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Inequalities via Generalized (p, r, h, η)-Convex Functions

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Abstract. The main aim of this paper is to introduce a new class of convex functions with respect to non-negative functions h and bifunction $\eta(.,.)$, which is called generalized (p, r, h, η)-convex functions. We derive some new integral inequalities for this class of functions. Some special cases are also discussed. The ideas and techniques of this paper may stimulate further research.

Keywords: Generalized convex function; (p, r, h, η)-convex function; Hermite-Hadamard type inequality; Fejer type integral inequality.

1. Introduction

Convexity theory has become a rich source of inspiration in pure and applied sciences. This theory had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general framework for studying a wide class of unrelated problems. For recent application- s, generalizations and other aspects of convex functions and their variant forms, see [1, 2, 3, 4, 5, 6, 7, 12, 13, 15, 17, 27, 20, 22, 28, 30, 33] and the references therein. Varosanec [25], introduced the class of h-convex functions with respect to an arbitrary non-negative function h. It has shown that this class contains some previously known classes of convex functions as special cases. Zhang et al. [34] introduced and studied a class of nonconvex functions which is called p-convex functions. Motivated by this ongoing research, Noor et al. [15] have derived several inequalities for differentiable p-convex functions.

Let I be an interval. A function $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is a p-convex function, if and only if,

$$\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
\leq\; & \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)\right] \\
\leq\; & \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x \leq \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{f(a)+f(b)}{2}\right] \\
\leq\; & \frac{1}{2}[f(a)+f(b)] \qquad (1)
\end{aligned}$$

This double inequality is known as the Hermite-Hadamard inequality for p-convex functions, which may be regarded as a refinement of the concept of convexity. The inequality (1) holds in reversed direction if f is a p-concave function. If $p = 1$, then inequality (1) is known as Hermite-Hadamard inequality for convex functions.

M. Tunc et al. [31] defined a new concept of beta-convex function and established some inequalities. One of the most recent significant generalizations of convex functions is generalized convex (ϕ-convex function), introduced by Gordji et al. [5]. These functions are non-convex functions. The class of harmonic convex function was introduced by Anderson [1] and Iscan [9]. It has been shown [18] that the minimum of the differentiable harmonic convex functions can be characterized by a class of variational inequalities, which is known as harmonic variational inequalities. See [12, 13, 14, 15, 19, 16, 17, 26, 27] for the recent developments in variational inequalities. Noor et al. [23] introduced and investigated new class of convex functions, which is called relative harmonic (s, η)-convex functions. They discussed some basic results of harmonic (s, η)-convex functions and also derived the Hermite-Hadamard and Fejer type inequalities for this class of functions. For recent developments, see [3, 6, 23, 25] and the references therein.

Motivated and inspired by the ongoing research on convex functions, we introduce concept of generalized (p, r, h)-convex functions. We establish some basic results regarding inequalities related to generalized (p, r, h)-convex functions. Several special cases are discussed which can be obtained from our main results. Our results can be viewed as significant and important refinement of well-known results for inequalities.

2. Preliminaries

In this section, we recall some basic concepts. Let $\eta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous bifunction.

Definition 2.1. [11] *A set $I = [a, b] \subset \mathbb{R}$ is said to be a convex set, if*

$$(1-t)x + ty \in I, \quad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. [11] *A function $f : I = [a, b] \to \mathbb{R}$ is said to be a convex function, if*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 2.3. [29] *Let $h : J = [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be an h-convex function, if*

$$f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 2.4. [34] *A set $I = [a, b] \subseteq \mathbb{R}\backslash\{0\}$ is said to be p-convex set, if*

$$[(1-t)x^p + ty^p]^{\frac{1}{p}} \in I, \quad \forall x, y \in I, t \in [0, 1].$$

Some special cases of the p-convex sets are:

(i) *If $p = 1$, then p-convex set is a convex set.*
(ii) *If $p = -1$, then p-convex set becomes a harmonic convex set.*
(iii) *If $p = 0$, then p-convex set collapses to the geometrically convex set.*

This shows that the concept of p-convex sets is quite general and unifying one.

Definition 2.5. [34] *A function $f : I = [a, b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be p-convex function, where $p \neq 0$, if*

$$f\left([(1-t)x^p + ty^p]^{\frac{1}{p}}\right) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0, 1]. \tag{2}$$

We note that, if $p = 0$, then p-convex functions reduce to geometrically convex functions [11]*, that is,*

$$f\left(x^{1-t}y^{t}\right) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in I, t \in [0,1].$$

For different and appropriate choices of p, one can show that the p-convex functions include the convex functions, harmonic convex functions and geometrically convex functions as special cases.

Definition 2.6. [4] *A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ is (r)-convex, if f is positive and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f((1-t)x+ty) = \begin{cases} ((1-t)[f(x)]^r + t[f(y)]^r)^{\frac{1}{r}} & \text{if } r \neq 0, \\ (f(x))^{1-t}(f(y))^t & \text{if } r = 0. \end{cases}$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right) \leq \begin{cases} \left(\frac{[f(x)]^r+[f(y)]^r}{2}\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ \sqrt{(f(x))(f(y))} & \text{if } r = 0. \end{cases}$$

It is clear that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

Definition 2.7. [3] *A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized convex (ϕ-convex) function with respect to the bifunction $\eta(.,.) : H \times H \to R$, such that,*

$$f((1-t)x+ty) \leq f(x) + t\eta(f(y), f(x)), \quad \forall x, y \in I, t \in [0,1].$$

Definition 2.8. [12] *A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized r-convex with respect to the bifunction $\eta(.,.) : H \times H \to R$,if f is positive and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f((1-t)x+ty) = \begin{cases} ((1-t)[f(x)]^r + t[f(x)+\eta(f(y),f(x))]^r)^{\frac{1}{r}} & \text{if } r \neq 0, \\ (f(x))^{1-t}(f(x)+\eta(f(y),f(x)))^t & \text{if } r = 0. \end{cases}$$

It is clear that generalized 0-convex functions are simply generalized log-convex functions [25] *and generalized 1-convex functions are generalized convex (ϕ-convex) functions, see* [3]*. We now introduce the concept of generalized (p,r,h)-convex functions.*

Definition 2.9. *Let $h : J = [0,1] \to \mathbb{R}$ be a nonnegative function. A function $f : I = [a,b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be generalized $(p,r,h,\eta(,.,))$-convex* *function with respect to the bifunction* $\eta(.,.) : H \times H \to R$, *if*

$$f\left([(1-t)x^p + tb^p]^{\frac{1}{p}}\right) = \begin{cases} [h(1-t)[f(x)]^r + h(t)[f(x)+\eta(f(y),f(x))]^r]^{\frac{1}{r}} & \text{if } r \neq 0, \\ (f(x))^{h(1-t)}(f(x)+\eta(f(y),f(x)))^{h(t)} & \text{if } r = 0. \end{cases} \tag{3}$$

The function f *is said to be generalized* (p,r,h,η)*-concave function, if and only if,* $-f$ *is generalized* (p,r,h,η)*-convex function. For* $t = \frac{1}{2}$*, we have*

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) = \begin{cases} h\left(\frac{1}{2}\right)^{\frac{1}{r}}\left[[[f(x)]^r + [f(x)+\eta(f(y),f(x))]^r)^{\frac{1}{r}}\right] & \text{if } r \neq 0, \\ (f(x)(f(x)+\eta(f(y),f(x))))^{h\left(\frac{1}{2}\right)} & \text{if } r = 0. \end{cases} \tag{4}$$

The function f *is called generalized* (p,r,h,η)*-Jensen convex function. If* $\eta(f(y),f(x)) = f(y) - f(x)$ *in* (6)*, then it reduces to the class of* (p,r,h)*-convex functions.*

Now we discuss some special cases of generalized (p,r,h)-convex function, which appears to be new ones:

(i) If $h(t) = t$ in Definition 2.9, then it reduces to the definition of generalized (p,r)-convex functions.
(ii) If $r = 1$ in Definition 2.9, then it reduces to the definition of generalized (p,h) convex functions.
(iii) If $h(t) = t^s$ in Definition 2.9, then it reduces to the definition of Breckner type of generalized (p,r)-convex functions.
(iv) If $h(t) = t^p(1-t)^q$ in Definition 2.9, then it reduces to the definition of generalized (p,r)-beta convex functions.

Lemma 2.10. *Suppose that* $a,b,c \in \mathbb{R}$*. Then the following statements hold:*

(i) $\min\{a,b\} \leq \frac{a+b}{2}$.
(ii) *if* $c \geq 0, c.\min\{a,b\} = \min\{ca,cb\}$.

Generalized logarithmic means of order r *of positive numbers* x,y *is defined by:*

$$L_r(x,y) = \begin{cases} \frac{r}{r+1}\left(\frac{x^{r+1}-y^{r+1}}{x^r-y^r}\right) & \text{if } r \neq \{-1,0\}, x \neq y \\ \frac{x-y}{\ln x - \ln y} & \text{if } r = 0, x \neq y \\ xy\frac{\ln x - \ln y}{x-y} & \text{if } r = -1, x \neq y, \\ x & \text{if } x = y. \end{cases}$$

3. Main Results

In this section, we obtain Hermite-Hadamard type inequalities for generalized (p, r, h) convex functions.

Theorem 3.1. *Let $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a generalized (p, r, h, η)-convex function. If $f \in L[a, b]$, then*

$$\begin{aligned}
&\frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} \,\mathrm{d}x \\
&\le \min\left\{ [[f(a)]^r + [f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}} \left(\int_0^1 [h(t)]^{\frac{1}{r}} \,\mathrm{d}t \right),\right. \\
&\quad \left. [[f(b)]^r + [f(b) + \eta(f(a), f(b))]^r]^{\frac{1}{r}} \left(\int_0^1 [h(t)]^{\frac{1}{r}} \,\mathrm{d}t \right) \right\} \\
&\le \frac{1}{2} \Big\{ [[f(a)]^r + [f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}} \\
&\quad + [[f(b)]^r + [f(b) + \eta(f(a), f(b))]^r]^{\frac{1}{r}} \Big\} \left(\int_0^1 [h(t)]^{\frac{1}{r}} \,\mathrm{d}t \right). \qquad (5)
\end{aligned}$$

Proof. Let f be a generalized (p, r, h, η)-convex function. Then, $\forall x, y \in I, t \in [0, 1]$,

$$f\left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) \le [h(1-t)[f(a)]^r + h(t)[f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}},$$

and

$$f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \le [h(1-t)[f(b)]^r + h(t)[f(b) + \eta(f(a), f(b))]^r]^{\frac{1}{r}}.$$

Thus, we have

$$\begin{aligned}
&f\left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) + f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \\
&\le [h(1-t)[f(a)]^r + h(t)[f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}} \\
&\quad + [h(1-t)[f(b)]^r + h(t)[f(b) + \eta(f(a), f(b))]^r]^{\frac{1}{r}},
\end{aligned}$$

Integrating (5) over the interval $[0, 1]$ and using Minkowskis inequality, we have

$$\begin{aligned}
&\int_0^1 f\left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) \mathrm{d}t + \int_0^1 f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \mathrm{d}t \\
&\le \left[\left(\int_0^1 [h(1-t)]^{\frac{1}{r}} f(a) \mathrm{d}t \right)^r + \left(\int_0^1 [h(t)]^{\frac{1}{r}} [f(a) + \eta(f(b), f(a))] \mathrm{d}t \right)^r \right]^{\frac{1}{r}}
\end{aligned}$$

$$+\left[\left(\int_0^1 [h(1-t)]^{\frac{1}{r}} f(b)\mathrm{d}t\right)^r + \left(\int_0^1 [h(t)]^{\frac{1}{r}}[f(b)+\eta(f(a),f(b))]\mathrm{d}t\right)^r\right]^{\frac{1}{r}}$$
$$= \Big\{[[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}$$
$$+ [[f(b)]^r + [f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big\}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\ \mathrm{d}t\right).$$

This implies

$$\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\ \mathrm{d}x$$
$$\le \frac{1}{2}\Big\{[[f(a)]^r + [f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}$$
$$+ [[f(b)]^r + [f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\Big\}\left(\int_0^1 [h(t)]^{\frac{1}{r}}\ \mathrm{d}t\right),$$

which is the required result. ■

Corollary 3.2. *Under the assumptions of Theorem* 3.1 *with* $r = 1$*, we have*

$$\begin{aligned}\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\ \mathrm{d}x \le\ & \min\Big\{f(a)\int_0^1 [h[(1-t)]+h(t)]\mathrm{d}t \\ & +\eta(f(b),f(a))\int_0^1 h(t)\mathrm{d}t, f(b)\int_0^1 [h[(1-t)]+h(t)]\mathrm{d}t \\ & +\eta(f(a),f(b))\int_0^1 h(t)\mathrm{d}t\Big\} \\ \le\ & [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t \\ & +\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t)\mathrm{d}t.\end{aligned}$$

Theorem 3.3. *Let* $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a generalized* (p, r, h, η)*-convex function. If* $f \in L[a, b]$*, then*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\le \min\Big\{[h(\frac{1}{2})]^{\frac{1}{r}}\left(\left[f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right]^r + \left[f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right.\right.$$
$$\left.\left.+\eta\left(f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right), f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right)\right]^r\right)^{\frac{1}{r}},$$
$$\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}\left(\left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right]^r + \left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right.\right.$$

$$+\eta\left(f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right),f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right)\Big]^r\Big)^{\frac{1}{r}}\Bigg\}.$$

Proof. Let f be a generalized (p,r,h,η)-convex function. Then, taking $x = [(1-t)a^p+tb^p]^{\frac{1}{p}}$ and $y=[ta^p+(1-t)b^p]^{\frac{1}{p}}$ in (4), we have

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} &\left(\left[f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right]^r\right. \\ &+\Big[f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right) \\ &\left.+\eta\left(f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right),f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right)\Big]^r\right)^{\frac{1}{r}}, \end{aligned}$$

and

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} &\left(\left[f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right]^r\right. \\ &+\Big[f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right) \\ &\left.+\eta\left(f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right),f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right)\Big]^r\right)^{\frac{1}{r}}. \end{aligned}$$

Thus,

$$\begin{aligned} &f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\ &\le \min\Bigg\{[h(\frac{1}{2})]^{\frac{1}{r}}\left(\left[f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right]^r+\Big[f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right. \\ &\quad\left.+\eta\left(f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right),f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right)\Big]^r\right)^{\frac{1}{r}} \\ &\quad\times\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}\left(\left[f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right]^r+\Big[f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right. \\ &\quad\left.+\eta\left(f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right),f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right)\Big]^r\right)^{\frac{1}{r}}\Bigg\} \end{aligned}$$

The required result. ∎

Corollary 3.4. *Under the assumptions of Theorem* 3.3 *with* $r=1$, *we have*

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \min\Bigg\{h(\frac{1}{2})&\Big[2f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right) \\ &+\eta\left(f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right),f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right)\Big], \end{aligned}$$

$$h\left(\frac{1}{2}\right)\left[2f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right.$$
$$\left.\left.+\eta\left(f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right), f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right)\right]\right\}.$$

Theorem 3.5. *Let $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a generalized (p, r, h, η)-convex function. If $f \in L[a, b]$, then*

$$\frac{2^{\frac{r-1}{r}}}{h\left(\frac{1}{2}\right)}\left(f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right)^r$$
$$-\left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)+\eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right), f(x)\right)}{x^{1-p}}\,\mathrm{d}x\right)^r$$
$$\leq \left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x\right)^r$$
$$\leq \frac{1}{2^r}\left([[f(a)]^r+[f(a)+\eta(f(b), f(a))]^r]^{\frac{1}{r}}\right.$$
$$\left.+[[f(b)]^r+[f(b)+\eta(f(a), f(b))]^r]^{\frac{1}{r}}\right)^r\left(\int_0^1 [h(t)]^{\frac{1}{r}}\,\mathrm{d}t\right)^r.$$

Proof. Let f be a generalized (p, r, h, η)-convex function. Then, from inequality (6) and Lemma 2.10, we have

$$\frac{2}{\left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}}} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq \left[\left(\int_0^1 f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\mathrm{d}t\right)^r+\left(\int_0^1 f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\mathrm{d}t\right)^r\right.$$
$$+\left(\int_0^1 f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\mathrm{d}t+\int_0^1 \eta\left(f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right),\right.\right.$$
$$\left.\left.f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right)\mathrm{d}t\right)^r+\left(\int_0^1 f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\mathrm{d}t\right.$$
$$\left.\left.+\int_0^1 \eta\left(f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right), f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right)\mathrm{d}t\right)^r\right]^{\frac{1}{r}}$$
$$=\left[2\left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x\right)^r\right.$$
$$\left.+2\left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)+\eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right), f(x)\right)}{x^{1-p}}\,\mathrm{d}x\right)^r\right]^{\frac{1}{r}}.$$

This implies

$$\begin{aligned}
&\frac{2^{\frac{r-1}{r}}}{h\left(\frac{1}{2}\right)}\left(f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right)^r \\
&-\left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)+\eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right),f(x)\right)}{x^{1-p}}\,\mathrm{d}x\right)^r \\
\leq\ &\left(\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x\right)^r \\
\leq\ &\frac{1}{2^r}\left([[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]^{\frac{1}{r}}\right. \\
&\left.+[[f(b)]^r+[f(b)+\eta(f(a),f(b))]^r]^{\frac{1}{r}}\right)^r\left(\int_0^1 [h(t)]^{\frac{1}{r}}\,\mathrm{d}t\right)^r
\end{aligned}$$

which is the required result. ∎

Corollary 3.6. *Under the assumptions of Theorem* 3.3 *with* $r=1$*, we have*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)-\frac{ab}{2(b-a)}\int_a^b \frac{\eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right),f(x)\right)}{x^{1-p}}\,\mathrm{d}x \\
\leq\ &\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x \\
\leq\ &[f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t)\mathrm{d}t.
\end{aligned}$$

One can also obtain the Hermite-Hadamard inequality for generalized (p,r,h,η)*-convex functions as:*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}f^r\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&-\frac{p}{b^p-a^p}\int_a^b \frac{\left[f(x)+\eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right),f(x)\right)\right]^r}{x^{1-p}}\,\mathrm{d}x \\
\leq\ &\frac{p}{b^p-a^p}\int_a^b \frac{f^r(x)}{x^{1-p}}\,\mathrm{d}x \\
\leq\ &[[f(a)]^r+[f(a)+\eta(f(b),f(a))]^r]\left(\int_0^1 [h(t)]^{\frac{1}{r}}\,\mathrm{d}t\right)^r.
\end{aligned}$$

We now obtain some Fejer type integral inequalities for generalized (p,r,h,η)-convex functions.

Theorem 3.7. *Let $f, g : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be generalized (p, r, h)-convex functions. If $fg \in L[a, b]$, then*

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^{1-p}}\,\mathrm{d}x \\
&\leq \frac{1}{2}\int_a^b \left[h\left(\frac{b^p - x^p}{b^p - a^p}\right)[f(a)]^r + h\left(\frac{x^p - a^p}{b^p - a^p}\right)[f(a) + \eta(f(b),\right. \\
&\quad \left. f(a))]^r]^{\frac{1}{r}}\right] \frac{g(x)}{x^{1-p}}\,\mathrm{d}x + \frac{1}{2}\int_a^b \left[h\left(\frac{b^p - x^p}{b^p - a^p}\right)[f(b)]^r\right. \\
&\quad \left. + h\left(\frac{x^p - a^p}{b^p - a^p}\right)[f(b) + \eta(f(a), f(b))]^r\right]^{\frac{1}{r}}\Bigg] \frac{g(x)}{x^{1-p}}\,\mathrm{d}x,
\end{aligned}$$

where $g : [a, b] \subset \mathbb{R}\backslash\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x) = g\left([a^p + b^p - x^p]^{\frac{1}{p}}\right), \quad \forall x \in [a, b].$$

Proof. Let f be a generalized (p, r, h, η)-convex function. Then multiplying inequality (5) with $g\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)$ and integrating over t, we have

$$\begin{aligned}
&\int_0^1 \left[f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right) + f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)\right] g\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t \\
\leq &\int_0^1 [h(1-t)[f(a)]^r + h(t)[f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}}\Big] g\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t \\
+ &\int_0^1 [h(1-t)[f(b)]^r + h(t)[f(b) + \eta(f(a), f(b))]^r]^{\frac{1}{r}}\Big] g\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^{1-p}}\,\mathrm{d}x \\
\leq &\frac{1}{2}\int_a^b \left[h\left(\frac{b^p - x^p}{b^p - a^p}\right)[f(a)]^r + h\left(\frac{x^p - a^p}{b^p - a^p}\right)[f(a) + \eta(f(b), f(a))]^r\right]^{\frac{1}{r}}\Bigg] \frac{g(x)}{x^{1-p}}\mathrm{d}x \\
+ &\frac{1}{2}\int_a^b \left[h\left(\frac{b^p - x^p}{b^p - a^p}\right)[f(b)]^r + h\left(\frac{x^p - a^p}{b^p - a^p}\right)[f(b) + \eta(f(a), f(b))]^r\right]^{\frac{1}{r}}\Bigg] \frac{g(x)}{x^{1-p}}\mathrm{d}x,
\end{aligned}$$

the required result. ∎

Corollary 3.8. *Under the assumptions of Theorem 3.1 with $r = 1$, we have*

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{x^{1-p}}\, \mathrm{d}x \\
\leq\ & \frac{f(a)+f(b)}{2} \int_a^b \left[h\left(\frac{b^p-x^p}{b^p-a^p}\right) + h\left(\frac{x^p-a^p}{b^p-a^p}\right)\right] \frac{g(x)}{x^{1-p}}\, \mathrm{d}x \\
&+ \frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2} \int_a^b h\left(\frac{x^p-a^p}{b^p-a^p}\right) \frac{g(x)}{x^{1-p}}\, \mathrm{d}x.
\end{aligned}$$

Theorem 3.9. *Let $f, g : I = [a,b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be generalized (p,r,h)-convex functions. If $fg \in L[a,b]$, then*

$$\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \int_a^b \frac{g(x)}{x^{1-p}}\, \mathrm{d}x \\
\leq\ & \int_a^b \frac{g(x)}{x^{1-p}} \min\left\{ \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left([f(x)]^r + \left[f(x) + \eta\left(f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right),\right.\right.\right.\right. \\
& \left.\left.\left. f(x))\right]^r\right)^{\frac{1}{r}}, \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left(\left[f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right)\right]^r \right.\\
& \left.\left. + \left[f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right) + \eta\left(f(x), f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right)\right)\right]^r\right)^{\frac{1}{r}} \right\} \mathrm{d}x.
\end{aligned}$$

where $g : [a,b] \subset \mathbb{R}\backslash\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x) = g\left([a^p+b^p-x^p]^{\frac{1}{p}}\right), \quad \forall x \in [a,b].$$

Proof. Let f, g be generalized (p,r,h)-convex functions. Then multiplying (6) with $g\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)$ and integrating over t, we have

$$\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \int_0^1 g\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right) \mathrm{d}t \\
\leq\ & \int_0^1 g\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right) \min\left\{ [h(\frac{1}{2})]^{\frac{1}{r}} \left(\left[f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right]^r \right.\right.\\
& + \left[f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right) + \eta\left(f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right),\right.\right. \\
& \left.\left.\left. f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)\right)\right]^r\right)^{\frac{1}{r}} \left[h\left(\frac{1}{2}\right)\right]^{\frac{1}{r}} \left(\left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right]^r \right.\\
& + \left[f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) + \eta\left(f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right),\right.\right. \\
& \left.\left.\left.\left. f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right)\right]^r\right)^{\frac{1}{r}} \right\} \mathrm{d}t.
\end{aligned}$$

By the symmetry of g on $[a, b]$, we have

$$
\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \int_a^b \frac{g(x)}{x^{1-p}}\, \mathrm{d}x \\
\leq & \int_a^b \frac{g(x)}{x^{1-p}} \min \left\{ [h(\frac{1}{2})]^{\frac{1}{r}} \left([f(x)]^r + \left[f(x) + \eta \left(f \left([a^p+b^p-x^p]^{\frac{1}{p}} \right) \right. \right. \right. \right. , \\
& \left. \left. \left. f(x) \right) \right]^r \right)^{\frac{1}{r}} \left[h\left(\frac{1}{2}\right) \right]^{\frac{1}{r}} \left(\left[f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) \right]^r \right. \\
& \left. \left. + \left[f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) + \eta \left(f(x), f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) \right) \right]^r \right)^{\frac{1}{r}} \right\} \mathrm{d}x,
\end{aligned}
$$

which is the required result. ■

Corollary 3.10. *Under the assumptions of Theorem* 3.5 *with* $r = 1$*, we have*

$$
\begin{aligned}
& \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \int_a^b \frac{g(x)}{x^{1-p}}\, \mathrm{d}x \\
\leq & \int_a^b \frac{g(x)}{x^{1-p}} \min \left\{ f(x) + \frac{1}{2}\eta \left(f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right), f(x) \right), \right. \\
& \left. f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) + \frac{1}{2}\eta \left(f(x), f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) \right) \right\} \mathrm{d}x \\
\leq & \int_a^b \frac{f(x)g(x)}{x^{1-p}}\, \mathrm{d}x + \frac{1}{2} \int_a^b \frac{g(x)}{x^{1-p}} \left[\eta \left(f(x), f\left([a^p+b^p-x^p]^{\frac{1}{p}} \right) \right) \right] \mathrm{d}x.
\end{aligned}
$$

Theorem 3.11. *Let* $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a generalized* (p, r, η)*-convex function. If* $f \in L[a, b]$*, then*

$$
\begin{aligned}
& \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}}\, \mathrm{d}x \\
\leq & \begin{cases}
\frac{r}{r+1} \left(\frac{[f(a)]^{r+1} - [f(a)+\eta(f(b),f(a))]^{r+1}}{f f(a)]^r - [f(a)+\eta(f(b),f(a))]^r} \right) & r \neq \{-1, 0\}, f(a) \neq f(b) \\
\frac{\eta(f(b),f(a))}{\ln[f(a)+\eta(f(b),f(a)] - \ln f(a)} & r = 0, f(a) \neq f(b) \\
f(a)[f(a)+\eta(f(b), f(a))] \frac{\ln[f(a)+\eta(f(b),f(a))] - \ln f(a)}{\eta(f(b),f(a))} & r = -1, f(a) \neq f(b) \\
f(a) & f(a) = f(b).
\end{cases}
\end{aligned}
$$

Proof.

$$
f\left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) \leq [(1-t)[f(a)]^r + [f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}},
$$

Case 1. $r \neq \{-1, 0\}, f(a) \neq f(b)$.

$$\begin{aligned}\frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}\, \mathrm{d}x &= \int_0^1 f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\leq \int_0^1 [(1-t)[f(a)]^r + t[f(a) + \eta(f(b), f(a))]^r]^{\frac{1}{r}}\ \mathrm{d}t \\ &= \frac{r}{r+1}\left(\frac{[f(a)]^{r+1} - [f(a) + \eta(f(b), f(a))]^{r+1}}{[f(a)]^r - [f(a) + \eta(f(b), f(a))]^r}\right).\end{aligned}$$

Case 2. $r = 0, f(a) \neq f(b)$.

$$\begin{aligned}\frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}\, \mathrm{d}x &= \int_0^1 f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\leq \int_0^1 [f(a)]^{1-t} + [f(a) + \eta(f(b), f(a))]^t\ \mathrm{d}t \\ &= \frac{\eta(f(b), f(a))}{\ln[f(a) + \eta(f(b), f(a))] - \ln f(a)}.\end{aligned}$$

Case 3. $r = -1, f(a) \neq f(b)$.

$$\begin{aligned}&\frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}\, \mathrm{d}x \\ &= \int_0^1 f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\leq \int_0^1 \left[(1-t)[f(a)]^{-1} + t[f(a) + \eta(f(b), f(a))]^{-1}\right]^{-1}\ \mathrm{d}t \\ &= \frac{f(a)[f(a) + \eta(f(b), f(a))]}{\eta(f(b), f(a))}\int_{f(a)}^{f(a)+\eta(f(b),f(a))} \frac{1}{u}\ \mathrm{d}u \\ &= f(a)[f(a) + \eta(f(b), f(a))]\frac{\ln[f(a) + \eta(f(b), f(a))] - \ln f(a)}{\eta(f(b), f(a))}.\end{aligned}$$

■

4. Conclusion and Future Research

In this paper we, have introduced and studied a new class of generalized convex functions involving an arbitrary function h and bifunction $\eta(.,.)$. Several new integral inequalities for this class of functions are obtained. Applications of our results are discussed. Result of this paper can be extended for the new class. To be more precise, let $h : J = [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : I = [a, b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be generalized (p, r, h, ζ)-convex function, if

$$\begin{aligned}&f\left([(1-t)x^p + tb^p]^{\frac{1}{p}}\right) \\ &= \begin{cases} [h(1-t)[f(x)]^r + h(t)[f(x) + \zeta(f(y) - f(x))]^r]^{\frac{1}{r}} & \text{if } r \neq 0, \\ (f(x))^{h(1-t)}(f(x) + \eta(f(y), f(x)))^{h(t)} & \text{if } r = 0. \end{cases}\end{aligned}$$

where $\zeta(f(y) - f(x)$ is a bifunction $\zeta(.,.) : H \times H \to R$. Note that the bifunctions $\eta(f(y), f(x)$ and $\zeta(f(y) - f(x))$ are quite different from other. In fact $\eta(f(y), f(x) \neq \zeta(f(y) - f(x))$. The ideas and techniques of this paper may stimulate further research.

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