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An Investigation on Coannihilators and Coannulets of a Residuated Lattice

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Abstract. The notions of coannihilator and coannulet in a residuated lattice are investigated. For a residuated lattice $\mathfrak{A}$, it is shown that $\gamma(\mathfrak{A})$, the set of coannulets of $\mathfrak{A}$, is a sublattice of $\Gamma(\mathfrak{A})$, the Boolean lattice of coannihilators of $\mathfrak{A}$. It is observed that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\Gamma(\mathfrak{A})$ if and only if $\mathfrak{A}$ is quasicomplemented and $\gamma(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$, the filter lattice of $\mathfrak{A}$, if and only if $\mathfrak{A}$ is normal. Finally, it is shown that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is generalized Stone. During this research, some facts about coannihilators, coannulets and dual coannulets of a residuated lattice are also obtained which are given in the paper.

Keywords: Quasicomplemented residuated lattice; Normal residuated lattice; Generalized Stone residuated lattice; Coannihilator; Coannulet.

1. Introduction

In the ring theory the annihilator of a set is a concept generalizing torsion and orthogonality. Also, Baer rings and Rickart rings are various attempts to give an algebraic analogue of Von Neumann algebras, using axioms about annihilators of various sets. The theory of relative annihilators was introduced in lattices by Mandelker who characterized distributive lattices in terms of their relative annihilators in [17]. Later, many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. Speed [28] and Cornish [2, 4] made an extensive study on annihilators in distributive lattices. Filipoiu [6] and Leustean [15] used this notion for Baer extensions of MV-algebras and BL-algebras, respectively. Halaš [11] and Kondo [14] applied this notion to study of BCK algebras. Recently, Rasouli [18] generalized this notion for residuated lattices and studied its properties. This work is motivated by the above works and a desire to extend these investigations to residuated lattices. Our findings show that the results obtained by [2, 4] can also be reproduced via residuated lattices.

This paper is organized in four sections. In Section 2, some definitions and facts about residuated lattices that we use in the sequel are recalled. In Section 3, the notion of coannihilators, coannulets and dual coannulets are introduced and investigated. It is established a connection between coannihilators and Galois connection theory and coannihilators are characterized by means of minimal prime filters and it is observed that any prime coannihilator is minimal prime. For a residuated lattice $\mathfrak{A}$, it is shown that $\gamma(\mathfrak{A})$, the set of coannulets of $\mathfrak{A}$, and $\lambda(\mathfrak{A})$ is a sublattice of $\Gamma(\mathfrak{A})$, the Boolean lattice of coannihilators of $\mathfrak{A}$. This section ends with a kind of Chinese remainder theorem for the lattice of coannulets. In Section 4, the notions of quasicomplemented, normal and generalized Stone residuated lattices are introduced and characterized. It is observed that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\Gamma(\mathfrak{A})$ if and only if $\mathfrak{A}$ is quasicomplemented, $\gamma(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is normal and $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is generalized Stone.

2. A Brief Excursion into Residuated Lattices

In this section, we recall some definitions, properties, and results relative to residuated lattices, which will be used in the following.

An algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *residuated lattice* if $\ell(\mathfrak{A}) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice, $(A; \odot, 1)$ is a commutative monoid and $(\odot, \rightarrow)$ is an adjoint pair. A residuated lattice $\mathfrak{A}$ is called an *MTL algebra* [7] if it satisfies the *pre-linearity condition* (denoted by pprel):

(*prel*) $(x \rightarrow y) \vee (y \rightarrow x) = 1.$

In a residuated lattice $\mathfrak{A}$, for any $a \in A$, we put $\neg a := a \rightarrow 0$. It is well-known that the class of residuated lattices is equational [12], and so it forms a variety. For a survey of residuated lattices we refer to [8].

Remark 2.1. [1, Proposition 2.6] Let $\mathfrak{A}$ be a residuated lattice. The following conditions are satisfied for any $x, y, z \in A$:

(i) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(ii) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Example 2.2. Let $B_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 1. Routine calculation shows that $\mathfrak{B}_6 = (B_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, in which the commutative operation "$\odot$" is given by Table 1, and the operation "$\rightarrow$" is defined by $x \rightarrow y = \bigvee\{a \in B_6 | x \odot a \leq y\}$, for any $x, y \in B_6$.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
	a	a	0	a	0	a
		b	0	0	b	b
			c	a	b	c
				d	d	d
					1	1

Table 1: Cayley table for "$\odot$"

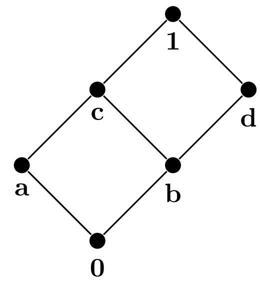

Figure 1

Example 2.3. Let $B_7 = \{0, a, b, c, d, e, 1\}$ be a lattice whose Hasse diagram is given by Figure 2. Routine calculation shows that $\mathfrak{B}_7 = (B_7; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, in which the commutative operation "$\odot$" is given by Table 2, and the operation " $\rightarrow$ is defined by $x \rightarrow y = \bigvee\{a \in B_7 | x \odot a \leq y\}$, for any $x, y \in B_7$.

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
	a	a	0	a	a	a	a
		b	b	b	b	b	b
			c	c	c	c	c
				d	d	c	d
					e	e	e
						1	1

Table 2: The Cayley table for "$\odot$" of $\mathfrak{B}_7$

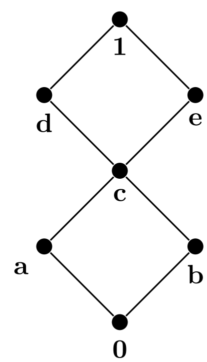

Figure 2

Let $\mathfrak{A}$ be a residuated lattice. A non-void subset F of A is called a *filter* of $\mathfrak{A}$ if $x, y \in F$ implies $x \odot y \in F$ and $x \vee y \in F$ for any $x \in F$ and $y \in A$. The set of filters of $\mathfrak{A}$ is denoted by $\mathcal{F}(\mathfrak{A})$. A filter F of $\mathfrak{A}$ is called *proper* if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$. For any subset X of A the *filter of* $\mathfrak{A}$ *generated by* X is denoted by $\mathcal{F}(X)$. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $\mathcal{F}(x)$ and it is called *principal filter*. The set of principal filters is denoted by $\mathcal{PF}(\mathfrak{A})$. Let $\mathcal{F}$ be a collection of filters of $\mathfrak{A}$. Set $\veebar\mathcal{F} = \mathcal{F}(\cup\mathcal{F})$. For the basic facts concerning filters of a residuated lattice we refer to [24, 26, 20]. According to [8], $(\mathcal{F}(\mathfrak{A}); \cap, \veebar, \{1\}, A)$ is a frame and so it is a complete Heyting algebra.

Example 2.4. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.2 and the residuated lattice $\mathfrak{B}_7$ from Example 2.3. The set of their filters is presented in Table 3.

	filters
$\mathfrak{B}_6$	$\{1\}, \{a, c, 1\}, \{d, 1\}, B_6$
$\mathfrak{B}_7$	$\{1\}, \{d, 1\}, \{e, 1\}, \{c, d, e, 1\}, \{a, c, d, e, 1\}, \{b, c, d, e, 1\}, B_7$

Table 3

The following proposition has a routine verification.

Proposition 2.5. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x, y \in A$:*

(1) $\mathcal{F}(F, x) := F \veebar \mathcal{F}(x) = \{a \in A | f \odot x^n \leq a, \exists f \in F \wedge \exists n \in \mathbb{N}\}$;
(2) $x \leq y$ *implies* $\mathcal{F}(F, y) \subseteq \mathcal{F}(F, x)$.
(3) $\mathcal{F}(F, x) \cap \mathcal{F}(F, y) = \mathcal{F}(F, x \vee y)$;
(4) $\mathcal{F}(x) \veebar \mathcal{F}(y) = \mathcal{F}(x \odot y)$;
(5) $\mathcal{PF}(\mathfrak{A})$ *is a sublattice of* $\mathcal{F}(\mathfrak{A})$.

The following lemma is an important property of principal filters of residuated lattices which must be compared with its lattice version [10, Lemma 105].

Lemma 2.6. *Let F, G and H be filters of a residuated lattice $\mathfrak{A}$. If $G \cap H = \mathcal{F}(F, x)$ and $G \veebar H = \mathcal{F}(F, y)$, then there exist $u, w \in A$ such that $G = \mathcal{F}(F, u)$ and $H = \mathcal{F}(F, w)$.*

Proof. $y \in G \veebar H$ implies that $g \odot h \leq y$, for some $g \in G$ and $h \in H$. So $\mathcal{F}(F, y) \subseteq \mathcal{F}(F, g) \veebar \mathcal{F}(F, h) \subseteq G \veebar H = \mathcal{F}(F, y)$ and this shows that $\mathcal{F}(F, y) = \mathcal{F}(F, g) \veebar \mathcal{F}(F, h)$. So, for any $z \in G$, we have $z \in \mathcal{F}(F, g) \veebar (\mathcal{F}(F, h) \cap \mathcal{F}(F, z)) \subseteq \mathcal{F}(F, g \odot x)$ and this state that $G \subseteq \mathcal{F}(F, g \odot x)$. The inverse inclusion is evident and so $G = \mathcal{F}(F, g \odot x)$. By symmetry, we can obtain the other case. ∎

A proper filter of a residuated lattice $\mathfrak{A}$ is called *maximal* if it is a maximal element in the set of all proper filters. The set of all maximal filters of $\mathfrak{A}$ is denoted by $Max(\mathfrak{A})$. A proper filter P of $\mathfrak{A}$ is called *prime*, if for any $x, y \in A$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. The set of all prime filters of $\mathfrak{A}$ is denoted by

$Spec(\mathfrak{A})$. Since $\mathcal{F}(\mathfrak{A})$ is a distributive lattice, so $Max(\mathfrak{A}) \subseteq Spec(\mathfrak{A})$. By Zorn's lemma follows that any proper filter is contained in a maximal filter and so in a prime filter.

A non-empty subset $\mathcal{C}$ of $\mathfrak{A}$ is called $\vee$-*closed* if it is closed under the join operation, *i.e.* $x, y \in \mathcal{C}$ implies $x \vee y \in \mathcal{C}$.

Theorem 2.7. [23, Theorem 3.18] *If $\mathcal{C}$ is a $\vee$-closed subset of $\mathfrak{A}$ which does not meet the filter F, then F is contained in a filter P which is maximal with respect to the property of not meeting $\mathcal{C}$; furthermore P is prime.*

Let $\mathfrak{A}$ be a residuated lattice and X be a subset of A. A prime filter P is called a *minimal prime filter belonging to* X or X-*minimal prime filter* if P is a minimal element in the set of prime filters containing X. The set of X-minimal prime filters of $\mathfrak{A}$ is denoted by $Min_X(\mathfrak{A})$. A prime filter P is called a *minimal prime* if $P \in Min_{\{1\}}(\mathfrak{A})$. The set of minimal prime filters of $\mathfrak{A}$ is denoted by $Min(\mathfrak{A})$.

Example 2.8. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.2 and the residuated lattice $\mathfrak{B}_7$ from Example 2.3. The set of their maximal, prime and minimal prime filters is presented in Table 4.

	prime filters	
	Maximal	**Minimal**
$\mathfrak{B}_6$	{d,1},{a,c,1}	{d,1},{a,c,1}
$\mathfrak{B}_7$	{a,c,d,e,1},{b,c,d,e,1}	{d,1},{e,1}

Table 4

The following proposition states some properties of minimal prime filters.

Proposition 2.9. [27, Corollary 2.8] *Let F be a filter of a residuated lattice $\mathfrak{A}$ and X be a subset of A. The following assertions hold:*

(1) *If $X \nsubseteq F$, there exists an F-minimal prime filter $\mathfrak{m}$ such that $X \nsubseteq \mathfrak{m}$;*
(2) $\mathcal{F}(X) = \bigcap Min_X(\mathfrak{A})$.

3. Coannihilators and Coannulets

In this section, the notions of coannihilator and coannulet in a residuated lattice are recalled and investigated. The results in the this section are original, excepting those that we cite from other papers. For the basic facts concerning coannihilators of a residuated lattice we refer to [18].

Definition 3.1. [18] *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. For any subset X of A the coannihilator of X belonging to F (or, F-coannihilator of X) is denoted by $(F : X)$ and defined as follow:*

$$(F : X) = \{a \in A | x \vee a \in F, \forall x \in X\}.$$

If $X = \{x\}$, we write $(F : x)$ instead of $(F : X)$.

Example 3.2. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.2. Let $F = \{a, c, 1\}$. We have $(F : 0) = (F : b) = (F : d) = F$ and $(F : a) = (F : c) = (F : 1) = B_6$.

Example 3.3. Consider the residuated lattice $\mathfrak{B}_7$ from Example 2.3. Let $F = \{d, 1\}$. We have $(F : 0) = (F : a) = (F : b) = (F : c) = (F : e) = F$ and $(F : d) = (F : 1) = B_7$.

Proposition 3.4. [18, Proposition 3.1] *Let $\mathfrak{A}$ be a residuated lattice. The following assertions hold for any $X, Y \subseteq A$ and $F, G \in \mathcal{F}(\mathfrak{A})$:*

(1) *$X \subseteq (F : Y)$ implies $Y \subseteq (F : X)$;*
(2) *$F \subseteq (F : X)$;*
(3) *$F \subseteq G$ implies $(F : X) \subseteq (G : X)$;*
(4) *$(F : X) = A$ if and only if $X \subseteq F$. In particular, $(F : \emptyset) = (F : 1) = (F : F) = A$;*
(5) *$X \cap (F : X) \subseteq F$;*
(6) *if $F \subseteq X$, then $X \cap (F : X) = F$. In particular, $(F : A) = F$;*
(7) *$(F : X) \cap (F : (F : X)) = F$;*
(8) *$((F : X) : Y) = ((F : Y) : X) = (F : X \vee Y)$, where $X \vee Y = \{x \vee y | x \in X, y \in Y\}$;*
(9) *$(F : X)$ is a filter of $\mathfrak{A}$.*

We know that the set of filters of a residuated lattice forms a Heyting algebra. The next proposition characterizes relative pseudocomplements of this Heyting algebra in terms of coannihilators.

Proposition 3.5. *Let $\mathfrak{A}$ be a residuated lattice and F, G be two filters of $\mathfrak{A}$. Then $(F : G)$ is the relative pseudocomplement of G with respect to F.*

Proof. By Proposition 3.4(5), we have $G \cap (F : G) \subseteq F$. Now, assume that $G \cap H \subseteq F$. Let $a \in H$ and $b \in G$. Since $a, b \leq a \vee b$ so $a \vee b \in G \cap H \subseteq F$. Thus $a \in (F : G)$ and it shows that $H \subseteq (F : G)$. ■

We recall that a bounded lattice $(A; \wedge, \vee, 0, 1)$ is a Heyting algebra if for all $x, y \in A$, there exists the relative pseudocomplement of x with respect to y which it is denoted by $x \to y$. For any $x \in A$, we set $\neg x = x \to 0$. An element of A is called *regular* if $\neg\neg x = x$. It is well-known that a Heyting algebra is a Boolean algebra if and only if each its element is regular.

In the following, If F is a filter of a residuated lattice $\mathfrak{A}$, then $\mathcal{F}_F(\mathfrak{A})$ stands for the set of filters of $\mathfrak{A}$ containing F.

Corollary 3.6. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold:*

(1) *$(\mathcal{F}_F(\mathfrak{A}); \cap, \veebar, F, A)$ is a Heyting algebra;*
(2) *$\mathcal{F}_F(\mathfrak{A})$ is a Boolean algebra if and only if $(F : (F : G)) = G$, for any $G \in \mathcal{F}_F(\mathfrak{A})$.*

Proof. (1) It is obvious that $(\mathcal{F}_F(\mathfrak{A}); \cap, \veebar, F, A)$ is a bounded lattice. Let $G, H \in \mathcal{F}_F(\mathfrak{A})$. Applying Proposition 3.4, it follows that $(G : H) \in \mathcal{F}_F(\mathfrak{A})$. Also, Proposition 3.5 implies that $\mathcal{F}_F(\mathfrak{A})$ is closed under relative pseudo-complements and so it is a Heyting algebra.

(2) It follows by Cor. 3.6. ∎

Let $\mathcal{A} = (A; \leq)$ and $\mathcal{B} = (B; \preccurlyeq)$ be two posts. Recalling that, a pair (f, g) is called a *(contravariant or antitone) Galois connection* between posets $\mathcal{A}$ and $\mathcal{B}$, where $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are two functions such that for all $a \in A$ and $b \in B$, $a \leq g(b)$ if and only if $b \preccurlyeq f(a)$. It is well known that (f, g) is a Galois connection if and only if gf, fg are inflationary and f, g are antitone [9, Theorem 2]. In the following, if Cl is a closure operator on a set X, the set of closed elements of Cl shall be denoted by $\mathcal{C}_{Cl}$.

Proposition 3.7.*[9] Let (f, g) be a Galois connection between posets $\mathcal{A}$ and $\mathcal{B}$. The following assertions hold:*

(1) *$fgf = f$ and $gfg = g$;*
(2) *if $\vee X$ exists for some $X \subseteq A$ then $\wedge f(X)$ exists and $\wedge f(X) = f(\vee X)$;*
(3) *gf is a closure operator on $\mathcal{A}$ and $\mathcal{C}_{gf} = g(B)$.*

Proposition 3.8. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Consider the following map:*

$$\begin{aligned} \mathcal{C}_F : \mathcal{P}(A) &\longrightarrow \mathcal{P}(A) \\ X &\longmapsto (F : X). \end{aligned}$$

Then, the pair $(\mathcal{C}_F, \mathcal{C}_F)$ is a Galois connection on $\mathcal{P}(A)$.

Proof. It follows obviously from Proposition 3.4. ∎

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Set $\Gamma_F(\mathfrak{A}) = \{(F : X) | X \subseteq A\}$.

Corollary 3.9. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. For any $X, Y \subseteq A$ the following assertions hold:*

(1) $X \subseteq (F : (F : X))$;
(2) $X \subseteq Y$ *implies* $(F : Y) \subseteq (F : X)$;
(3) $(F : X) = (F : (F : (F : X)))$;
(4) $\mathcal{C}_F\mathcal{C}_F$ *is a closure operator on* $\mathcal{P}(A)$ *and* $\mathcal{C}_{\mathcal{C}_F\mathcal{C}_F} = \Gamma_F(\mathfrak{A})$.

Proof. It is a direct consequence of Props. 3.7 and 3.8. ■

As an application of Proposition 3.8, the next proposition gives some properties of coannihilators.

Proposition 3.10. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $X \subseteq A$:*

(1) $(F : \cup_{i\in I} X_i) = \cap_{i\in I}(F : X_i)$, *for any* $\{X_i\}_{i\in I} \subseteq \mathcal{P}(A)$;
(2) $(F : X) = \cap_{x\in X}(F : x)$;
(3) *for any filter* $G \subseteq F$ *we have* $(F : \mathcal{F}(G, X)) = (F : X)$;
(4) $(F : \mathcal{F}(X)) = (F : X)$. *In particular,* $(F : 0) = F$;
(5) *for any filter* $G \subseteq F$ *we have* $\mathcal{F}(G, X) \cap (F : X) \subseteq F$;
(6) $\mathcal{F}(F, X) \cap (F : X) = F$;
(7) $(F : X) = (F : X - F)$.

Proof. (1) It is straightforward by Props. 3.7 and 3.8.

(2) By taking $X = \cup_{x\in X}\{x\}$ it follows by Prop. 3.10.

(3) Let G be a filter contained in F. Applying Proposition 3.9, it follows that $(F : \mathcal{F}(G, X)) \subseteq (F : X)$. Assume that $a \in (F : X)$. By Proposition 3.4, we obtain that $G \cup X \subseteq (F : a)$. Since $(F : a)$ is a filter it states that $\mathcal{F}(G, X) \subseteq (F : a)$. Thus we have $a \in (F : \mathcal{F}(G, X))$.

(4) By taking $G = \{1\}$ it follows by Props. 2.5 and 3.4. By Prop. 3.4 it follows that $(F : 0) = (F : \mathcal{F}(0)) = (F : A) = F$.

(5) It follows by Props. 3.4 and 3.10.

(6) It follows by Props. 3.4 and 3.10.

(7) By Proposition 3.4, we have $(F : X) = (F : (X - F) \cup (X \cap F)) = (F : X - F) \cap (F : X \cap F)$. Also, by Proposition 3.4, we have $(F : X \cap F) = A$. It states that $(F : X) = (F : X - F)$. ■

Proposition 3.11. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. We have*

$$\Gamma_F(\mathfrak{A}) = \{(F : G) | F \subseteq G \in \mathcal{F}(\mathfrak{A})\}.$$

Proof. It is obvious that $\{(F : G) | F \subseteq G \in \mathcal{F}(\mathfrak{A})\} \subseteq \Gamma_F(\mathfrak{A})$. Let $H = (F : X)$ for some $X \subseteq A$. By Corollary 3.9 follows $(F : (F : H)) = H$ and by Proposition

3.4 follows $F \subseteq (F : H) \in \mathcal{F}(\mathfrak{A})$. It shows that $\Gamma_F(\mathfrak{A}) \subseteq \{(F : G) | G \in \mathcal{F}_F(\mathfrak{A})\}$. ■

Example 3.12. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.2. By notations of Example 3.2, we have $\Gamma_F(\mathfrak{B}_6) = \{F, B_6\}$.

Example 3.13. Consider the residuated lattice $\mathfrak{B}_7$ from Example 2.3. By notations of Example 3.3, we have $\Gamma_F(\mathfrak{B}_7) = \{F, B_7\}$.

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. By [18, Proposition 3.13] follows that $(\Gamma_F(\mathfrak{A}); \cap, \vee^{\Gamma_F}, F, A)$ is a complete Boolean lattice, where for any $\mathcal{G} \subseteq \Gamma_F(\mathfrak{A})$ we have $\vee^{\Gamma_F}\mathcal{G} = (F : (F : \cup\mathcal{G}))$.

The following proposition characterizes coannihilators by means of minimal prime filters.

Proposition 3.14. [25, Proposition 2.10] *Let Π be a collection of prime filters in a residuated lattice $\mathfrak{A}$. For any subset X of A, we have*

$$(\bigcap \Pi : X) = \bigcap\{P \in \Pi |\ X \nsubseteq P\}.$$

Corollary 3.15. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. For any subset X of A, we have*

$$(F : X) = \bigcap\{\mathfrak{m} \in Min_F(\mathfrak{A}) |\ X \nsubseteq \mathfrak{m}\}.$$

Proof. It follows by Props. 2.9 and 3.14. ■

Corollary 3.16. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. We have*

$$Spec(\mathfrak{A}) \cap \Gamma_F(\mathfrak{A}) \subseteq Min_F(\mathfrak{A}).$$

Proof. It is evident by Corollary 3.15. ■

Proposition 3.17. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If P is a prime filter of $\mathfrak{A}$ containing F, then either $(F : (F : P)) = A$ or $P \in \Gamma_F(\mathfrak{A})$.*

Proof. Let $(F : (F : P)) \neq A$. By Proposition 3.4 follows that $(F : P) \nsubseteq F$. Consider $x \in (F : (F : P)) \setminus P$. Then $x \vee y \in F$, for any $y \in (F : P)$. As $x \notin P$, we have $y \in P$. However, Proposition 3.4 yields that $y \in F$. Consequently, $(F : P) \subseteq F$; a contradiction. The rest follows immediately from Corollary 3.16. ■

Theorem 3.18. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If for a prime filter P which containing F we have $(F:(F:P)) \neq A$, then P is an F-minimal prime filter.*

Proof. It follows by Corollary 3.16 and Proposition 3.17. ∎

Proposition 3.19. [18, Proposition 3.15] *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x, y \in A$:*
(1) $x \leq y$ *implies* $(F:x) \subseteq (F:y)$;
(2) $(F:x) \cap (F:y) = (F:x \odot y)$;
(3) $(F:x) \veebar (F:y) \subseteq (F:x) \vee^{\Gamma_F} (F:y) = (F:x \vee y)$;
(4) $(F:(F:x)) \cap (F:(F:y)) = (F:(F:x \vee y))$;
(5) $(F:(F:x)) \vee^{\Gamma_F} (F:(F:y)) = (F:(F:x \odot y))$.

Let $\mathfrak{A}$ be a residuated lattice. We set $\gamma_F(\mathfrak{A}) = \{(F:x) | x \in A\}$ and $\lambda_F = \{(F:(F:x)) : x \in A\}$. The elements of $\gamma_F(\mathfrak{A})$ are called *F-coannulets* of $\mathfrak{A}$ and the elements of $\lambda_F(\mathfrak{A})$ are called *dual F-coannulets* of $\mathfrak{A}$.

Corollary 3.20. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then $\gamma_F(\mathfrak{A})$ and $\lambda_F(\mathfrak{A})$ are sublattices of $\Gamma_F(\mathfrak{A})$.*

Proof. It follows by Proposition 3.19. ∎

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. A subset X of A is called *F-dense* if $(F:X) = F$. The set of all F-dense elements of $\mathfrak{A}$ shall be denoted by $\mathfrak{de}_F(\mathfrak{A})$. By Proposition 3.19 follows that $\mathfrak{de}_F(\mathfrak{A})$ is an ideal of $\ell(\mathfrak{A})$. Also, by Props. 3.4 and 3.10 follows that a filter of $\mathfrak{A}$ is F-dense provided that it contains an F-dense element.

Proposition 3.21. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then any non F-dense prime filter of $\mathfrak{A}$ containing F is an F-coannulet.*

Proof. Let P be a non F-dense prime filter containing F. So $(F:P) \neq F$. Let $x \in (F:P) \setminus F$. Thus $P \subseteq (F:(F:P)) \subseteq (F:x)$. Otherwise, $y \in (F:x)$ implies that $x \vee y \in F \subseteq P$. But $x \notin P$, since $x \in P$ states that $x \in P \cap (F:P) = F$ and so $x \in F$; a contradiction. So $y \in P$ and it shows that $P = (F:x)$. ∎

We end this section with a kind of Chinese remainder theorem for coannulets, inspired by the one obtained for bounded distributive lattices by [3, Proposition 3.6].

Proposition 3.22. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$ and $a_1, \cdots, a_n, x_1, \cdots, x_n \in A$. Suppose that $(F:(F:a_1)) \veebar \cdots \veebar (F:(F:a_1)) = A$*

and $x_i \equiv_{(F:a_i \vee a_j)} x_j$ *for all* $i, j \in \{1, \cdots, n\}$*. Then there exist* $u, v \in A$ *such that* $u \to x_i, x_i \to v \in (F : a_i)$ *for all* $i \in \{1, \cdots, n\}$*.*

Proof. Since $(F : (F : a_1)) \veebar \cdots \veebar (F : (F : a_1)) = A$, so for all $i \in \{1, \cdots, n\}$ there exist $y_i \in (F : (F : a_i))$ such that $y_1 \odot \cdots \odot y_n = 0$. For all $i, j \in \{1, \cdots, n\}$ we set $u_{ij} = x_i \to x_j$. By hypothesis, we have $u_{ij} \in (F : a_i \vee a_j)$. It implies that $u_{ij} \vee a_i \in (F : a_j) \subseteq (F : y_j)$ and it states that $u_{ij} \vee y_j \in (F : a_i)$. Analogously, we deduce that $u_{ij} \vee y_i \in (F : a_j)$. Set $u = (x_1 \vee y_1) \odot \cdots \odot (x_n \vee y_n)$ and $v = (x_1 \vee y_1) \wedge \cdots \wedge (x_n \vee y_n)$. In a similar manner of [16, Proposition 2.2.17] we obtain that $u \to x_i \geq (u_{1i} \vee y_1) \odot (u_{2i} \vee y_2) \odot \cdots \odot (u_{ni} \vee y_n) \in (F : a_i)$ and $x_i \to v \in (u_{i1} \vee y_1) \odot (u_{i2} \vee y_2) \odot \cdots \odot (u_{in} \vee y_n) \in (F : a_i)$ for all $i \in \{1, \cdots, n\}$. ■

4. The Lattice of Coannulets

In this section, the notions of quasicomplemented, normal and generalized Stone residuated lattice with respect to a filter are introduced and investigated. For a residuated lattice $\mathfrak{A}$ and a filter F of $\mathfrak{A}$, it is shown that $\gamma_F(\mathfrak{A})$ is a Boolean sublattice of $\Gamma_F(\mathfrak{A})$ if and only if $\mathfrak{A}$ is F-quasicomplemented and $\gamma_F(\mathfrak{A})$ is a sublattice of $\mathcal{F}_F(\mathfrak{A})$ if and only if $\mathfrak{A}$ is F-normal. Furthermore, it is observed that $\gamma_F(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}_F(\mathfrak{A})$ if and only if $\mathfrak{A}$ is generalized F-Stone. For the basic facts concerning quasicomplemented residuated lattices we refer to [21, 22] and for the basic facts concerning normal residuated lattices we refer to [27, 19].

Definition 4.1. *Let* $\mathfrak{A}$ *be a residuated lattice and* F *be a filter of* $\mathfrak{A}$*.* $\mathfrak{A}$ *is called quasicomplemented with respect to* F *(or,* F*-quasicomplemented) provided that for any* $x \in A$*, there exists* $y \in A$ *such that* $(F : (F : x)) = (F : y)$*, i.e.* $\lambda_F(\mathfrak{A}) \subseteq \gamma_F(\mathfrak{A})$*.*

Remark 4.2. Following by [21, Definition 3.1], a residuated lattice $\mathfrak{A}$ is quasicomplemented if it is quasicomplemented with respect to $\{1\}$.

Proposition 4.3. *Let* $\mathfrak{A}$ *be a residuated lattice and* F *be a filter of* $\mathfrak{A}$*. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is* F*-quasicomplemented;*
(2) *for any* $x \in A$*, there exists* $y \in A$ *such that* $x \odot y \in \mathfrak{de}_F(\mathfrak{A})$ *and* $x \vee y \in F$*;*
(3) $\gamma_F(\mathfrak{A})$ *is a Boolean lattice.*

Proof. (1)⇒(2): Consider $x \in A$. So there exists $y \in A$ such that $(F : (F : x)) = (F : y)$. Applying Propositions 3.4 and 3.19, it follows that $x \odot y$ is an F-dense element and Proposition 3.9 shows that $x \in (F : (F : x)) = (F : y)$ and so $x \vee y \in F$.

(2)⇒(3): By applying Proposition 3.19, it is evident.

(3)⇒(1): Let $x \in A$. So there exists some $y \in A$ such that $(F : x) \cap (F : y) = F$ and $(F : x) \vee^{\Gamma_F} (F : y) = A$. Applying Proposition 3.5, the former states $(F : y) \subseteq (F : (F : x))$, and the latter states the reverse inclusion. ∎

The following proposition derives a sufficient condition for a residuated lattice to become quasicomplemented.

Proposition 4.4. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. $\mathfrak{A}$ is F-quasicomplemented provided that $\gamma_F(\mathfrak{A}) \subseteq \mathcal{PF}_F(\mathfrak{A})$.*

Proof. Consider $x \in A$. So there exist $y \in A$ such that $(F : x) = \mathcal{F}(F, y)$. Using Proposition 3.10, it follows that $(F : (F : x)) = (F : y)$ and so $\mathfrak{A}$ is F-quasicomplemented. ∎

Corollary 4.5. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}/F$ is a finite residuated lattice, then $\mathfrak{A}$ is F-quasicomplemented. In particular, any finite residuated lattice is quasicomplemented.*

Proof. Since in finite residuated lattices any filter is principal, so it follows by Proposition 4.4. ∎

Corollary 4.6. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}/F$ is a finite residuated lattice, then $\gamma_F(\mathfrak{A})$ is a Boolean lattice. In particular, in any finite residuated lattice $\gamma(\mathfrak{A})$ is a Boolean lattice.*

Proof. It is an immediate consequence of Proposition 4.3 and Corollary 4.5. ∎

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. For any prime filter P of $\mathfrak{A}$ we set

$$D_F(P) = \{a \in A | (F : a) \nsubseteq P\}.$$

Following by [27], $D_F(P)$ is a filter of $\mathfrak{A}$ containing F. The following theorem characterizes minimal prime filters belonging to a filter.

Theorem 4.7. [27, Proposition 3.14] *Let $\mathfrak{A}$ be a residuated lattice, F be a filter and P be a prime filter containing F. The following assertions are equivalent:*

(1) *P is an F-minimal prime filter;*

(2) *$P = D_F(P)$;*

(3) *for any $x \in A$, P contains precisely one of x or $(F : x)$.*

Corollary 4.8. [27, Proposition 3.21] *Let $\mathfrak{A}$ be a residuated lattice, F be a filter and P be a prime filter. We have*

$$D_F(P) = \bigcap\{\mathfrak{m} | \mathfrak{m} \in Min_F(\mathfrak{A}),\ \mathfrak{m} \subseteq P\}.$$

The following proposition by applying Theorem 4.7 gives some necessary and sufficient conditions for any residuated lattice to become quasicomplemented.

Proposition 4.9. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is F-quasicomplemented;*
(2) *any prime filter containing F which not contains any F-dense element is F-minimal prime;*
(3) *any filter containing F which not contains any F-dense element is contained in an F-minimal prime filter.*

Proof. (1)⇒(2): Let P be a prime filter containing F such that $P \cap \mathfrak{de}_F(\mathfrak{A}) = \emptyset$. Consider $x \in P$. Applying Proposition 4.3, there exists a $y \in A$ such that $x \odot y$ is F-dense and $x \vee y \in F$. It shows that $x \odot y \notin P$ and so $y \notin P$. Hence, $x \in D_F(P)$ and it states that $P = D_F(P)$. So the result holds by Theorem 4.8.

(2)⇒(3): It follows by Theorem 2.7.

(3)⇒(1): Let $x \in A$. By Theorem 4.7 follows that $\mathcal{F}(F, x) \veebar (F : x)$ cannot be contained in any F-minimal prime filter and so it contains an F-dense element like d. Hence, there are $a \in \mathcal{F}(F, x)$ and $b \in (F : x)$ such that $a \odot b \leq d$. So for some integer n follows that $x^n \odot b$ is F-dense. Let $u \in (F : b)$ and $v \in (F : x)$. Thus we have $(u \vee v) \vee b \in F$ and $(u \vee v) \vee x^n \in F$. And by using Remark 2.1 (ii) we deduce that $(u \vee v) \vee (x^n \odot b) \in F$. It shows that $u \vee v \in F$ and it means that $(F : b) \subseteq (F : (F : x))$. The other inclusion is evident by Proposition 3.4, and so the result holds. ■

Corollary 4.10. *Let $\mathfrak{A}$ be a finite residuated lattice and P be a prime filter of $\mathfrak{A}$ which is not minimal prime. Then P contains a dense element.*

Proof. It follows by Corollary 4.5 and Proposition 4.9. ■

Cornish [2] studied distributive lattices with 0 in which each prime ideal contains a unique minimal prime ideal under the name "*normal lattices*". Rasouli and Kondo [27] generalized this notion to residuated lattices.

Definition 4.11. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. $\mathfrak{A}$ is called normal with respect to F (or, F-normal) provided that any prime filter containing F contains a unique F-minimal prime filter.*

Remark 4.12. Following by [27, Definition 4.3], a residuated lattice $\mathfrak{A}$ is normal if it is normal with respect to $\{1\}$.

The following proposition gives some necessary and sufficient conditions for any residuated lattice to become normal.

Proposition 4.13. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions are equivalent:*

(1) *Any two distinct F-minimal prime filters are comaximal;*
(2) *$\mathfrak{A}$ is F-normal;*
(3) *for any prime filter P, $D_F(P)$ is prime;*
(4) *for any $x, y \in A$, $x \vee y \in F$ implies $(F : x) \veebar (F : y) = A$;*
(5) *for any $x, y \in A$, $x \vee y \in F$ implies that there exist $u \in (F : x)$ and $v \in (F : y)$ such that $u \odot v = 0$;*
(6) *$\gamma_F(\mathfrak{A})$ is a sublattice of $\mathfrak{F}(\mathfrak{A})$;*
(7) *for any $x, y \in A$, $(F : x \vee y) = A$ implies $(F : x) \veebar (F : y) = A$;*

Proof. (1)⇒(2) is trivial and (2)⇒(3) is a direct consequence of Corollary 4.8.

(3)⇒(4): Let $x, y \in A$ such that $x \vee y \in F$. If $(F : x) \veebar (F : y) \neq A$, then there exists a prime filter P containing $(F : x) \veebar (F : y)$. So by Theorem 4.7 follows that $x, y \notin D_F(P)$; a contradiction.

(4)⇒(5): Let $x, y \in A$ such that $x \vee y \in F$. Hence $(F : x) \veebar (F : y) = A$ and so by Proposition 2.5 there exist $u \in (F : x)$ and $v \in (F : y)$ such that $u \odot v = 0$.

(5)⇒(6): Let $a \in (F : x \vee y)$. This implies that $(a \vee x) \vee y \in F$ and so there exist $u \in (F : a \vee x)$ and $v \in (F : y)$ such that $u \odot v = 0$. By Remark 2.1 (ii) it follows that $a = a \vee (u \odot v) \geq (a \vee u) \odot (a \vee v)$. On the other hand, $a \vee u \in (F : x)$ and $a \vee v \in (F : y)$. This means that $a \in (F : x) \veebar (F : y)$. The converse inclusion follows by Proposition 3.19.

(6)⇒(7): It is trivial.

(7)⇒(1): Let $x_1 \in \mathfrak{m}_2 - \mathfrak{m}_1$ and $x_2 \in \mathfrak{m}_1 - \mathfrak{m}_2$. Applying Theorem 4.7, there exists $y_2 \notin \mathfrak{m}_1$ such that $x_2 \vee y_2 \in F$. Let $x = x_1 \vee y_2$ and $y = x_2$. So $x \vee y \in F$, $x \notin \mathfrak{m}_1$ and $y \notin \mathfrak{m}_2$. Applying Theorem 4.7 and hypothesis follows that $(F : x) \veebar (F : y) = A$, $(F : x) \subseteq \mathfrak{m}_1$ and $(F : y) \subseteq \mathfrak{m}_2$. It holds the result. ∎

Example 4.14. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.3 and the residuated lattice $\mathfrak{B}_7$ from Example 2.4. By Example 2.8 follows that $\mathfrak{B}_6$ is a normal residuated lattice and $\mathfrak{B}_7$ is not a normal residuated lattice.

The notion of generalized Stone lattice is introduced by Katrinak [13] as a generalization of Stone lattices. Motivated by this notion, [2, Theorem 5.6] characterized generalized Stone lattices in terms of normal and quasicomplemented distributive lattices with 0. In the following, we generalize this notion to residuated lattices.

Definition 4.15. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. $\mathfrak{A}$ is called generalized Stone with respect to F (or, generalized F-Stone) provided that for any $x \in A$ we have $(F : x) \veebar (F : (F : x)) = A$. $\mathfrak{A}$ is called generalized Stone if it is generalized Stone with respect to $\{1\}$.*

Proposition 4.16. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}$ is generalized F-Stone, then $\mathfrak{A}$ is F-quasicomplemented.*

Proof. Let $x \in A$. So we have $(F : x) \cap (F : (F : x)) = F = \mathcal{F}(F, 1)$ and $(F : x) \veebar (F : (F : x)) = A = \mathcal{F}(F, 0)$. Using Lemma 2.6, it follows that there exists $y \in A$ such that $(F : x) = \mathcal{F}(F, y)$. By Proposition 3.10, it follows that $(F : (F : x)) = (F : y)$ and so the result holds. ∎

Proposition 4.17. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}$ is generalized F-Stone, then $\mathfrak{A}$ is F-normal.*

Proof. Let $x, y \in A$ and $a \in (F : x \vee y)$. So $a \vee y \in (F : x)$ and it implies that $\mathcal{F}(F, a \vee y) \cap (F : (F : x)) = F$. It means that $(\mathcal{F}(F, a) \cap \mathcal{F}(F, y)) \cap (F : (F : x)) = F$ and it states that $\mathcal{F}(F, a) \cap (F : (F : x)) \subseteq (F : \mathcal{F}(F, y)) = (F : y)$. Now we have the following sequence of formulas:

$$\begin{aligned}(F : x) \veebar (F : y) &\supseteq (\mathcal{F}(F, a) \cap (F : x)) \veebar (\mathcal{F}(F, a) \cap (F : (F : x)))\\ &= \mathcal{F}(F, a) \cap ((F : x) \veebar (F : (F : x)))\\ &= \mathcal{F}(F, a) \cap A = \mathcal{F}(F, a) \ni a.\end{aligned}$$

The other inclusion follows by Proposition 3.19. ∎

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. A filter G of $\mathfrak{A}$ is called a direct F-*summand* of $\mathfrak{A}$, if there exists a filter H such that $G \cap H = F$ and $G \veebar H = A$.

Theorem 4.18. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions are equivalent:*

(1) *$\mathfrak{A}$ is generalized F-Stone;*
(2) *any F-coannulet of $\mathfrak{A}$ is a direct F-summand of $\mathfrak{A}$.*

Proof. (1)⇒(2): It is evident by Proposition 3.4 and hypothesis.

(2)⇒(1): Let $x \in A$. So there exists a filter H such that $(F : x) \cap H = F$ and $(F : x) \veebar H = A$. Applying Proposition 3.5, we obtain that $H \subseteq (F : (F : x))$ and it implies that $(F : x) \veebar (F : (F : x)) = A$. ∎

In the following corollary gives the interrelation between the subclasses of quasicomplemented, normal and generalized stone residuated lattices (See Fig. 3).

Corollary 4.19. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is generalized F-Stone;*
(2) *$\mathfrak{A}$ is F-quasicomplemented and F-normal.*

Proof. (1)⇒(2): It is evident by Props. 4.16 and 4.17.

(2)⇒(1): Consider $x \in A$. So there exists $y \in A$ such that $(F : (F : x)) = (F : y)$ and it implies that $(F : x) \veebar (F : (F : x)) = (F : x) \veebar (F : y) = (F : x) \vee^{\Gamma_F} (F : y) = (F : x) \vee^{\Gamma_F} (F : (F : x)) = A$. ■

Corollary 4.20. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}/F$ is a finite residuated lattice, then $\mathfrak{A}$ is generalized F-Stone if and only if it is F-normal. In particular, any finite residuated lattice is generalized Stone if and only if it is normal.*

Proof. It follows from Corollaries 4.5 and 4.19. ■

Example 4.21. Consider the residuated lattice $\mathfrak{B}_6$ from Example 2.2 and the residuated lattice $\mathfrak{B}_7$ from Example 2.3. By Example 4.14 and Corollary 4.20 follows that $\mathfrak{B}_6$ is a generalized Stone residuated lattice and $\mathfrak{B}_7$ is not a generalized Stone residuated lattice.

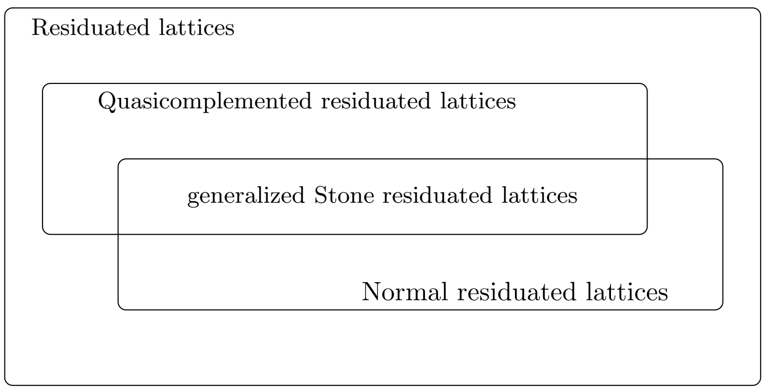

Figure 3

Corollary 4.22. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions are equivalent:*

(1) *$\mathfrak{A}$ is generalized F-Stone;*
(2) *$\gamma_F(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}_F(\mathfrak{A})$.*

Proof. It is an immediate consequence of Props 4.3, 4.13 and Corollary 4.19. ■

Corollary 4.23. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. If $\mathfrak{A}/F$ is a finite residuated lattice, then $\gamma_F(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\gamma_F(\mathfrak{A})$ is a sublattice of $\mathcal{F}_F(\mathfrak{A})$. In particular, in any finite residuated lattice $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\gamma(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$.*

Proof. It is an immediate consequence of Corollaries 4.22 and 4.23. ■

5. Conclusion

We have investigated the notions of coannihilator and coannulet in residuated lattices. For a residuated lattice $\mathfrak{A}$, we have established a connection between coannihilators and Galois connection theory (Proposition 3.10). We have shown that $\gamma(\mathfrak{A})$ and $\lambda(\mathfrak{A})$ are sublattices of $\Gamma(\mathfrak{A})$ (Corollary 3.20). Also, we have proved a kind of Chinese remainder theorem for the lattice of coannulets (Proposition 3.22). The notions of quasicomplemented, normal, and generalized Stone for residuated lattices are also generalized. It is observed that for a residuated lattice $\mathfrak{A}$ (i) $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\Gamma(\mathfrak{A})$ if and only if $\mathfrak{A}$ is quasicomplemented (Proposition 4.3) (ii) $\gamma(\mathfrak{A})$ is a sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is normal (Proposition 4.13) (iii) $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathcal{F}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is generalized Stone (Corollary 4.22). We have generalized the results by [2, 4] for residuated lattices as the most common structure among algebras of logics. It is worth considering that the residuated lattices do not fulfill the distributivity or modularity properties.

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