambda_{(\alpha_1,\beta_2)}[f\circ g]=\lambda_{(\alpha_2,\beta_2)}[g]$" by "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]\le \sigma_{(\alpha_1,\beta_1)}[f]<\infty$" and "$\lambda_{(\alpha_1,\beta_2)}[f\circ g]=\rho_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (i) remains valid with "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\big(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))\big)$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" respectively.

In Theorem 2.11 (ii), if we replace the conditions "$0<\overline{\tau}_{(\alpha_2,\beta_2)}[g]\le\tau_{(\alpha_2,\beta_2)}[g] <\infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\lambda_{(\alpha_2,\beta_2)}[g]$" by "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]\le\sigma_{(\alpha_1,\beta_1)}[f] <\infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f\circ g]=\rho_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (ii) remains valid with "$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp\big(\alpha_1(M_f(\beta_1^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))\big)$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))$" respectively.

In Theorem 2.11 (iii), if we replace the conditions "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" by "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (iii) remains valid with ""$\sigma_{(\alpha_1,\beta_1)}[f]$", "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" and "$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))$"instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\exp(\alpha_2(M_g(r)))$" respectively.

Remark 2.14. In Theorem 2.11 (i), if we replace the conditions "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\lambda_{(\alpha_1,\beta_2)}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" by "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \le \sigma_{(\alpha_2,\beta_2)}[g] < \infty$ " and "$\lambda_{(\alpha_1,\beta_2)}[f \circ g] = \rho_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (i) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" respectively.

In Theorem 2.11 (ii), if we replace the conditions "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" by "$0 < \overline{\sigma}_{(\alpha_2,\beta_2)}[g] \le \sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f \circ g] = \rho_{(\alpha_2,\beta_2)}[g]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (ii) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" respectively.

In Theorem 2.11 (iii), if we replace the conditions "$0 < \overline{\tau}_{(\alpha_2,\beta_2)}[g] \le \tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \lambda_{(\alpha_2,\beta_2)}[g]$" by "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] \le \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" and "$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f \circ g] = \rho_{(\alpha_1,\beta_1)}[f]$" respectively and other conditions remain the same, then the conclusion of Theorem 2.11 (iii) remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$", and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" respectively.

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