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Infinitely Many Homoclinic Solutions for Fourth-Order Differential Equations with Superquadratic or Combined Nonlinearities

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Abstract. In this paper, we are interested in the existence of infinitely many homoclinic solutions for fourth-order differential equations where the nonlinearity is superquadratic or involves a combination of superquadratic and subquadratic terms at infinity. By using some weaker conditions, our results extend and improve some existing results in the literature.

Keywords: Fourth-order differential equation; Homoclinic solutions; Variational methods; Critical point theory; superquadratic; Combined nonlinearities.

1. Introduction

In this paper, we consider the nonperiodic fourth-order differential equation

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)),\ \forall x \in \mathbb{R} \tag{1}$$

where ω is a constant, $a \in C(\mathbb{R})$ and $f \in C(\mathbb{R}^2)$ are real functions.

It is well-known that the mathematical modeling of important questions in different fields of research such as mechanical engineering, control systems, economics and many others, leads naturally to the consideration of the nonlinear differential equations. In particular, the fourth order differential equations, like (1) have been put forward as mathematical model for the study of pattern formation in physics and mechanics. For example, the well-known extended Fisher-Kolmogorov equation proposed by Coullet et al. in [5], in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [15], the Swift-Hohenberg equation which is a general model for pattern-forming process derived in [7] to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers [8].

As usual, we say that a solution u of equation (1) is homoclinic (to 0) if $u(x) \to 0$ as $x \longrightarrow \pm\infty$. In addiction, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

In recent years, based on critical point theory and variational methods, many researchers are interested in the existence of homoclinic solutions for the special case of $f(x,u) = c(x)u^2 + d(x)u^3$, where $a(x)$, $c(x)$ and $d(x)$ are independent of x or periodic in x, see [1,3,4,9,17] and the references cited therein. Li [9] extended these results to equation (1) in the general case where $a(x)$ and $f(x,u)$ are periodic in x. Precisely, let $F(x,u) = \int_0^u f(x,t)dt$, and by assuming that $f(x,u)$ satisfies the well-known Ambrosetti-Rabinowitz condition:

$(\mathcal{AR})$ There is a constant $\mu > 2$ such that

$$0 < \mu F(x,u) \leq f(x,u)u, \ \forall (x,u) \in \mathbb{R}^2,$$

some results on the existence of homoclinic solutions are obtained.

Compared to the periodic case, the nonperiodic case seems to be more difficult, because of the lack of compactness of the Sobolev embedding. In 2009, Li [10] dealt with the nonperiodic case of equation (1) and obtained the existence of nontrivial homoclinic solutions via using a compactness lemma and Mountain Pass Theorem. Since then, there is a few literature available for the case where $a(x)$ and $f(x,u)$ are nonperiodic in x, see [10–14, 18–22].

We notice that, for the case that equation (1) is nonperiodic, to obtain the existence of homoclinic solutions, the following coercive condition on a is often needed:

$(\mathcal{A}_0)$ $a : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and there exists a constant a_1 such that

$$\begin{aligned} &0 \ < \ a_1 \leq a(x) \longrightarrow +\infty \ as \ \ |x| \longrightarrow \infty; \\ &\omega \ \leq \ 2\sqrt{a_1}. \end{aligned}$$

This is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see [10, 11, 18, 20]. Moreover, most of the well-known results were obtained for the case where $F(x,u)$ is superquadratic at infinity in u satisfying the $(\mathcal{AR})-$condition.

The purpose of this paper is devoted to proving the existence of infinitely many homoclinic solutions for equation (1) when the function a may be negative on a bounded interval and the potential $F(x,u)$ is either superquadratic at infinity in the second variable and does not need to satisfy the $(\mathcal{AR})-$condition or involves a combination of subquadratic and superquadratic terms. To the best of our knowledge, it seems that no similar results are obtained in the literature for fourth-order differential equations. Firstly, we deal with the superquadratic case and we introduce the following hypotheses on the function $a(x)$ and the nonlinearity $f(x,u)$:

$(\mathcal{A})$ $\lim_{|x|\longrightarrow\infty} a(x) = +\infty$;

(F_1) There exist constants $a_0, b_0 > 0$ and $p > 2$ such that

$$|f(x,u)| \leq a_0 |u| + b_0 |u|^{p-1}, \ \forall (x,u) \in \mathbb{R}^2;$$

(F_2) $\lim_{|u|\longrightarrow\infty} \frac{F(x,u)}{|u|^2} = +\infty$ *uniformly in* $x \in \mathbb{R}$ and $F(x,u)$ is bounded from below;

(F_3) $F(x,-u) = F(x,u), \ \forall (x,u) \in \mathbb{R}^2$;

(F_4) There exist constants $c_0 > 0$, $r_0 \geq 0$ and $\sigma > 1$ such that

$$\begin{aligned} f(x,u)u - 2F(x,u) &\geq 0, \ \forall (x,u) \in \mathbb{R}\times\mathbb{R}, \\ |F(x,u)|^{\sigma} &\leq c_0 |u|^{2\sigma} [f(x,u)u - 2F(x,u)], \ \forall x \in \mathbb{R}, \ |u| \geq r_0; \end{aligned}$$

(F_4') There exist constants $\mu > 2$ and $\gamma > 0$ such that

$$\mu F(x,u) \leq f(x,u)u + \gamma u^2, \ \forall (x,u) \in \mathbb{R}\times\mathbb{R}.$$

Our first main results read as follows.

Theorem 1.1. *Assume that* $(\mathcal{A})$ *and* $(F_1) - (F_4)$ *hold. Then* (1) *has infinitely many nontrivial homoclinic solutions.*

Theorem 1.2. *Assume that* $(\mathcal{A})$, $(F_1)-(F_3)$ *and* (F_4') *hold. Then* (1) *has infinitely many nontrivial homoclinic solutions.*

Example 1.3. Let

$$F(x,u) = \theta(x)\Big[(4|u|^2 - 1)ln(\frac{1}{2} + |u|) - 2(\frac{1}{2} + |u|)^2 + 4|u| + \frac{1}{2} - ln2\Big],$$

where $\theta \in C(\mathbb{R},\mathbb{R})$ is such that $0 < \inf_{x\in\mathbb{R}} \theta(x) \leq \sup_{x\in\mathbb{R}} \theta(x) < +\infty$. It is clear that $F(x,u)$ satisfies $(F_1)-(F_3)$. It remains to verify (F_4). An easy computation shows that

$$f(x,u)u - 2F(x,u) = \theta(x)\Big[(4|u|^2 - 1)\frac{2|u|}{2|u|+1} - 2|u| + 2ln(\frac{1}{2} + |u|) + 2ln2\Big].$$

It is easy to see that $f(x,u)u - 2F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^2$. Moreover, for all $\sigma > 1$, we have

$$\Big(\frac{F(x,u)}{|u|^2}\Big)^{\sigma} [f(x,u)u - 2F(x,u)]^{-1} \cong_{\infty} (4\theta(x))^{\sigma-1} \frac{\Big(ln(\frac{1}{2} + |u|)\Big)^{\sigma}}{|u|^2},$$

which converges to 0 as $|u| \longrightarrow \infty$, uniformly in $x \in \mathbb{R}$. Hence there exist two positive constants r_0, c_0 such that

$$\Big(\frac{F(x,u)}{|u|^2}\Big)^{\sigma} \leq c_0[f(x,u)u - 2F(x,u)], \ \forall x \in \mathbb{R}, \ |u| \geq r_0.$$

Therefore (F_4) holds. By Theorem 1.1, the corresponding fourth-order differential equation (1) possesses infinitely many nontrivial homoclinic solutions.

Next, consider equation (1) involving a combination of superquadratic and subquadratic terms at infinty. More precisely, we take $f(x,u) = g(x,u)+h(x,u)$, where $g, h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. Consider the following conditions:

(F_5) There exist constants $1 < \gamma < 2$, $1 < \sigma < 2$ and functions $c_0, a_0 \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ and $b_0 \in L^{\frac{2}{2-\sigma}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$c_0(x)\,|u|^{\gamma} \leq g(x,u)u, \ |g(x,u)| \leq a_0(x)\,|u|^{\gamma-1} + b_0(x)\,|u|^{\sigma-1}, \ a.e. \ x \in \mathbb{R},$$

for $u \in \mathbb{R}$;

(F_6) $H(x,u) = \int_0^u h(x,s)ds \geq 0$ and there exist $\mu > 2$, $c \in L^2(\mathbb{R}, \mathbb{R}^+)$ and $d \in L^{\infty}(\mathbb{R}, \mathbb{R}^+)$ such that $|h(x,u)| \leq c(x) + d(x)\,|u|^{\mu-1}$, $a.e.\ x \in \mathbb{R}$, $\forall u \in \mathbb{R}$;

(F_7) There exist $\rho > 2$, $1 < \delta < 2$ and $\theta \in C(\mathbb{R}, \mathbb{R}^+) \bigcap L^{\frac{2}{2-\delta}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$\rho H(x,u) - h(x,u)u \leq \theta(x)\,|u|^{\delta}, \ a.e. \ x \in \mathbb{R}, \ \forall u \in \mathbb{R}.$$

Our third result reads as follows.

Theorem 1.4. *Assume that* $(\mathcal{A}_0)$, (F_3) *and* $(F_5) - (F_7)$ *are satisfied. Then equation* (1) *possesses infinitely many homoclinic solutions.*

Remark 1.5. Obviously, Theorem 1.4 generalizes Theorem 1.1 in [14] and Theorem 2 in [21]. In fact, let $g(x,u) = 0$. Then Theorem 1.4 generalizes Theorem 1.1 in [14]. Similarly, let $h(x,u) = 0$. Then Theorem 1.4 generalizes Theorem 2 in [21]. For example, the function $F(x,u) = G(x,u) + H(x,u)$ where

$$\begin{aligned} G(x,u) &= \Big(\frac{1}{1+|x|^2}\Big)^{\frac{1}{3}} |u|^{\frac{4}{3}} + \Big(\frac{1}{1+|x|^2}\Big)^{\frac{1}{6}} |u|^{\frac{5}{3}}, \\ H(x,u) &= \Big(\frac{1}{1+|x|^2}\Big) |u|^6 \end{aligned}$$

satisfies (F_3) and $(F_5) - (F_7)$. Hence the corresponding equation (1) possesses infinitely many homoclinic solutions.

2. Variational Setting and Preliminaries

To prove our main result via critical point theory, we need to establish the variational setting for (1). In the following, we shall use $\|.\|_s$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [2, \infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$\begin{aligned} < u, v >_{H^2} &= \int_{\mathbb{R}} [u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)]dx, \\ \|u\|_{H^2} &= < u, u >_{H^2}^{\frac{1}{2}} \end{aligned}$$

for all $u, v \in H^2(\mathbb{R})$.

In this Section, we will assume that the function a satisfies the condition $(\mathcal{A}_0)$.

Lemma 2.1. [4, Lemma 8] *Assume that a satisfies $(\mathcal{A}_0)$. Then there exists a constant $b > 0$ such that*

$$\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx \geq b \|u\|_{H^2}^2, \ \forall u \in H^2(\mathbb{R}).$$

By Lemma 2.1, we define

$$E = \left\{ u \in H^2(\mathbb{R}) / \int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx < \infty \right\}$$

with the inner product

$$< u, v >= \int_{\mathbb{R}} [u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm

$$\|u\| = (\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx)^{\frac{1}{2}}.$$

It is easy to verify that E is a Hilbert space.

In order to prove our results, the following compactness result is necessary.

Lemma 2.2. [18, Lemma 2.2] *Assume that a satisfies $(\mathcal{A}_0)$. Then E is compactly embedded in $L^s(\mathbb{R})$ for all $s \in [2, \infty]$. Moreover, for all $s \in [2, \infty]$, there exists $\eta_s > 0$ such that*

$$\|u\|_{L^s(\mathbb{R})} \leq \eta_s \|u\|, \ \forall u \in E. \tag{2}$$

To study the critical points of the variational functional associated with (1), we need to recall the following critical point theorems.

Definition 2.3. *Let E be a Banach space with the norm $\|.\|$. We say that $\psi \in C^1(E, \mathbb{R})$ satisfies the following conditions:*

(a) *$(PS)_c$-condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset E$ satisfying*

$$\psi(u_n) \longrightarrow c \text{ and } \psi'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence,

(b) *$(C)_c-$condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset E$ satisfying*

$$\psi(u_n) \longrightarrow c \text{ and } \|\psi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence.

Lemma 2.4. [16, Symmetric Mountain Pass Theorem] *Let E be an infinite dimensional Banach space, $E = Y \oplus Z$, where Y is finite dimensional space. Suppose that $\psi \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition and*

(a) $\psi(0) = 0$, $\psi(-u) = \psi(u)$, $\forall u \in E$;
(b) *There exist constants $\rho, \alpha > 0$ such that $\psi_{|\partial B_\rho \cap Z} \geq \alpha$;*
(c) *For any finite dimensional subspace $\widetilde{E} \subset E$, there is $R = R(\widetilde{E}) > 0$ such that $\psi(u) \leq 0$ on $\widetilde{E} \setminus B_R$, where $B_R = \{u \in E / \|u\| < R\}$.*

Then ψ possesses an unbounded sequence of critical values.

Remark 2.5. As shown in [2], a deformation lemma can be proved with $(C)_c-$condition replacing the $(PS)_c-$condition, and it turns out that Lemma 2.4 still holds true with the $(C)_c-$condition instead of the $(PS)_c-$condition.

Now, let E be a Banach space with the norm $\|.\|$ and $E = \overline{\oplus_{j \in \mathbb{N}} X_j}$, where X_j is a finite dimensional subspace of E. For each $k \in \mathbb{N}$, let $Y_k = \oplus_{j=0}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. The functional $\psi \in C^1(E, \mathbb{R})$ is said to satisfy the $(PS)^*$ condition if for any sequence (u_j) for which $(\psi(u_j))$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \longrightarrow \infty$ and $(\psi_{|Y_{k_j}})'(u_j) \longrightarrow 0$ as $j \longrightarrow \infty$, has a subsequence converging to a critical point of ψ.

Lemma 2.6. [6, Dual Fountain Theorem] *Suppose that the functional $\psi \in C^1(E, \mathbb{R})$ is even and satisfies the $(PS)^*$ condition. Assume that for each sufficiently large integer k, there exist $0 < r_k < \rho_k$ such that*

(a) $a_k = \inf_{u \in Z_k, \|u\| = \rho_k} \psi(u) \geq 0$;
(b) $b_k = \max_{u \in Y_k, \|u\| = r_k} \psi(u) < 0$;
(c) $d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} \psi(u) \longrightarrow 0$ *as $k \longrightarrow \infty$.*

Hence ψ has a sequence of negative critical values converging to zero.

3. Proof of Theorems 1.1 and 1.2

First, note that $(\mathcal{A})$ implies that there exists a constant $a_1 > 0$ such that $\widetilde{a}(x) = a(x) + 2a_1 \geq a_1$ for all $x \in \mathbb{R}$, $\omega \leq 2\sqrt{a_1}$. Let $\widetilde{f}(x,u) = f(x,u) + 2a_1 u$ for all $(x,u) \in \mathbb{R}^2$ and consider the following fourth-order differential equation

$$u^{(4)}(x) + \omega u''(x) + \widetilde{a}(x)u(x) = \widetilde{f}(x,u(x)),\ \forall x \in \mathbb{R}. \tag{3}$$

Then (3) is equivalent to (1). Moreover, it is easy to check that the hypotheses $(F_1)-(F_4)$ and (F_4') still hold for $\widetilde{f}(x,u)$ provided that those hold for $f(x,u)$ and the function $\widetilde{a}$ satisfies $(\mathcal{A}_0)$. Hence in what follows, we always assume without loss of generality that a satisfies $(\mathcal{A}_0)$.

Consider the variational functional ψ associated to equation (1):

$$\begin{aligned}\psi(u) &= \frac{1}{2}\int_{\mathbb{R}}[u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx - \int_{\mathbb{R}} F(x,u(x))dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)dx\end{aligned}$$

defined on the Hilbert space E introduced in Section 2. It is well known that, under assumption (F_1), $\psi \in C^1(E,\mathbb{R})$ and critical point of ψ are solutions of (1). Moreover, for all $u,v \in E$

$$\psi'(u)v \int_{\mathbb{R}}[u''v'' - \omega u'v' + auv]dx - \int_{\mathbb{R}} f(x,u)vdx =< u,v > - \int_{\mathbb{R}} f(x,u)vdx.$$

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of E. We set

$$Y_m = span\,\{e_1,...,e_m\}\,,\ Z_m = \overline{span\,\{e_{m+1},...\}},\ m \in \mathbb{N}.$$

Then $E = Y_m \oplus Z_m$.

Lemma 3.1. *Assume that $(\mathcal{A}_0)$ and (F_1) are satisfied. Then there exist positive constants m_0, α, ρ such that $\psi_{|\partial B_\rho \cap Z_{m_0}} \geq \alpha$.*

Proof. Note that by (F_1), we have

$$|F(x,u)| \leq \frac{a_0}{2}|u|^2 + \frac{b_0}{p}|u|^p,\ \forall (x,u) \in \mathbb{R}^2. \tag{4}$$

For any $m \in \mathbb{N}$, let

$$l_2(m) = \sup_{u\in Z_m\setminus\{0\}} \frac{\|u\|_2}{\|u\|}\ \textit{and}\ l_p(m) = \sup_{u\in Z_m\setminus\{0\}} \frac{\|u\|_p}{\|u\|}. \tag{5}$$

It is clear that $l_2(m+1) \leq l_2(m)$, so $l_2(m) \longrightarrow l \geq 0$ as $m \longrightarrow \infty$. For any $m \in \mathbb{N}$, there exists $u_m \in Z_m$ such that $\|u_m\| = 1$ and $\|u_m\|_2 \geq \frac{1}{2}l_2(m)$. By the definition of Z_m, $u_m \rightharpoonup 0$ in E, then $u_m \longrightarrow 0$ in $L^2(\mathbb{R})$. Hence, we have $l = 0$, that is $l_2(m) \longrightarrow 0$ as $m \longrightarrow \infty$. Similarly $l_p(m) \longrightarrow 0$ as $m \longrightarrow \infty$. Therefore, we can choose a larger integer m_0 such that

$$\|u\|_2^2 \leq \frac{1}{2a_0} \|u\|^2, \ \|u\|_p^p \leq \frac{p}{4b_0} \|u\|^p, \ \forall u \ Z_{m_0}. \tag{6}$$

Then by (4) and (6), we have

$$\begin{aligned} \psi(u) &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} F(x,u)dx \geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} (\frac{a_0}{2} |u|^2 + \frac{b_0}{p} |u|^p)dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{a_0}{2} \|u\|_2^2 - \frac{b_0}{p} \|u\|_p^p \geq \frac{1}{2} \|u\|^2 - \frac{1}{4} \|u\|^2 - \frac{1}{4} \|u\|^p \\ &= \frac{1}{4}(\|u\|^2 - \|u\|^p) = \frac{2^{p-2}-1}{2^{p+2}} = \alpha, \ \forall u \in Z_{m_0}, \ \|u\| = \frac{1}{2} = \rho, \end{aligned}$$

which finish the proof. ■

To apply Lemma 2.4, we will take $E = Y \oplus Z$ with $Y = Y_{m_0}$ and $Z = Z_{m_0}$, where m_0 is introduced in Lemma 3.1.

Lemma 3.2. *Assume that $(\mathcal{A}_0)$, (F_1) and (F_2) are satisfied. Then for any finite dimensional subspace $\tilde{E} \subset E$, there is a constant $R = R(\tilde{E}) > 0$ such that*

$$\psi(u) \leq 0, \ \forall u \in \tilde{E}, \ \|u\| \geq R. \tag{7}$$

Proof. In order to prove (7), we only need to prove

$$\psi(u) \longrightarrow -\infty \ as \ \|u\| \longrightarrow \infty, \ u \in \tilde{E}. \tag{8}$$

Assume by contradiction that there exists a sequence $(u_n) \subset \tilde{E}$ with $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\psi(u_n) \geq -M$ for some constant $M > 0$, $\forall n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Going to a subsequence if necessary, we can assume that $v_n \rightharpoonup v$ in E. Since $\tilde{E}$ is finite dimensional, we have $v_n \longrightarrow v$ in E, and thus $\|v\| = 1$. Let

$$\begin{aligned} \Lambda_n(c,d) &= \{x \in \mathbb{R}/c \leq |u_n(x)| < d\}, \ 0 \leq c < d, \\ \Lambda &= \{x \in \mathbb{R}/v(x) \neq 0\}. \end{aligned}$$

For any $x \in \Lambda$, we have $\lim_{n \longrightarrow \infty} |u_n(x)| = \lim_{n \longrightarrow \infty} \|u_n\| |v_n(x)| = +\infty$. Hence $\Lambda \subset \Lambda_n(r, \infty)$ for all integer n large enough. Property (4), Lemma 2.2, assumption (F_2) and Fatou's lemma imply

$$
\begin{aligned}
0 &= \lim_{n\longrightarrow\infty} \frac{-M}{\|u_n\|^2} \leq \lim_{n\longrightarrow\infty} \frac{\Phi(u_n)}{\|u_n\|^2} = \lim_{n\longrightarrow\infty} [\frac{1}{2} - \int_{\mathbb{R}} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx] \\
&= \lim_{n\longrightarrow\infty} [\frac{1}{2} - \int_{\Lambda_n(0,r)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx - \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx] \\
&\leq \limsup_{n\longrightarrow\infty} [\frac{1}{2} + (\frac{a_0}{2} + \frac{b_0}{p} r^{p-2}) \int_{\mathbb{R}} |v_n|^2 \, dx - \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx] \qquad (9) \\
&\leq \frac{1}{2} + (\frac{a_0}{2} + \frac{b_0}{p} r^{p-2}) \eta_2^2 - \liminf_{n\longrightarrow\infty} \int_{\mathbb{R}} \frac{F(x,u_n)}{|u_n|^2} \chi_{\Lambda_n(r,\infty)} |v_n|^2 \, dx \\
&\leq \frac{1}{2} + (\frac{a_0}{2} + \frac{b_0}{p} r^{p-2}) \eta_2^2 - \int_{\mathbb{R}} \liminf_{n\longrightarrow\infty} \frac{F(x,u_n)}{|u_n|^2} \chi_{\Lambda_n(r,\infty)} |v_n|^2 \, dx = -\infty.
\end{aligned}
$$

It is a contradiction. Hence (8) is satisfied and the proof is finished. ∎

Lemma 3.3. *Assume that* $(\mathcal{A}_0)$, (F_1), (F_2) *and* (F_4) *are satisfied. Then* ψ *satisfies the* $(C)_c-$*condition for any level* $c > 0$.

Proof. Let c be a positive real and $(u_n) \subset E$ be a $(C)_c-$sequence, that is

$$\psi(u_n) \longrightarrow c \text{ and } \|\psi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Assume by contradiction that (u_n) is not bounded. Then up to a subsequence, we can assume that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Taking a subsequence if necessary, then $v_n \rightharpoonup v$ in E and Lemma 2.2 implies that $v_n \longrightarrow v$ in $L^q(\mathbb{R})$ for $q = 2, p, \frac{2\sigma}{\sigma-1}$ and $v_n \longrightarrow v$ a.e. on $\mathbb{R}$.
If $v \neq 0$, we have

$$0 = \lim_{n\longrightarrow\infty} \frac{\psi(u_n)}{\|u_n\|^2} = \lim_{n\longrightarrow\infty} [\frac{1}{2} - \int_{\mathbb{R}} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx] \leq -\infty,$$

which is a contradiction. So (u_n) is bounded.

If $v = 0$, then $v_n \longrightarrow 0$ in $L^q(\mathbb{R})$ for $q = 2, p, 2\sigma' = \frac{2\sigma}{\sigma-1}$. On one hand, since $\psi(u_n) \longrightarrow c$ and $\|u_n\| \longrightarrow \infty$, it is easy to see that

$$\limsup_{n\longrightarrow\infty} \int_{\mathbb{R}} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx = \frac{1}{2}. \qquad (10)$$

On the other hand, (4) implies

$$
\begin{aligned}
\int_{\Lambda_n(0,r)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 \, dx &\leq (\frac{a_0}{2} + \frac{b_0}{p} r^{p-2}) \int_{\Lambda_n} |v_n|^2 \, dx \\
&\leq (\frac{a_0}{2} + \frac{b_0}{p} r^{p-2}) \|v_n\|_2^2 \longrightarrow 0.
\end{aligned} \qquad (11)
$$

Now, for all integer n large enough, we have

$$\int_{\mathbb{R}}[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)]dx = \psi(u_n) - \frac{1}{2}\psi'(u_n)u_n \leq c+1$$

which with Hölder's inequality and assumption (F_4) implies

$$\begin{aligned}
&\int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx \\
\leq&\Big(\int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2})^{\sigma}dx\Big)^{\frac{1}{\sigma}}\Big(\int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'}\,dx\Big)^{\frac{1}{\sigma'}} \\
\leq&(2c_0)^{\frac{1}{\sigma}}\Big(\int_{\Lambda_n(r,\infty)}[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)]dx\Big)^{\frac{1}{\sigma}}\Big(\int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'}\,dx\Big)^{\frac{1}{\sigma'}} \\
\leq&(2c_0(c+1))^{\frac{1}{\sigma}}\|v_n\|^2_{2\sigma'} \longrightarrow 0 \ as\ n\longrightarrow\infty.
\end{aligned} \tag{12}$$

Combining (11) and (12) yields

$$\begin{aligned}
\int_{\mathbb{R}} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx &= \int_{\Lambda_n(0,r)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx + \int_{\Lambda_n(r,\infty)} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2\,dx \\
&\longrightarrow 0
\end{aligned}$$

as $n\longrightarrow\infty$, which contradicts (10). Hence (u_n) is bounded. Up to a subsequence, we can assume that $u_n \longrightarrow u$ in both $L^2(\mathbb{R})$ and $L^p(\mathbb{R})$. It follows from (F_1) and Hölder's inequality that

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(x,u_n)(u_n-u)dx\right| &\leq \int_{\mathbb{R}}(a_0|u_n| + b_0|u_n|^{p-1})|u_n-u|\,dx \\
&\leq a_0\int_{\mathbb{R}}|u_n||u_n-u|\,dx + b_0\int_{\mathbb{R}}|u_n|^{p-1}|u_n-u|\,dx \\
&\leq a_0\|u_n\|_2\|u_n-u\|_2 + b_0\|u_n\|_p^{p-1}\|u_n-u\|_p \longrightarrow 0,
\end{aligned}$$

as $n\longrightarrow\infty$. Therefore, we have

$$\begin{aligned}
0 &= \lim_{n\longrightarrow\infty}\psi'(u_n)(u_n-u) \\
&= \lim_{n\longrightarrow\infty} < u_n, u_n-u > - \lim_{n\longrightarrow\infty}\int_{\mathbb{R}} f(x,u_n)(u_n-u)dx \\
&= \lim_{n\longrightarrow\infty}\|u_n\|^2 - \|u\|^2.
\end{aligned}$$

That is $\lim_{n\longrightarrow\infty}\|u_n\|^2 = \|u\|^2$, which with $u_n \rightharpoonup u$ in E implies

$$\|u_n-u\|^2 =< u_n-u, u_n-u >\longrightarrow 0 \ as\ n\longrightarrow\infty.$$

Hence (u_n) possesses a convergent subsequence in E. Thus ψ satisfies the $(C)_c-$condition. The proof is completed. ■

Proof of Theorem 1.1. By (F_3) and Lemmas 3.1–3.3, ψ satisfies the conditions of Lemma 2.4. It remains to prove the Cerami's condition.

Consequently, Lemma 2.4 with Remark 2.5 imply that the functional ψ possesses an unbounded sequence of critical points. Therefore, the fourth-order differential equation (1) possesses infinitely many homoclinic solutions. The proof is finished. ∎

Lemma 3.4. *Assume that $(\mathcal{A}_0)$, (F_1), (F_2) and (F_4') are satisfied. Then ψ satisfies the $(C)_c-$condition for all positive constant c.*

Proof. Let c be a positive real and $(u_n) \subset E$ be a $(C)_c-$sequence, that is

$$\psi(u_n) \longrightarrow c \textit{ and } \|\psi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0 \textit{ as } n \longrightarrow \infty.$$

Assume by contradiction that (u_n) is not bounded. Then up to a subsequence, we can assume that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. By (F_4'), for n large enough, we have

$$\begin{aligned} c+1 &\geq \psi(u_n) - \frac{1}{\mu}\psi'(u_n)u_n \\ &= \frac{\mu-2}{2\mu}\|u_n\|^2 + \int_{\mathbb{R}}[\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)]dx \\ &\geq \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\gamma}{\mu}\|u_n\|_2^2. \end{aligned}$$

It follows that

$$\limsup_{n\longrightarrow\infty} \|v_n\|_2^2 \geq \frac{\mu-2}{2\gamma}. \tag{13}$$

Since $\|v_n\| = 1$, passing to a subsequence, $v_n \rightharpoonup v$ in E and Lemma 2.2 implies that $v_n \longrightarrow v$ in $L^2(\mathbb{R})$, which with (3.11) implies that $v \neq 0$. Similar to (9), we get a contradiction. Therefore (u_n) is bounded. As in the end of the proof of Lemma 3.3, we conclude that (u_n) possesses a convergent subsequence. Hence ψ satisfies the $(C)_c-$condition. The proof is completed. ∎

Proof of Theorem 1.2. We conclude as in the proof of Theorem 1.1 that the functional ψ possesses an unbounded sequence of critical points and the proof is finished. ∎

Lemma 3.5. *Assume that $(\mathcal{A}_0)$, (F_5) and (F_6) are satisfied. If $u_n \rightharpoonup u$ in E, then*

$$f(.,u_n) \longrightarrow f(.,u) \textit{ in } L^2(\mathbb{R}). \tag{14}$$

Proof. Arguing indirectly, by Lemma 2.2, we may assume that

$$u_{n_k} \longrightarrow u \textit{ in both } L^2(\mathbb{R}) \textit{ and } L^{2(\mu-1)}(\mathbb{R}) \textit{ and } u_{n_k} \longrightarrow u \textit{ a.e. in } \mathbb{R} \tag{15}$$

as $k \longrightarrow \infty$ and

$$\int_{\mathbb{R}} |f(x, u_{n_k}(x)) - f(x, u(x))|^2 \, dx \geq \epsilon_0, \ \forall k \in \mathbb{N}, \tag{16}$$

for some positive constant ϵ_0. By (15) and up to a subsequence if necessary, we can assume that

$$\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{L^2} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_{n_k} - u\|_{L^{2(\mu-1)}} < \infty.$$

Let $w(x) = \sum_{k=1}^{\infty} |u_{n_k}(x) - u(x)|$ for all $x \in \mathbb{R}$. Then $w \in L^2(\mathbb{R}) \bigcap L^{2(\mu-1)}(\mathbb{R})$. By (F_5) and (F_6), there holds for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\begin{aligned}
& |f(x, u_{n_k}) - f(x, u)|^2 \\
\leq & \Big(|f(x, u_{n_k})| + |f(x, u)| \Big)^2 \\
\leq & \Big[|g(x, u_{n_k})| + |h(x, u_{n_k})| + |g(x, u)| + |h(x, u)| \Big]^2 \\
\leq & \Big[a_0 |u_{n_k}|^{\gamma-1} + b_0 |u_{n_k}|^{\sigma-1} + a_0 |u|^{\gamma-1} + b_0 |u|^{\sigma-1} \\
& +2c + d |u_{n_k}|^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
\leq & \Big[a_0(|u_{n_k} - u| + |u|)^{\gamma-1} + b_0(|u_{n_k} - u| + |u|)^{\sigma-1} \\
& +a_0 |u|^{\gamma-1} + b_0 |u|^{\sigma-1} \\
& +2c + d(|u_{n_k} - u| + |u|)^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
\leq & \Big[a_0(w + |u|)^{\gamma-1} + b_0(w + |u|)^{\sigma-1} + a_0 |u|^{\gamma-1} + b_0 |u|^{\sigma-1} \\
& +2c + d(w + |u|)^{\mu-1} + d |u|^{\mu-1} \Big]^2 \\
\leq & c_1 \Big[a_0^2 w^{2(\gamma-1)} + a_0^2 |u|^{2(\gamma-1)} + b_0^2 w^{2(\sigma-1)} + b_0^2 |u|^{2(\sigma-1)} \\
& +c^2 + d^2 w^{2(\mu-1)} + d^2 |u|^{2(\mu-1)} \Big] = k(x)
\end{aligned} \tag{17}$$

where c_1 is a positive constant. It is easy to see that $k \in L^1(\mathbb{R})$. Hence, combining (15) and (17), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k \longrightarrow \infty} \int_{\mathbb{R}} |f(x, u_{n_k}(x)) - f(x, u(x))|^2 \, dx = 0,$$

which contradicts with (16). Hence (14) is true. ∎

Lemma 3.6. *Suppose that $(\mathcal{A}_0)$, (F_5) and (F_6) are satisfied. Then $\psi \in C^1(E, \mathbb{R})$ and for all $u, v \in E$*

$$\psi'(u)v = \int_{\mathbb{R}} [u''v'' - \omega u'v' + auv - f(x, u)v] dx. \tag{18}$$

Moreover, $\psi' : E \longrightarrow E^*$ is compact and any critical point of ψ on E is a classical solution for equation (1).

Proof. Let

$$\psi_1(u) = \int_{\mathbb{R}} F(x, u(x))dx.$$

By (F_5) and (F_6), for any $s \in [0,1]$ and $u, v \in E$, we have

$$\begin{aligned} &|f(x, u + sv)v| \\ \le &|g(x, u + sv)v| + |h(x, u + sv)v| \\ \le &a_0 |u + sv|^{\gamma-1} |v| + b_0 |u + sv|^{\sigma-1} |v| + c |v| + d |u + sv|^{\mu-1} |v| \\ \le &a_0(|u| + |v|)^{\gamma-1} |v| + b_0(|u| + |v|)^{\sigma-1} |v| + c |v| \\ &+ d(|u| + |v|)^{\mu-1} |v| = l(x). \end{aligned} \tag{19}$$

It is easy to check, by Hölder's inequality that $l \in L^1(\mathbb{R})$. Hence, by (19), the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get for all $u, v \in E$

$$\lim_{s \longrightarrow 0} \frac{\psi(u + sv) - \psi(u)}{s} = \lim_{s \longrightarrow 0} \int_{\mathbb{R}} f(x, u + sv)vdx = \int_{\mathbb{R}} f(x, u)vdx = J(u, v).$$

Moreover, it follows from (F_5), (F_6) and Hölder's inequality that

$$\begin{aligned} |J(u, v)| \le &\int_{\mathbb{R}} |f(x, u)| |v| dx \le \int_{\mathbb{R}} \left[a_0 |u|^{\gamma-1} + a_0 |u|^{\sigma-1} + c + d |u|^{\mu-1}\right] |v| dx \\ \le &\Big(\int_{\mathbb{R}} a_0^{\frac{2}{2-\gamma}} dx\Big)^{\frac{2-\gamma}{2}} \Big(\int_{\mathbb{R}} (|u|^{\gamma-1} |v|)^{\frac{2}{\gamma}} dx\Big)^{\frac{\gamma}{2}} \\ &+\Big(\int_{\mathbb{R}} b_0^{\frac{2}{2-\sigma}} dx\Big)^{\frac{2-\sigma}{2}} \Big(\int_{\mathbb{R}} (|u|^{\sigma-1} |v|)^{\frac{2}{\sigma}} dx\Big)^{\frac{\sigma}{2}} \\ &+\Big(\int_{\mathbb{R}} c^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} |v|^2 dx\Big)^{\frac{1}{2}} + \|d\|_\infty \int_{\mathbb{R}} |u|^{\mu-1} |v| dx \\ \le &\left[\|a_0\|_{\frac{2}{2-\gamma}} \|u\|_2^{\gamma-1} + \|b_0\|_{\frac{2}{2-\sigma}} \|u\|_2^{\sigma-1} + \|c\|_2 + \|d\|_\infty \|u\|_{2(\mu-1}^{\mu-1}\right] \|v\|_2 \\ \le &\Big[\|a_0\|_{\frac{2}{2-\gamma}} \eta_2^{\gamma} \|u\|^{\gamma-1} + \|b_0\|_{\frac{2}{2-\sigma}} \eta_2^{\sigma} \|u\|^{\sigma-1} \\ &+ \|c\|_2 + \|d\|_\infty \eta_{2(\mu-1)}^{\mu-1} \eta_2 \|u\|_{2(\mu-1)}^{\mu-1}\Big] \|v\| . \end{aligned}$$

Therefore, $J(u, .)$ is linear and bounded, and $J(u, .)$ is the Gâteaux-derivative of ψ at u. Next, we prove that $J(u, .)$ is weakly continuous in u. Let $u_n \rightharpoonup u$ in E. Then by Lemma 3.5, we have $f(., u_n) \longrightarrow f(., u)$ in $L^2(\mathbb{R})$. By Hölder's inequality, we have

$$\begin{aligned} \|J(u_n, .) - J(u, .)\|_{E^*} = &\sup_{\|v\|=1} \int_{\mathbb{R}} (f(x, u_n) - f(x, u))vdx \\ \le &\eta_2\Big(\int_{\mathbb{R}} |f(x, u_n) - f(x, u)|^2 dx\Big)^{\frac{1}{2}} \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow \infty$. This means that $u \longmapsto J(u,.)$ is weakly continuous and then it is continuous in E. Therefore $\psi_1 \in C^1(E,\mathbb{R})$ and

$$\psi_1'(u)v = \int_{\mathbb{R}} f(x,u)v dx.$$

On the other hand, the function $\psi_2 : u \longmapsto \int_{\mathbb{R}}[u''^2 - \omega u'^2 + au^2]dx$ is a continuous quadratic form. Then $\psi_2 \in C^1(E,\mathbb{R})$ and

$$\psi_2'(u)v = \int_{\mathbb{R}} [u''v'' - \omega u'v' + auv]dx$$

So $\psi = \psi_1 + \psi_2 \in C^1(E,\mathbb{R})$ and (18) is verified. Furthermore, ψ' is compact by the weak continuity of ψ' since E is reflexive. Finally, it is a standard argument that critical points of ψ on E are solutions of equation (1).

In the next, we shall prove our Theorem 1.4 by applying Lemma 2.6. Choose a completely orthonormal basis (e_j) of E and define $X_j = \mathbb{R}e_j$. Then Y_k and Z_k can be defined as in Section 2. By (F_3) and Lemma 3.6, $\psi \in C^1(E,\mathbb{R})$ is even. In the following, we will check that all conditions of Lemma 2.6 are satisfied. ■

Lemma 3.7. *Assume that* $(\mathcal{A}_0)$, (F_5) *and* (F_7) *are satisfied. Then* ψ *satisfies the* $(PS)^*-$*condition.*

Proof. Let (u_j) be a $(PS)^*-$sequence, that is, $(\psi(u_j))$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \longrightarrow \infty$ and $(\psi_{|Y_{k_j}})'(u_j) \longrightarrow 0$ as $j \longrightarrow \infty$. Now, we show that (u_j) is bounded in E. By virtue of (2), (F_5) and (F_7), there exists a constant $M > 0$ such that

$$\begin{aligned}
\rho M + M\|u_j\| &\geq \rho\psi(u_j) - \psi'(u_j)u_j \\
&= (\frac{\rho}{2} - 1)\|u_j\|^2 + \int_{\mathbb{R}}[f(x,u_j)u_j - \rho F(x,u_j)]dx \\
&= (\frac{\rho}{2} - 1)\|u_j\|^2 + \int_{\mathbb{R}}[g(x,u_j)u_j - \rho G(x,u_j)]dx \\
&\quad + \int_{\mathbb{R}}[h(x,u_j)u_j - \rho H(x,u_j)]dx \\
&\geq (\frac{\rho}{2} - 1)\|u_j\|^2 - \int_{\mathbb{R}}[a_0(x)|u_j|^\gamma + b_0(x)|u_j|^\sigma]dx \\
&\quad - \rho\int_{\mathbb{R}}[\frac{a_0(x)}{\gamma}|u_j|^\gamma + \frac{b_0(x)}{\sigma}|u_j|^\sigma]dx - \int_{\mathbb{R}}\theta(x)|u_j|^\delta dx \\
&\geq (\frac{\rho}{2} - 1)\|u_j\|^2 - (1 + \frac{\rho}{\gamma})\|a_0\|_{\frac{2}{2-\gamma}}\|u_j\|_2^\gamma \\
&\quad - (1 + \frac{\rho}{\sigma})\|b_0\|_{\frac{2}{2-\sigma}}\|u_j\|_2^\sigma - \|\theta\|_{\frac{2}{2-\delta}}\|u_j\|_2^\delta \\
&\geq (\frac{\rho}{2} - 1)\|u_j\|^2 - (1 + \frac{\rho}{\gamma})\eta_2^\gamma\|a_0\|_{\frac{2}{2-\gamma}}\|u_j\|^\gamma \\
&\quad - (1 + \frac{\rho}{\sigma})\eta_2^\sigma\|b_0\|_{\frac{2}{2-\sigma}}\|u_j\|^\sigma - \|\theta\|_{\frac{2}{2-\delta}}\eta_2^\delta\|u_j\|^\delta .
\end{aligned}$$

Since $\rho > 2$ and $\gamma, \sigma, \delta < 2$, it follows that (u_j) is bounded in E. From the reflexivity of E and up to a subsequence if necessary, we may assume that $u_j \rightharpoonup u$ in E, for some $u \in E$. Now, we have

$$\|u_j - u\|^2 = (\psi'(u_j) - \psi'(u))(u_j - u) + \int_{\mathbb{R}} (f(x,u_j) - f(x,u))(u_j - u)dx. \quad (20)$$

It is clear that

$$(\psi'(u_j) - \psi'(u))(u_j - u) \longrightarrow 0 \ as \ j \longrightarrow \infty. \quad (21)$$

By Hölder's inequality, (2) and Lemma 3.5, one has

$$\begin{aligned} &\left| \int_{\mathbb{R}} (f(x,u_j) - f(x,u))(u_j - u)dx \right| \\ &\leq \|f(.,u_j) - f(.,u)\|_2 \|u_j - u\|_2 \\ &\leq \eta_2 \|f(.,u_j) - f(.,u)\|_2 \|u_j - u\| \longrightarrow 0 \end{aligned} \quad (22)$$

as $j \longrightarrow \infty$. Combining (20)-(22), we deduce that $u_j \longrightarrow u$ in E and the proof is completed. ■

Lemma 3.8. *Assume that* $(\mathcal{A}_0)$, (F_5) *and* (F_6) *are satisfied. Then for any sufficiently large* $k \in \mathbb{N}$, *there exist* $0 < r_k < \rho_k$ *such that*

$$a_k = \inf_{u \in Z_k, \|u\| = \rho_k} \psi(u) \geq 0. \quad (23)$$

Proof. Let $l_2(k)$ be defined as in the proof of Lemma 3.1. By (F_5), (F_6) and (2), we have for any $u \in Z_k$

$$\begin{aligned} \psi(u) =& \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} F(x,u)dx \\ \geq& \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} [a_0 |u|^\gamma + b_0 |u|^\sigma]dx - \int_{\mathbb{R}} [c|u| + d|u|^\mu]dx \\ \geq& \frac{1}{2} \|u\|^2 - \|a_0\|_{\frac{2}{2-\gamma}} \|u\|_2^\gamma - \|b_0\|_{\frac{2}{2-\sigma}} \|u\|_2^\sigma - \|c\|_2 \|u\|_2 - \|d\|_\infty \|u\|_\mu^\mu \\ \geq& \frac{1}{2} \|u\|^2 - l_2^\gamma(k) \|a_0\|_{\frac{2}{2-\gamma}} \|u\|^\gamma - l_2^\sigma(k) \|b_0\|_{\frac{2}{2-\sigma}} \|u\|^\sigma \\ &- l_2(k) \|c\|_2 \|u\| - \eta_\mu^\mu \|d\|_\infty \|u\|^\mu . \end{aligned} \quad (24)$$

In view of (24), $\mu > 2$ and $\gamma, \sigma > 1$, one has

$$\psi(u) \geq \frac{1}{4} \|u\|^2 - \left(l_2^\gamma(k) \|a_0\|_{\frac{2}{2-\gamma}} + l_2^\sigma(k) \|b_0\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \right) \|u\| \quad (25)$$

for $\|u\| \leq \inf \left\{ 1, \left(\frac{\|d\|_\infty \eta_\mu^\mu}{4} \right)^{\frac{1}{\mu-2}} \right\}$. Let $\rho_k = 8\Big(l_2^\gamma(k) \|a_0\|_{\frac{2}{2-\gamma}} + l_2^\sigma(k) \|b_0\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \Big)$. It is easy to see that $\rho_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, for sufficiently large integer k, (25) implies $a_k \geq \frac{1}{8}\rho_k^2 > 0$. ■

Lemma 3.9. *Assume that $(\mathcal{A}_0)$, (F_5) and (F_6) are satisfied. Then*

$$d_k = \inf_{u\in Z_k, \|u\|\le\rho_k} \psi(u) \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{26}$$

Proof. By (25), for any $u \in Z_k$, we have

$$\psi(u) \ge -\Big(l_2^{\gamma}(k)\,\|a_0\|_{\frac{2}{2-\gamma}} + l_2^{\sigma}(k)\,\|b_0\|_{\frac{2}{2-\sigma}} + l_2(k)\,\|c\|_2\Big)\,\|u\|\,. \tag{27}$$

Therefore, we get with $\|u\| \le \rho_k$

$$0 \ge d_k \ge -\Big(l_2^{\gamma}(k)\,\|a_0\|_{\frac{2}{2-\gamma}} + l_2^{\sigma}(k)\,\|b_0\|_{\frac{2}{2-\sigma}} + l_2(k)\,\|c\|_2\Big)\rho_k. \tag{28}$$

Since $l_2(k), \rho_k \longrightarrow 0$ as $k \longrightarrow \infty$, one has $d_k \longrightarrow 0$ as $k \longrightarrow \infty$. ■

Lemma 3.10. *Assume that $(\mathcal{A}_0)$, (F_5) and (F_6) are satisfied. Then*

$$b_k = \inf_{u\in Y_k, \|u\|=r_k} \psi(u) < 0,\ \forall k \in \mathbb{N}. \tag{29}$$

Proof. Firstly, we claim that there exists $\epsilon > 0$ such that

$$meas(\{x \in \mathbb{R}/c_0(x)\,|u(x)|^{\gamma} \ge \epsilon\,\|u\|^{\gamma}\}) \ge \epsilon,\ \forall u \in Y_k \setminus \{0\}\,. \tag{30}$$

If not, there exists a sequence $(u_n) \subset Y_k$ with $\|u_n\| = 1$ such that

$$meas(\Big\{x \in \mathbb{R}/c_0(x)\,|u_n(x)|^{\gamma} \ge \frac{1}{n}\Big\}) \le \frac{1}{n}. \tag{31}$$

Since $dim Y_k < \infty$, it follows from the compactness of the unit sphere of Y_k that there exists a subsequence, say (u_n) such that (u_n) converges to some $u \in Y_k$. Hence, we have $\|u\| = 1$. Since all norms are equivalent in the finite-dimensional space Y_k, we have $u_n \longrightarrow u$ in $L^2(\mathbb{R})$. By Hölder's inequality, one has

$$\int_{\mathbb{R}} c_0(x)\,|u_n - u|^{\gamma}\,dx \le \|c_0\|_{\frac{2}{2-\gamma}} \Big(\int_{\mathbb{R}} |u_n - u|^2\,dx\Big)^{\frac{\gamma}{2}} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{32}$$

Thus, there exists $\epsilon_0 > 0$ such that

$$meas(\{x \in \mathbb{R}/c_0(x)\,|u(x)|^{\gamma} \ge \epsilon_0\}) \ge \epsilon_0. \tag{33}$$

In fact, if not, we have for all $n \in \mathbb{N}$

$$meas(\Big\{x \in \mathbb{R}/c_0(x)\,|u(x)|^{\gamma} \ge \frac{1}{n}\Big\}) \le \frac{1}{n}.$$

Let $n \in \mathbb{N}$. Then for all integer $m \ge n$

$$meas(\Big\{x \in \mathbb{R}/c_0(x)\,|u(x)|^{\gamma} \ge \frac{1}{n}\Big\}) \le meas(\Big\{x \in \mathbb{R}/c_0(x)\,|u(x)|^{\gamma} \ge \frac{1}{m}\Big\}) \le \frac{1}{m}$$

which implies

$$meas(\left\{x \in \mathbb{R}/c_0(x)\,|u(x)|^\gamma \geq \frac{1}{n}\right\}) = 0.$$

So

$$\begin{aligned}\int_{\mathbb{R}} c_0(x)\,|u|^{\gamma+2}\,dx &= \int_{\left\{x\in\mathbb{R}/c_0(x)|u(x)|^\gamma \leq \frac{1}{n}\right\}} c_0(x)\,|u|^{\gamma+2}\,dx \\ &\leq \frac{1}{n^\gamma}\int_{\mathbb{R}} |u|^2\,dx \leq \frac{\eta_2}{n^\gamma}\,\|u\|^2 = \frac{\eta_2}{n^\gamma} \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow \infty$. Hence $u = 0$, which contradicts $\|u\| = 1$. Therefore (33) holds. Thus, define

$$\Omega_0 = \{x \in \mathbb{R}/c_0(x)\,|u(x)|^\gamma \geq \epsilon_0\}\,,\ \Omega_n = \left\{x \in \mathbb{R}/c_0(x)\,|u_n(x)|^\gamma \leq \frac{1}{n}\right\}.$$

Combining (31) and (33), we obtain

$$\begin{aligned}meas(\Omega_0 \bigcap \Omega_n) &= meas(\Omega_0 \setminus (\Omega_n^c \bigcap \Omega_0)) \\ &\geq meas(\Omega_0) - meas(\Omega_n^c \bigcap \Omega_0) \geq \epsilon_0 - \frac{1}{n},\ \forall n \in \mathbb{N}.\end{aligned}$$

Let n be an integer large enough such that $\epsilon_0 - \frac{1}{n} \geq \frac{1}{2}\epsilon_0$ and $\frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n} \geq \frac{\epsilon_0}{2^\gamma}$. We get

$$\begin{aligned}\int_{\mathbb{R}} c_0(x)\,|u_n - u|^\gamma\,dx &\geq \int_{\Omega_0 \bigcap \Omega_n} c_0(x)\,|u_n - u|^\gamma\,dx \\ &\geq (\frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n})meas(\Omega_0 \bigcap \Omega_n) \geq \frac{\epsilon_0^2}{2^{\gamma+1}}\end{aligned}$$

for all large integer n, which is a contradiction with (32). Therefore (30) holds.

For the ϵ given in (30), let

$$\Omega_u = \{x \in \mathbb{R}/c_0(x)\,|u(x)|^\gamma \geq \epsilon\,\|u\|^\gamma\}\,,\ \forall u \in Y_k \setminus \{0\}\,. \tag{34}$$

By (30), we obtain

$$meas(\Omega_u) \geq \epsilon,\ \forall u \in Y_k \setminus \{0\}\,. \tag{35}$$

For any $u \in Y_k$, by (F_5), (F_6), (34) and (35), one has

$$\begin{aligned}\psi(u) &= \frac{1}{2}\,\|u\|^2 - \int_{\mathbb{R}} F(x,u)dx \leq \frac{1}{2}\,\|u\|^2 - \frac{1}{\gamma}\int_{\mathbb{R}} c_0(x)\,|u|^\gamma\,dx - \int_{\mathbb{R}} H(x,u)dx \\ &\leq \frac{1}{2}\,\|u\|^2 - \frac{1}{\gamma}\int_{\Omega_u} c_0(x)\,|u|^\gamma\,dx \leq \frac{1}{2}\,\|u\|^2 - \frac{\epsilon}{\gamma}\,\|u\|^\gamma\,meas(\Omega_u) \\ &\leq \frac{1}{2}\,\|u\|^2 - \frac{\epsilon^2}{\gamma}\,\|u\|^\gamma\,.\end{aligned}$$

Choose $0 < r_k < \inf\left\{\rho_k, (\frac{\epsilon^2}{\gamma})^{\frac{1}{2-\gamma}}\right\}$. Direct computation shows that

$$b_k = \inf_{u\in Y_k, \|u\|=r_k} \psi(u) \leq \frac{1}{2}r_k^2 - r_k^{2-\gamma} r_k^\gamma = -\frac{1}{2}r_k^2 < 0.$$

The proof is completed. ∎

Proof of Theorem 1.4. Consider the functional ψ associated to equation (1)

$$\begin{aligned}\psi(u) &= \frac{1}{2}\int_{\mathbb{R}} [u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x)u(x)^2]dx - \int_{\mathbb{R}} F(x,u)dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)dx\end{aligned}$$

defined on the space E introduced in Section 2.

The functional ψ satisfies all the conditions of Lemma 2.6. Hence ψ has infinitely many nontrivial critical points, that is, equation (1) possesses infinitely many homoclinic solutions. ∎

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