Southeast Asian Bulletin of Mathematics (2023) 47: 35–48

Southeast Asian Bulletin of Mathematics


# Properties of Modules and Rings Satisfying Certain Chain Conditions

Avanish Kumar Chaturvedi and Surya Prakash*
Department of Mathematics, University of Allahabad, Prayagraj-211002, India
Email: akchaturvedi.math@gmail.com; akc99@rediffmail.com;
suryaprakash.maths@gmail.com

Received 18 May 2020
Accepted 20 December 2020

Communicated by Nguyen Van Sanh

**AMS Mathematics Subject Classification(2020):** 16P20, 16P40, 16P60, 16P70, 16D50, 16D60

**Abstract.** In 2016, Facchini and Nazemian defined the notion of iso-Artinian and iso-Noetherian modules and rings. We discuss some new properties of iso-Artinian and iso-Noetherian rings and modules. We also generalize the notion of iso-Artinian (iso-Noetherian) and introduced the notion of mono-Artinian (mono-Noetherian) rings and modules. We investigate several properties of mono-Artinian and mono-Noetherian modules and rings.

**Keywords:** Iso-Noetherian rings; Iso-Noetherian modules; Iso-Artinian rings; Iso-Artinan modules; Mono-Artinian rings; Mono-Noetherian rings.

## 1. Introduction

Some notions in the "Theory of Rings and Modules" made a deep impact on us. The one example of it is the notions of chain conditions, introduced by the great mathematicians Noether and Artin. We can realize the importance of these by the fact that many people are still working on these notions and their generalizations (for some examples, see [2, 12, 13, 15, 16, 17, 22, 23, 24, 25, 27, 28, 29], etc.). Therefore, during the period a rich theory has been developed. As an example of recent developments, the term iso-Noetherian (iso-Artinian) was coined by Facchini and Nazemian in [15]. They studied a class of modules (rings), where any chain is stationary in the sense of isomorphism (i.e. instead of equality of submodules as in the case of ACC and DCC). In [12], authors generalized these

*Corresponding Author

notions to epi-ACC (epi-DCC) and studied the class of modules in which the chain is stationary in the sense of epimorphism. In [23], we call these notions as epi-Noetherian (epi-Artinian) and studied various properties over these notions. In the present work, we continued our study of these modules. By the motivation, in the final section, we have introduced the idea of mono-Artinian and mono-Noetherian rings and modules.

In the Section 2, we discuss some properties of iso-Noetherian and iso-Artinian rings. In [6], authors defined the notion of virtually semisimple module. We show that every semiprime right iso-Artinian ring is generated by iso-retractable right ideals (Theorem 2.5 (4)). In general, iso-Noetherian modules need not be virtually semisimple and vice versa. We provide some examples. We know that the notions of iso-Noetherian and Noetherian are not equivalent in general. But, if we take virtually semisimple module then both notions are same; i.e. if $M$ is virtually semisimple then $M$ is iso-Noetherian if and only if $M$ is Noetherian (Proposition 2.6). We finish this section with the results involving some properties related to fe-module that is a module with finitely many essential submodules. In Theorem 2.9, we show that if $M$ is fe-module such that $soc(M)$ is iso-Noetherian, then $M$ is Noetherian.

In the Section 3, we define the notion of mono-Artinian and mono-Noetherian rings and modules that are dual to epi-Artinian and epi-Noetherian. Also, they generalize the notions of iso-Artinian and iso-Noetherian. We investigate some general properties of mono-Artinian (mono-Noetherian) rings and modules. We show the existance of mono-Artinian (mono-Noetherian) modules over hereditary ring. We show that a finite direct product of mono-Artinian (mono-Noetherian) rings is mono-Artinian (mono-Noetherian). But the direct sum of two mono-Artinian (mono-Noetherian) modules is not necessarily mono-Artinian (mono-Noetherian). Over a commutative ring $R$, if the sum of annihilator ideals of two mono-Artinian (mono-Noetherian) modules is the ring $R$, then their direct sum is the mono-Artinian (mono-Noetherian) module. If $R$ is a mono-Artinian ring, then we show that $R$ satisfies ACC on right (left) annihilator ideals, $Z(R_R)$ is nilpotent and every nonzero nil left (right) ideal of $R$ contains a nonzero nilpotent left (right) ideal.

Trivially, every iso-Artinian (iso-Noetherian) modules are examples of mono-Artinian (mono-Noetherian). But every mono-Artinian (mono-Noetherian) need not be iso-Artinian (iso-Noetherian). For example, every compressible module which is not iso-retractable is mono-Artinian but not iso-Artinian. We show the existance of mono-Artinian (mono-Noetherian) modules over a hereditary ring. We also prove that the direct sum of two $R$-modules $M$ and $K$ is mono-Artinian (mono-Noetherian) if $M$ is mono-Artinian (mono-Noetherian) and $K$ is projective iso-retractable or $K$ is iso-retractable and $M \oplus K$ is distributive. We find the structure of an essential ideal of a mono-Artinian ring. Further, we discuss some properties of semiprime mono-Artinian (mono-Noetherian) rings. We prove that a right mono-Artinian (mono-Noetherian) integral domain is a right ore domain.

The notations $N \leq M$, $N \leq_e M$ means $N$ is a submodule, an essential submodule of $M$, respectively and $I$-$soc(M)$, $I$-$rad(R)$, $u.dim(M)$ denotes iso-socle of $M$, iso-radical of $R$, uniform dimention of $M$, respectively. Unless otherwise stated, we assume rings as associative with unity and modules as unitary right modules. We refer [1, 20, 30] for undefined terms and notions.

## 2. Iso-Noetherian and Iso-Artinian Rings and Modules

In the following, we discuss structure of any essential submodule of an iso-Noetherian module in terms of uniform submodules.

**Theorem 2.1.** *Let $M$ be a nonzero iso-Noetherian $R$-module. Then the following statements hold:*

(1) *Every submodule of $M$ contains a uniform submodule.*

(2) *If $M$ is injective, then $M$ is a direct sum of finitely many indecomposable injective modules.*

*Proof.* (1). Since every submodule of an iso-Noetherian module is iso-Noetherian, it is sufficient to show that $M$ contains a uniform submodule. If $M$ is uniform, then nothing to prove. If not, $M$ contains a direct sum of two nonzero submodules, say $M = M_0 \supseteq M_1 \oplus M_1'$. If either of $M_1$ or $M_1'$ is uniform, then nothing to prove. If not, we repeat this argument for $M_1$. We get $M_2, M_3, M_4, \dots$ and a direct sum $M_1' \oplus M_2' \oplus M_3' \oplus \dots$. Since $M$ is iso-Noetherian, $u.dim(M) < \infty$ by [15, Proposition 5.1]. Due to the finite uniform dimension this process must stop after $k$ steps and the submodule $M_k$ is uniform.

(2). Since $M$ is iso-Noetherian, $u.dim(M)$ is finite. It follows from [20, Proposition 6.12] that $M$ is a direct sum of finite indecomposable injective modules. ∎

In [9, 8], first author defined the notion of iso-retractable modules and calls a module $M$ *iso-retractable* if every nonzero submodule of $M$ is isomorphic to $M$. In [15], authors call this notion by isosimple and in [6] virtually simple. In [14], authors defined the notion of $I$-$soc(M)$ as the sum of isosimple submodules of an $R$-module $M$. In the following result we provide a characterization of $I$-$soc(M)$ when it is essential.

**Proposition 2.2.** *Let $M$ be an $R$-module. Every nonzero submodule of $M$ contains an iso-retractable submodule if and only if $I$-$soc(M) \leq_e M$.*

*Proof.* Let $N$ be a nonzero submodule of $M$. Then $N$ contains an iso-retractable submodule. Thus $N \cap I$-$soc(M) \neq 0$. Therefore, $I$-$soc(M) \leq_e M$. Conversely suppose $I$-$soc(M) \leq_e M$. Let $N$ be a nonzero submodule of $M$. Then $N \cap I$-$soc(M) \neq 0$. Thus $N \cap I$-$soc(M)$, being a submodule of $I$-$soc(M)$, has an iso-retractable submodule. It follows that $N$ has an iso-retractable submodule. ∎

**Corollary 2.3.** *Let $M$ be an iso-Artinian $R$-module. Then $I$-$soc(M) \leq_e M$.*

*Proof.* Since $M$ is iso-Artinian, therefore every submodule of $M$ is iso-Artinian. Thus every submodule of $M$ contains an iso-retractable submodule. Now the result follows from Proposition 2.2. ■

**Definition 2.4.** [6, Definition 1.1] *An $R$-module $M$ is virtually semisimple if each submodule of $M$ is isomorphic to a direct summand of $M$. If each submodule of $M$ is virtually semisimple module, we call $M$ completely virtually semisimple.*

Recall, a ring $R$ is *semiprime* if $(0)$ is a semiprime ideal of $R$. If we consider the ring $\mathbb{Z}_4$, then trivially $\mathbb{Z}_4$ is iso-Artinian (iso-Noetherian) but it is not semiprime. A ring $R$ is said to be a *right Goldie ring* if it satisfies the ascending chain condition on right annihilators and $u.dim(R_R) < \infty$. We call a ring $R$ *Goldie ring* if it is both left and right Goldie. We note that for a semiprime ring $l.ann(A) = r.ann(A) = ann(A)$ for any ideal $A$ of $R$. We know that every right Artinian ring is right Noetherian. We have the following theorem:

**Theorem 2.5.** *Let $R$ be a semiprime right iso-Artinian ring. Then*

(1) *$R$ is a right Noetherian ring.*
(2) *Every projective $R$-module $M$ is completely virtually semisimple.*
(3) *$I$-$soc(M) = M$, for every $R$-module $M$.*
(4) *$R$ is generated by iso-retractable right ideals and $I$-$rad(R) = 0$.*

*Proof.* (1). Since $R$ is a semiprime right iso-Artinian ring, $R$ satisfies ACC on annihilators. Therefore $R$ satisfies ACC on right annihilators because every right annihilator is two sided annihilator. It follows by [20, Proposition 11.43] that $R$ satisfies ACC on right complement. Therefore, $u.dim(R_R) < \infty$ (see [20, Proposition (6.30)']). So, $R$ is a right Goldie ring and hence $R$ is a finite direct sum of iso-retractable right ideals. Since every iso-retractable right ideal is right Noetherian, $R$ is right Noetherian because $R$ is direct sum of finitely many right Noetherian.

(2). Since $R$ is a semiprime right iso-Artinian ring, by [23, Proposition 2.7], $R$ is a direct sum of iso-retractable right ideals. It follows by [6, Theorem 3.11] that $R$ is a right completely virtually semisimple ring. Hence by [6, Prop. 3.3], every projective right $R$-module is completely virtually semisimple.

(3). Since $R$ is a semiprime right iso-Artinian ring, it is right nonsingular (see [15, Lemma 4.3]). Therefore, $R$ is a direct sum of iso-retractable right ideals. By [14, Lemma 4.4], $I$-$soc(R_R) = R_R$. Thus, $R$ is generated by iso-retractable right ideals. Further due to $R_R$ projective, $I$-$rad(R) = 0$ by [14, Remark 4.3(5)].

(4). Since $R$ is a semiprime right iso-Artinian ring, it is a direct sum of iso-retractable right ideals. It follows by Theorem 2.5 (3) and [14, Theorem 4.9], $I$-$soc(M) = M$, for every right $R$-module $M$. ■

In general, iso-Noetherian modules need not be virtually semisimple. For example, consider $\mathbb{Z}_{p^n}$, $n > 1$ as a $\mathbb{Z}$-module. $\mathbb{Z}_{p^n}$ is iso-Noetherian but not virtually semisimple. Also, virtually semisimple module need not be iso-Noetherian. For example, $\oplus_{i=1}^{\infty}\mathbb{Z}_{p_i}$, where $p_i$'s are distinct primes, as a $\mathbb{Z}$-module is virtually semisimple but not iso-Noetherian.

**Proposition 2.6.** *Every nonzero virtually semisimple iso-Noetherian module is Noetherian.*

*Proof.* Since $M$ is iso-Noetherian, therefore $u.dim(M) < \infty$. It follows by [6, Proposition 2.8] that $M$ is Noetherian. ■

In [6], authors proved that $R$ is a left completely virtually semisimple ring if and only if ${}_RR$ is a direct sum of iso-retractable ideals. In the following, we prove this result in case of module.

**Proposition 2.7.** *Let $M$ be a completely virtually semisimple right $R$-module. If $M$ is an iso-Noetherian $R$-module then $M$ is a direct sum of iso-retractable $R$-modules and $M$ is Noetherian.*

*Proof.* Let $M$ be right iso-Noetherian. Then by [15, Prop. 5.1], $u.dim(M_R) < \infty$. By [6, Proposition 2.8], $M$ is finitely generated. Thus [6, Proposition 2.9] implies that $M$ is a direct sum of iso-retractable $R$-modules. It is clear that the direct sum is finite because $u.dim(M_R) < \infty$. Since every iso-retractable module is Noetherian and finite direct sum of Noetherian modules is Noetherian, therefore $M$ is Noetherian. ■

**Corollary 2.8.** *Over a right completely virtually semisimple ring, every iso-Noetherian projective right $R$-module is finite direct sum of iso-retractable right $R$-modules.*

*Proof.* Let $R$ be a right completely virtually semisimple ring. By [6, Prop. 3.2] every projective $R$-module is right completely virtually semisimple. It follows by Proposition 2.7 that $M$ is a finite direct sum of iso-retractable $R$-modules. ■

Recall by [3] that a right $R$-module $M$ is said to be a *fe-module* if, $M_R$ has only finitely many essential submodules. A ring $R$ is *right fe-ring* if $R_R$ is fe-module. In general, iso-Noetherian modules need not be fe-module. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is iso-Noetherian but not a fe-module.

**Theorem 2.9.** *A fe-module $M$ is Noetherian if any one of the following statements holds:*

(1) *A submodule $N$ of $M$ and the quotient module $M/N$ both are iso-Noetherian.*

(2) *$M$ is iso-Noetherian.*
(3) *$soc(M)$ is iso-Noetherian.*

*Proof.* (1). Let $M$ be a fe-module. Then $N$ and $M/N$ both are fe-module by [3, Prop. 1.3]. Since $N$ and $M/N$ both are iso-Noetherian, both have finite uniform dimension by [15, Proposition 5.1]. It follows by [3, Corollary 1.2] that $N$ and $M/N$ both are Noetherian. Thus $M$ is Noetherian.

(2). It follows from [15, Proposition 5.1] and [3, Corollary 1.2].

(3). Suppose $M$ is a fe-module and $soc(M)$ is iso-Noetherian. It follows from [3, Theorem 1.1] that $M/soc(M)$ has finitely many submodules. Therefore $M/soc(M)$ is Noetherian. By [3, Proposition 1.3, Corollary 1.2], $soc(M)$ is Noetherian. This implies that $M$ is Noetherian. ∎

**Corollary 2.10.** *If $R$ is semiprime fe-ring such that $I$ and $R/I$ both are iso-Artinian, then $R$ is Noetherian.*

Recall [2], an $R$-module $M$ is a *ue-module* if $M$ has an unique proper essential submodule. Finally we prove a result analogous to [3, Theorem 1.7].

**Theorem 2.11.** *If an $R$-module $M$ is iso-Artinian, iso-Noetherian ue-module, then $M = N \oplus L$, where $N$ is iso-Artinian, iso-Noetherian cyclic ue-module with unique maximal submodule and $L$ is finite sum of iso-retractable $R$-modules or $L = 0$.*

*Proof.* Assume $M$ as an iso-Artinian module. By Corollary 2.3, $I$-$soc(M)$ is essential in $M$. Also, $M$ is an ue-module implies that $I$-$soc(M)$ is maximal submodule of $M$. If $I$-$soc(M)$ is an unique maximal submodule, then $M$ is cyclic and hence we are done by [3, Lemma 1.6, Corollary 1.2]. If not, let $M_1$ be a maximal submodule of $M$ different from $I$-$soc(M)$. Then $M_1$ is not essential because $M$ is an ue-module. Hence, it must be a direct summand. Thus $M = M_1 \oplus K_1$, where $K_1$ is an iso-retractable submodule of $M$. By [18, Theorem 15], $M_1$ is an ue-module. If $M_1$ has unique maximal submodule, then nothing to prove. If not, let $M_2$ be a maximal submodule of $M_1$ which is a direct summand of $M_1$. Therefore, there exists an iso-retractable submodule $K_2$ of $M_1$ such that $M_1 = M_2 \oplus K_2$. Thus $M = M_2 \oplus K_2 \oplus K_1$. Since $M$ is iso-Noetherian hence $u.dim(M) < \infty$. Therefore, this process must stop after finitely many steps, say $n$, i.e. we have $M = M_n \oplus K_n \oplus \ldots \oplus K_1$, where $M_n$ is an ue-module with a unique maximal submodule and each $K_i$ is an iso-retractable submodule. Since $M$ is iso-Artinian, $M_n$ is iso-Artinian. So, it follows by previous steps that $M_n$ is cyclic. Take $N = M_n$ and $L = K_n \oplus \ldots \oplus K_1$. This completes the proof. ∎

**Corollary 2.12.** *If $R$ is a semiprime iso-Artinian ue-ring, then $R = I \oplus L$, where $I$ is a semiprime iso-Artinian cyclic ue-module with the unique maximal ideal and $L$ is a finite direct sum of iso-retractable ideals or $L = 0$.*

*Proof.* If $R$ is a semiprime iso-Artinian ring then the $u.dim(R) < \infty$. The rest of proof is on the same line as of the Theorem 2.11. ■

## 3. Mono-Artinian and Mono-Noetherian Rings

**Definition 3.1.** *A right $R$-module $M$ is mono-Artinian if for every descending chain $M_1 \geq M_2 \geq M_3 \geq \dots$ of submodules of $M$, there exists $n \in \mathbb{N}$ such that $M_i$ embeds in $M_{i+1}$, for all $i \geq n$. A ring $R$ is right mono-Artinian if the right $R$-module $R$ is mono-Artinian. A ring $R$ is said to be mono-Artinian if it is both left as well as right mono-Artinian. Similarly, we can define mono-Noetherian modules and rings.*

The classes of mono-Noetherian and mono-Artinian modules are dual to that of epi-Noetherian and epi-Artinian modules, respectively. We find that none of the classes imply one another, in support we give the following examples.

*Example 3.2.* Consider $\mathbb{Z}_4$ as $\mathbb{Z}_4$-module. It follows by [12, Example 3.1] that $\mathbb{Z}_4^{(\mathbb{N})}$ is epi-Noetherian. Clearly, $\mathbb{Z}_4$ is not mono-Noetherian.

*Example 3.3.* Let $R$ be a domain, which is not a principal right ideal domain (PRID). Clearly, every domain is compressible (see [26]), therefore $R$ is a mono-Artinian (mono-Noetherian) ring. By [12, Corollary 4.8], $R_R$ is not epi-Artinian.

*Remark 3.4.*

(1) Every iso-Noetherian (iso-Artinian) module is mono-Noetherian (mono-Artinian) but the converse need not be true. For example, we consider Example 3.3, in which $R_R$ is mono-Artinian but not epi-Artinian. This implies that $R_R$ is not iso-Artinian.

(2) Let $M$ be a nonzero virtually semisimple $R$-module. Then $M$ is mono-Noetherian if and only if $M$ is Noetherian.

(3) If $R$ is a ring such that all $R$-modules are compressible, then all $R$-modules are mono-Noetherian (mono-Artinian).

Now we find existance of mono- Artinian modules over hereditary rings.

**Proposition 3.5.** *Over a hereditary ring every finitely generated projective completely virtually semisimple module is mono-Artinian (mono-Noetherian).*

*Proof.* Let $M$ be a finitely generated completely virtually semisimple $R$-module. By [6, Proposition 2.9], $M$ is the finite direct sum of iso-retractable $R$-modules, say $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$, where each $M_i$ is iso-retractable. Since $R$ is a hereditary ring and $M$ is projective, therefore $M_1$ is projective. Next, $M_1$

is iso-Artinian (iso-Noetherian) and $M_2$ is projective iso-retractable. It follows by [15, Proposition 2.3] that $M_1 \oplus M_2$ is iso-Artinian (iso-Noetherian). Now $M_1 \oplus M_2$ is iso-Artinian (iso-Noetherian) and $M_3$ is projective iso-retractable hence $M_1 \oplus M_2 \oplus M_3$ is iso-Artinian (iso-Noetherian). Thus by induction on $n$, $M$ is iso-Artinian (iso-Noetherian). Obviously, $M$ is mono-Artinian (mono-Noetherian). ■

We have a straightforward characterization of mono-Artinian and mono-Noetherian modules.

**Lemma 3.6.** *Let $M$ be an $R$-module. The following statements are equivalent:*

(1) *$M$ is mono-Artinian (mono-Noetherian).*
(2) *For every non-empty set $F$ of submodules of $M$, there exists $N \in F$ such that, for every submodule $K \leq N$ ($K \geq N$), if $K \in F$, then $N$ embeds in $K$ ($K$ embeds in $N$).*
(3) *For every non-empty chain $C$ of submodules of $M$, there exists $N \in C$ such that, for every submodule $K \leq N$ ($K \geq N$), if $K \in C$, then $N$ embeds in $K$ ($K$ embeds in $N$).*

Now we study some general properties of right mono-Artinian (mono-Noetherian) rings. The following result is dual to the [12, Lemma 5.4].

**Proposition 3.7.** *A finite direct product of right mono-Artinian (mono-Noetherian) rings is right mono-Artinian (mono-Noetherian).*

*Proof.* Let $R_1, R_2, \dots, R_n$ be right mono-Artinian rings and $R = R_1 \times R_2 \times \dots \times R_n$. Let $I_1 \geq I_2 \geq I_3 \geq \dots$ be a descending chain of right ideals of $R$. For each $j \in \mathbb{N}$, $I_j$ is of the form $I_j = A_{j1} \times A_{j2} \times \dots \times A_{jn}$, where each $A_{jk}$ is a right ideal of $R_j$. Since for each $k \in \{1, 2, 3, \dots, n\}$, $R_k$ is right mono-Artinian, there exists $m \in \mathbb{N}$ such that for each $j \geq m$ there is a monomorphism, say $\psi_{jk} : A_{jk} \to A_{(j+1)k}$. For each $j \geq m$, we define a map $\psi_j : I_j \to I_{j+1}$ by $\psi_j(a_{j1}, a_{j2}, \dots, a_{jn}) = (\psi_{j1}(a_{j1}), \psi_{j2}(a_{j2}), \dots, \psi_{jn}(a_{jn}))$, for every $(a_{j1}, a_{j2}, \dots, a_{jn}) \in I_j$. Since each $\psi_{jk}$ is a monomorphism, therefore $\psi_j$ is a monomorphism. Hence $R$ is right mono-Artinian. In case of right mono-Noetherian, if we consider ascending chain of right ideals then the proof is on the same line. ■

*Remark 3.8.* We observe that the direct product of finitely many mono-Artinian (mono-Noetherian) modules over different rings is again mono-Artinian (mono-Noetherian) over the product of rings. Let $R$ be the product of $n$ rings $R_i$, $i = 1, 2, \dots, n$ and $M$ be the product of $R_i$-modules $M_i$, $i = 1, 2, \dots, n$. Then $M$ is a mono-Artinian (mono-Noetherian) $R$-module if and only if each $M_i$ is a mono-Artinian (mono-Noetherian) $R_i$-module. Because any submodule of an

$R$-module $M$ is of the form $M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where $M_i$ is $R_i$-module for $i = 1, 2, \ldots, n$.

In general, the direct sum of two mono-Artinian (mono-Noetherian) modules need not be mono-Artinian (mono-Noetherian). In the following results, we provide sufficient conditions for the direct sum to be mono-Artinian (mono-Noetherian).

**Lemma 3.9.** *Let $R$ be a commutative ring. Let $M_1$ and $M_2$ be two $R$-modules such that $M = M_1 \oplus M_2$. If $ann_R(M_1) + ann_R(M_2) = R$, then any submodule of $M$ is of the form $K_1 \oplus K_2$ for some submodules $K_1$ of $M_1$ and $K_2$ of $M_2$.*

*Proof.* Let $K$ be a submodule of $M$. Since $RK = K$, therefore $RK = (ann_R(M_1)+ann_R(M_2))K = ann_R(M_1)K+ann_R(M_2)K = K_2+K_1 = K_1+K_2$, where $K_i$ are submodules of $M_i$ for $i = 1, 2$. ∎

**Proposition 3.10.** *Let $R$ be a commutative ring. Let $M_1$ and $M_2$ be two $R$-modules such that $M = M_1 \oplus M_2$. If $ann_R(M_1) + ann_R(M_2) = R$, then $M$ is mono-Artinian (mono-Noetherian) if and only if $M_1$ and $M_2$ are mono-Artinian (mono-Noetherian).*

*Proof.* Let $K_1 \geq K_2 \geq K_3 \geq \ldots$ be a descending chain of submodules of $M$. Thus each $K_i$ is of the form $K_i = K_{i1} \oplus K_{i2}$. So, we have a chain of the form $K_{11} \oplus K_{12} \geq K_{21} \oplus K_{22} \geq K_{31} \oplus K_{32} \geq \ldots$. Since $M_1$ and $M_2$ both are mono-Artinian, there exist indices $n_1$ and $n_2$ such that $K_{i1}$ embeds in $K_{(i+1)1}$, for all $i \geq n_1$ and $K_{i2}$ embeds in $K_{(i+1)2}$, for all $i \geq n_2$. Let $n = max\{n_1, n_2\}$. Then $K_{i1} \oplus K_{i2}$ embeds in $K_{(i+1)1} \oplus K_{(i+1)2}$, for all $i \geq n$. Therefore $K_i$ embeds in $K_{i+1}$, for all $i \geq n$. Thus $M$ is mono-Artinian. The converse is obvious. ∎

**Corollary 3.11.** *Let $R$ be a commutative ring. Let $M_1, M_2, \ldots, M_n$ be $R$-modules such that $ann_R(M_1)+ann_R(M_2)+\ldots+ann_R(M_n) = R$. Then $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is mono-artinian if and only if each $M_i, i = 1, 2, \ldots, n$ is mono-artinian.*

**Theorem 3.12.** *Let $R$ be a mono-Artinian ring. Then the following statements hold:*

(1) *$R$ satisfies ACC on right (left) annihilator ideals.*
(2) *$Z(R_R)$ is nilpotent.*
(3) *Every nonzero nil left (right) ideal of $R$ contains a nonzero nilpotent left (right) ideal.*

*Proof.* (1). Let $A_1 \leq A_2 \leq A_3 \leq \ldots$ be an ascending chain of right annihilator ideals. Then we have a descending chain $l.ann(A_1) \geq l.ann(A_2) \geq l.ann(A_3) \geq \ldots$ in $R$. Since $R$ is mono-Artinian, there exists an index $n$ such that $l.ann(A_i)$ embeds in $l.ann(A_{i+1})$, for all $i \geq n$. Put $L_i = l.ann(A_i)$, for all $i$. Then $L_i$ embeds in $L_{i+1}$, for all $i \geq n$. Let $f_i : L_i \to L_{i+1}$ be embeddings. Then for

$i \geq n$, we have $L_i \cong f(L_i) \leq L_{i+1}$. Taking right annihilators we get $r.ann(L_i) = r.ann(f(L_i)) \geq r.ann(L_{i+1})$, for all $i \geq n$. Since $r.ann(L_i) = A_i$ for all $i$. Thus $A_i \geq A_{i+1}$, for all $i \geq n$. Therefore $R$ satisfies the ACC on right annihilator ideals. Similarly by taking ascending chain of left annihilator ideals we can show that $R$ satisfies the ACC on left annihilator ideals.

(2). Let $r.ann(Z(R_R)) \leq r.ann(Z(R_R)^2) \leq r.ann(Z(R_R)^3) \leq \dots$ be the ascending chain of right annihilator ideals. By (1) there exists an index $m$ such that $r.ann(Z(R_R)^m) = r.ann(Z(R_R)^{m+1})$. We claim that $Z(R_R)^{m+1} = 0$. Suppose that $Z(R_R)^{m+1} \neq 0$. Choose $x \in Z(R_R)$ with $Z(R_R)^m x \neq 0$ so that $r.ann(x)$ is as large as possible. If $y \in Z(R_R)$, then $r.ann(y) \leq_e R$, so $xR \cap r.ann(y) \neq 0$. Hence $yxr = 0$, for some $r \in R$ with $xr \neq 0$. This implies that $r.ann(yx) \supset r.ann(x)$ and this is a contradiction to the choice of $x$ except $Z(R_R)^m yx = 0$. This shows that $Z(R_R)^{m+1}x = 0$ and so by the choice of $m$, $Z(R_R)^m x = 0$. Hence $Z(R_R)^{m+1} = 0$. Thus $Z(R_R)$ is nilpotent.

(3). Let $N$ be a nil left ideal of $R$. Since $R$ is a mono-Artinian ring, $N$ is also mono-Artinian. It follows that $N$ satisfies ACC on annihilator ideals. So, we can choose $0 \neq x \in N$ such that $ann_R(x)$ is maximal in the collection of right annihilators of nonzero elements of $N$. If $y \in R$, then $(yx)^n = 0$, for some $n \geq 1$ and $(yx)^{n-1} \neq 0$. So, $ann_R(yx)^{n-1} = ann_R(x)$, by the maximality of right annihilators. Since $yx$ annihilates $(yx)^{n-1}$, therefore it annihilates $x$. It follows that $xyx = 0$. Thus $xRx = 0$ and so $Rx$ is nilpotent. ∎

Recall by [29] that a nonzero $R$-module $M$ is *quasi-polysimple* if every submodule of $M$ contains a uniform submodule. In [26], Smith introduced the notion of compressible modules. Clearly, every mono-Artinian module contains a compressible module. In the following result, we discuss structure of an essential submodule of a mono-Artinian module in terms of compressible modules.

**Theorem 3.13.** *Let $M$ be a nonzero mono-Artinian $R$-module. Then the following statements hold:*

1. *$M$ contains an essential submodule that is a direct sum of compressible modules.*
2. *If $M$ is semiprime then $M$ is quasi-polysimple.*
3. *If $K$ is a projective iso-retractable, then $M \oplus K$ is a mono-Artinian.*
4. *If $K$ is iso-retractable $R$-module and $M \oplus K$ is distributive, then $M \oplus K$ is mono-Artinian.*

*Proof.* (1). Let $\Omega$ be the set of all families of independent compressible submodules of $M$. Since a mono-Artinian module contains a compressible module, hence $\Omega$ is nonempty. By the Zorn's Lemma, $\Omega$ has a maximal member $L = \{U_\alpha : \alpha \in \Lambda\}$, where $\Lambda$ is some index set. Consider, $U = \oplus_{\alpha \in \Lambda} U_\alpha$. Suppose $U$ is not essential in $M$. Then $U \cap N = 0$, for some submodule $N$ of $M$. Since submodule of a mono-Artinian module is mono-Artinian, so is $N$. This implies that $N$ contains a nonzero compressible module $K$. Thus $\{L, K\}$ is a member of $\Omega$. This contradicts the maximality of $L$. Therefore $U = \oplus_{\alpha\in\Lambda} U_\alpha$ is essential in $M$.

(2). Let $M$ be a nonzero semiprime mono-Artinian $R$-module. Then $u.dim(M) < \infty$. So every submodule of $M$ is of finite uniform dimension. Thus by [20, Lemma 2.7], every submodule of $M$ contains a uniform submodule. Therefore $M$ is quasi-polysimple.

(3). It is analogous to [15, Proposition 2.3].

(4). Assume that $M$ is mono-Noetherian and $L_1 \leq L_2 \leq L_3 \leq \ldots$ is an ascending chain of submodules of $M \oplus K$. If $L_i \cap K = 0$, for each $i$, then every member of the chain is in $M$. Since $M$ is mono-Artinian, there exists $n \in \mathbb{N}$ such that $L_{i+1}$ embeds in $L_i$, for each $i \geq n$. Next, suppose there exists $i_0$ such that $K \cap L_{i_0} \neq 0$. Then for every $i \geq i_0$, $K \cap L_i \cong K$. Now $L_i = L_i \cap (K \oplus M) = (L_i \cap K) \oplus (L_i \cap M) \cong K \oplus (L_i \cap M)$, and $L_{i+1} \cong K \oplus (L_{i+1} \cap M)$. Consider a chain $L_{i_0} \cap M \leq L_{i_0+1} \cap M \leq \ldots$ of submodules of $M$. By the mono-Noetherianness of $M$, there exists $k \geq i_0$ such that $L_{i+1} \cap M$ embeds in $L_i \cap M, \forall i \geq k$. Now $K \oplus (L_{i+1} \cap M)$ embeds in $K \oplus (L_i \cap M)$. Thus $L_{i+1}$ embeds in $L_i, \forall i \geq k$. Therefore $K \oplus M$ is mono-Noetherian. ∎

**Proposition 3.14.** *Being mono-Artinian (mono-Noetherian) is a Morita invariant property of modules.*

*Proof.* It follows by the definition of mono-Artinian (mono-Noetherian) modules and [1, Proposition 21.7]. ∎

If we consider the ring $\mathbb{Z}_4$, then trivially $\mathbb{Z}_4$ is mono-Artinian (mono-Noetherian) but it is not semiprime. If we consider mono-Artinian (mono-Noetherian) ring which is semiprime, then we have the following theorem:

**Theorem 3.15.** *Let $R$ be a semiprime mono-Artinian (mono-Noetherian) ring. Then*

(1) *$R$ has finite uniform dimension.*
(2) *$R$ is a Goldie ring.*
(3) *$R$ has finitely many minimal prime ideals.*
(4) *$R/P$ is a Goldie ring, for each minimal prime ideal $P$ of $R$.*
(5) *$R$ has finitely many annihilator ideals.*

*Proof.* (1). Since $R$ is a semiprime mono-Artinian ring, $R$ satisfies ACC on annihilator ideals. It follows from [21, Theorem 2.15] that $R$ has finite uniform dimension.

(2). Since $R$ is semiprime mono-Artinian, it satisfies ACC on annihilator ideals and has finite uniform dimension. Thus $R$ is a Goldie ring.

(3). Let $u.dim(R) = n$. Then there are uniform ideals $U_1, U_2, \ldots U_n$ such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n \leq_e R$. If $P_i = ann(U_i)$, then $P_i$ is a minimal prime ideal by [21, Proposition 2.14(iv)]. Also, $\cap P_i = 0$. Therefore, these are all minimal prime ideals.

(4). Let $R$ be a semiprime ring. Since $R$ is mono-Artinian, it satisfies the ascending chain condition on annihilator ideals. This implies, by [20, Theorem 11.43] that $R$ has finitely many minimal prime ideals. By Theorem 3.13 (2), $R$ is a Goldie ring. It follows from [20, Corollary 11.44], $R/P$ is a Goldie ring for each minimal prime ideal of $R$.

(5). It follows from Theorem 3.15 (2) and [21, Proposition 2.14(iii)]. ■

**Theorem 3.16.** *Let $R$ be a semiprime mono-Artinian ring. Then*

(1) *$R$ is direct sum of finitely many compressible ideals.*
(2) *$R$ is right nonsingular.*
(3) *$R$ is right Noetherian.*
(4) *$R$ satisfies ACC and DCC on complements.*

*Proof.* (1). We know that a ring $R$ is mono-Artinian if and only if ${}_RR_R$ is a mono-Artinian module. Suppose ${}_RR_R$ is a mono-Artinian module. By Theorem 3.13 (1), ${}_RR_R$ has an ideal $I$ which is essential in ${}_RR_R$ and is a direct sum of compressible submodules. Let $R$ be a semiprime ring. Then $R$ is a Goldie ring by Theorem 3.15 (2). It follows by [12, Proposition 5.6] that ${}_RR_R$ is isomorphic to $I$. Also, the direct sum in $I$ is finite because $R$ is a Goldie ring.

(2). By Theorem 3.12 (2), $Z(R_R)$ is nilpotent. Since $R$ is semiprime, therefore $Z(R_R) = 0$. Thus $R$ is right nonsingular.

(3). Let $R$ be a semiprime right mono-Artinian ring. Then $R$ is right Goldie ring. By [15, Theorem 4.6], $R$ is right Noetherian.

(4). Let $R$ be a semiprime mono-Artinian ring. Then $u.dim(R) < \infty$. Thus by [20, Proposition (6.30)'], $R$ satisfies ACC and DCC on complements. ■

By Hopkins theorem every Artinian ring is a Noetherian ring. In general, a right iso-Artinian ring need not be right Noetherian. In [14], authors state that they were not able to prove or disprove whether a semiprime right iso-Artinian ring is right Noetherian. We have the following results:

**Corollary 3.17.** *A semiprime right iso-Artinian ring is right Noetherian.*

**Corollary 3.18.** [23, Proposition 2.9] *Let $R$ be a semiprime iso-Noetherian (iso-Artinian) ring. Then $R/P$ is a right Goldie ring for each minimal prime ideal $P$ of $R$.*

**Theorem 3.19.** *A right mono-Artinian (mono-Noetherian) integral domain is a right Ore domain.*

*Proof.* Let $a, b \in R$. We claim that $aR \cap bR \neq 0$. Suppose if possible, $aR \cap bR = 0$.

It follows from [21, Example 1.2.11(ii)] that $\sum b^n aR$ is direct. This contradicts that $u.dim(R) < +\infty$. Thus $aR \cap bR \neq 0$. Hence, $R$ is a right Ore domain. ∎

**Question 3.20.**

(1) What are the properties of the endomorphism ring of a mono-Artinian (mono-Noetherian) module ?

(2) How to characterize mono-Artinian (mono-Noetherian) modules in terms of their endomorphism rings ?

Various generalizations of injectivity have been studied in [4, 5, 7, 10, 11]. We can also study some properties of these chain conditions on such classes of modules.

**Acknowledgement.** The authors want to thank the referee for the carefully reading of this paper. The second author is grateful to the CMP Degree College for their support.

## References

[1] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Graduate Text in Math. **13**, Springer-Verlag Inc., New York, 1974.

[2] E.P. Armendariz, Rings with dcc on essential left ideals, *Comm. Algebra* **8** (1980) 299–308.

[3] A. Azarang and F. Shahrisvand, Rings with only finitely many essential right ideals, *Comm. Algebra* **47** (7) (2019) 2843–54.

[4] S. Baupradist, H.D. Hai, N.V. Sanh, A general form of pseudo-p-injectivity, *Southeast Asian Bull. Math.* **35** (2011) 927–933.

[5] S. Baupradist, H.D. Hai, N.V. Sanh, On pseudo-P-injectivity, *Southeast Asian Bull. Math.* **35** (2011) 21–27.

[6] M. Behboodi, A. Daneshvar, M.R. Vedadi, Virtually semisimple modules and a generalization of the Wedderburn- Artin theorem, *Comm. Algebra* **46** (6) (2018) 2384–2395.

[7] S. Chairat, C. Somsup, K.P. Shum, N.V. Sanh, A generalization of Azumaya's theorem on M-injective modules, *Southeast Asian Bull. Math.* **29** (2) (2005) 277–281.

[8] A.K. Chaturvedi, Iso-retractable modules and rings, *Asian-Eur J. Math.* **12** (1) (2019) 1950013-20.

[9] A.K. Chaturvedi, On iso-retractable modules and rings, In: *Proc. Algebra and its Applications (ICAA Aligarh, India, 2014)*, Springer, 2016. DOI: 10.1007/978-981-10-1651-6_24.

[10] A.K. Chaturvedi, QP-injective and QPP-injective modules, *Southeast Asian Bull. Math.* **38** (2) (2014) 191–194.

[11] A.K. Chaturvedi, B.M. Pandeya, A.J. Gupta, Quasi-c-irincipally injective modules and self-c-principally injective rings, *Southeast Asian Bull. Math.* **33** (4) (2009) 685–702.

[12] R. Dastanpour and A. Ghorbani, Modules with epimorphisms on chain of submodules, *J. Algebra Appl.* **16** (6) (2017) 1750101-18.

[13] N.V. Dung, D.V. Huyng, R. Wisbauer, Quasi-injective modules with acc or dcc on essential submodules, *Archiv der Mathematik* **53** (1989) 252–255.

[14] A. Facchini and Z. Nazemian, Artinian dimension and isoradical of modules, *J. Algebra* **484** (2017) 66–87.

[15] A. Facchini and Z. Nazemian, Modules with chain conditions up to isomorphism, *J. Algebra* **453** (2016) 578–601.

[16] A. Facchini and Z. Nazemian, On iso-Noetherian and iso-Artinian modules, *Contemp. Math.* **730** (2019). doi: 10.1090/conm/14707.

[17] D. Jonah, Rings with the minimum condition for principal right ideals have the maximum condition for principal left ideals, *Math. Z.* **113** (1970) 106–112.

[18] O.A.S. Karamzadeh, M. Motamedi, S.M. Shartash, On rings with a unique proper essential right ideals, *Fundam. Math.* **183** (3) (2004) 229–244.

[19] O.A.S. Karamzadeh, M. Motamedi, S.M. Shartash, Erratum to On rings with a unique proper essential right ideals, *Fundam. Math.* **205** (3) (2009) 289–291.

[20] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics **139**, Springer-Verlag, New York, Berlin, 1998.

[21] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics **30**, New York, 1987.

[22] B.M. Pandeya, A.K. Chaturvedi, A.J. Gupta, Applications of epi-retractable modules, *Bull. Iranian Math. Soc.* **38** (2) (2012) 469–477.

[23] S. Prakash and A.K. Chaturvedi, Iso-Noetherian rings and modules, *Comm. Algebra* **47** (2) (2019) 676–683.

[24] N.V. Sanh and S. Asawasamrit, On prime and semiprime goldie modules, *Asian-Eur J. Math.* **4** (2) (2011) 321–334.

[25] S. Sanpinij and N.V. Sanh, On serial Artinian modules and their endomorphism rings, *Southeast Asian Bull. Math.* **37** (2013) 401–404.

[26] P.F. Smith, Compressible and related modules, In: *Abelian Groups, Rings, Modules and Homological Algebra*, Ed. by P. Goeters and O.M.G. Jenda Chapman and Hall, CRC, 2006.

[27] J. Soontharanon, N.D.H. Nghiem, N.V. Sanh, A note on Noetherian modules, *Southeast Asian Bull. Math.* **44** (2020) 143–147.

[28] L.P. Thao and N.V. Sanh, A generalization of Hopkins-Levitzki theorem, *Southeast Asian Bull. Math.* **37** (2013) 591–600.

[29] A.K. Tiwary and B.M. Pandeya, Modules whose nonzero endomorphisms are monomorphisms, In: *Proc. Int. Symp. Algebra and its Applications (New Delhi, 1981)*, Lect. Notes Pure Appl. Math. **91**, Marcel Dekker, Inc., New York, 1984.

[30] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Reading, 1991.