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Subclass of Harmonic Univalent Functions Associated with the Differential Operator

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Abstract. In the present paper, we study a new subclasses of harmonic univalent functions by using differential operator in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also we obtain the coefficient bounds, convex combination, extreme points and convolution conditions.

Keywords: Harmonic functions; Univalent functions; Differential operator.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l \tag{1}$$

defined in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f_z(0) - 1 = 0$. Let the class of all normalized analytic univalent functions in the unit disc $\mathbb{U}$ is denoted by S.

A continuous complex valued function $f = u + iv$ defined in the simply connected domain $D \subset \mathbb{C}$ (Complex plane) is said to be harmonic in D if both u and v are real harmonic in D. Clunie and Shiel-Small [8] showed that in any simple connected domain D, we can write $f = h + \overline{g}$, where both h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D ([8]).

The class of functions $f(z) = h(z) + \overline{g(z)}$ which are harmonic univalent and sense-preserving in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$ is denoted by S_H. Each $f(z) \in S_H$, can be written as

$$f(z) = h(z) + \overline{g(z)}, \tag{2}$$

where

$$h(z) = z + \sum_{l=2}^{\infty} a_l z^l, \quad g(z) = \sum_{l=1}^{\infty} b_l z^l, |b_1| < 1$$

are analytic in $\mathbb{U}$.

If we take $g(z) = 0$ in (2), then the class S_H reduces to the class S. Also S_H^* is subclass of S_H consisting of function that map $\mathbb{U}$ onto starlike domain. In 1984, Clunie and Sheil-Small [8] investigated the class S_H and its geometric subclasses and obtain some coefficient bounds. Since then, many authors Sheil-Small [8], Silverman [16], Silverman and Silvia [17], and Jahangiri [8] have studied the subclasses of harmonic univalent functions. Ahuja [1] presented a systematic and unified study of harmonic univalent functions. Recently, many authors investigated various subclasses of harmonic univalent functions [2, 6, 7, 16]. Furthermore we refer to Duren [8], Ponnusamy [13] and their references for basic result on the subject.

In 2016, Makinde [12] introduced the differential operator $F^m : \mathcal{A} \to \mathcal{A}$ and defined as

$$F^m f(z) = z + \sum_{l=2}^{\infty} C_{lm} a_l z^l, \quad C_{lm} = \frac{l!}{|(l-m)|!}, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Later, Bharavi Sharma et al. [5] defined the differential operator $F^m : S_H \to \mathcal{S}_H$ as

$$F^m f(z) = F^m h(z) + (-1)^m \overline{F^m g(z)}, m \in \mathbb{N}_0, \tag{3}$$

where

$$F^m h(z) = z + \sum_{l=2}^{\infty} C_{lm} a_l z^l, \quad F^m g(z) = \sum_{l=1}^{\infty} C_{lm} b_l z^l.$$

In 2001, Rosy et al. [15] defined the subclass $\mathcal{G}_H(\beta)$ consisting of harmonic univalent functions $f(z)$ of the form (2), and $f(z)$ satisfies the following condition:

$$Re\left\{(1+e^{i\alpha})\frac{zf'(z)}{f(z)}-e^{i\alpha}\right\}\geq\beta,\quad 0\leq\beta<1,\alpha\in\mathbb{R},z\in\mathbb{U}.$$

In 2012, Patak et al.[14] defined the subclass $\mathcal{G}_H(m,\lambda,\rho,\beta)$ consist of harmonic univalent functions $f(z)$ of the form (2), and $f(z)$ satisfies the condition

$$Re\left\{(1+\rho e^{i\alpha})\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}-\rho e^{i\alpha}\right\}>\beta,\ 0\leq\beta<1,\rho\geq 0,\alpha\in\mathbb{R},z\in\mathbb{U},$$

where D_λ^m operator defined by Al-Shaksi and Darus [3], and is given by

$$D_\lambda^m f(z)=D_\lambda^m h(z)+(-1)^m\overline{D_\lambda^m g(z)},m,\lambda\in\mathbb{N}_0.$$

Motivated from this work, we defined the subclass $\mathcal{G}_H(m,\rho,\beta)$ consist of harmonic univalent functions $f(z)$ of the form (2), and $f(z)$ satisfies the condition

$$Re\left\{(1+\rho e^{i\alpha})\frac{F^{m+1}f(z)}{F^m f(z)}-\rho e^{i\alpha}\right\}>\beta, \tag{4}$$

for $m\in\mathbb{N}_0, 0\leq\beta<1,\rho\geq 0,\alpha\in\mathbb{R},(z\in\mathbb{U})$, where $F^m f(z)$ is defined by (3).

Let $\mathcal{G}_{\overline{H}}(m,\rho,\beta)$ denote the subclass of $\mathcal{G}_H(m,\rho,\beta)$ consisting of harmonic functions of the form

$$f_m(z)=h(z)+\overline{g_m(z)}, \tag{5}$$

where

$$h(z)=z-\sum_{l=2}^{\infty}|a_l|z^l\ \ and\ \ g_m(z)=(-1)^m\sum_{l=1}^{\infty}|b_l|z^l.$$

For $\rho=0$, the class $\mathcal{G}_H(m,\rho,\beta)$ reduced to the class $\mathcal{B}_H(m,\beta)$, studied by Bharavi Sharma et al. ([5]). Also, for $\rho=1,m=0$, the class $\mathcal{G}_H(m,\rho,\beta)$ reduced to the class $\mathcal{G}_H(\beta)$, studied by Rosy ([15]).

The aim of the present paper is, to obtain sufficient condition for functions $f(z)\in\mathcal{G}_H(m,\rho,\beta)$ of the form (2) and to obtain the necessary and sufficient condition for functions $f_m(z)\in\mathcal{G}_{\overline{H}}(m,\rho,\beta)$ of the form (5). Also, the aim is to obtain convolution, Convex combination and extreme points for functions $f_m(z)\in\mathcal{G}_{\overline{H}}(m,\rho,\beta)$ of the form (5).

2. Main Results

Theorem 2.1. *Let $f(z)=h(z)+\overline{g(z)}$ be given by* (2). *If*

$$\begin{aligned}&\sum_{l=2}^{\infty}\left[|l-m|-\beta+(|l-m|-1)\rho\right]C_{lm}|a_l|\\&+\sum_{l=1}^{\infty}\left[|l-m|+\beta+(|l-m|+1)\rho\right]C_{lm}|b_l|\leq 1-\beta,\end{aligned} \tag{6}$$

where $0 \leq \beta < 1, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha \in \mathbb{R}, \rho \geq 0$ *and* $C_{lm} = \frac{l!}{|(l-m)|!}$. *Then* f *is sense-preserving, harmonic univalent in* $\mathbb{U}$ *and* $f \in \mathcal{G}_H(m, \rho, \beta)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{l=1}^{\infty} b_l(z_1^l-z_2^l)}{(z_1-z_2)+\sum\limits_{l=2}^{\infty} a_l(z_1^l-z_2^l)}\right| \\
&> 1-\frac{\sum\limits_{l=1}^{\infty} l|b_l|}{1-\sum\limits_{l=2}^{\infty} l|a_l|} \\
&\geq 1-\frac{\sum\limits_{l=1}^{\infty} \frac{[|l-m|+\beta+(|l-m|+1)\rho]C_{lm}|b_l|}{1-\beta}}{1-\sum\limits_{l=2}^{\infty} \frac{[|l-m|-\beta+(|l-m|-1)\rho]C_{lm}|a_l|}{1-\beta}} \\
&\geq 0.
\end{aligned}$$

Hence, $f(z)$ is univalent in $\mathbb{U}$. $f(z)$ is sense-preserving in $\mathbb{U}$ because

$$\begin{aligned}
|h^{'}(z)| &\geq 1-\sum_{l=2}^{\infty} l|a_l||z|^{l-1} \\
&> 1-\sum_{l=2}^{\infty} \frac{[|l-m|-\beta+(|l-m|-1)\rho]\, C_{lm}|a_l|}{1-\beta} \\
&\geq \sum_{l=1}^{\infty} \frac{[|l-m|+\beta+(|l-m|+1)\rho]\, C_{lm}|b_l|}{1-\beta} \\
&> \sum_{l=1}^{\infty} \frac{[|l-m|+\beta+(|l-m|+1)\rho]\, C_{lm}|b_l||z|^{l-1}}{1-\beta} \\
&\geq \sum_{l=1}^{\infty} l|b_l||z|^{l-1} \geq |g^{'}(z)|.
\end{aligned}$$

Now, we show that $f(z) \in \mathcal{G}_H(m, \rho, \beta)$, using the fact that $Re(\alpha) > \beta$ if and only if $|1-\beta+\alpha| \geq |1+\beta-\alpha|$. It is suffices to show that

$$|1-\beta+(1+\rho e^{i\alpha})\frac{F^{m+1}f(z)}{F^m f(z)}-\rho e^{i\alpha}|-|1+\beta-(1+\rho e^{i\alpha})\frac{F^{m+1}f(z)}{F^m f(z)}+\rho e^{i\alpha}| \geq 0.$$

Now,

$$
\begin{aligned}
&|(1-\beta-\rho e^{i\alpha})F^m f(z)+(1+\rho e^{i\alpha})F^{m+1}f(z)| \\
&-|(1+\beta+\rho e^{i\alpha})F^m f(z)-(1+\rho e^{i\alpha})F^{m+1}f(z)| \\
=\;& |(2-\beta)z+\sum_{l=2}^{\infty}\left[|l-m|+|l-m|\rho e^{i\alpha}+(1-\beta-\rho e^{i\alpha})\right]C_{lm}a_l z^l \\
&-(-1)^m\sum_{l=1}^{\infty}\left[|l-m|+|l-m|\rho e^{i\alpha}-(1-\beta-\rho e^{i\alpha})\right]C_{lm}\overline{b_l z^l}| \\
&-|\beta z-\sum_{l=2}^{\infty}\left[|l-m|+|l-m|\rho e^{i\alpha}-(1+\beta+\rho e^{i\alpha})\right]C_{lm}a_l z^l \\
&+(-1)^m\sum_{l=1}^{\infty}\left[|l-m|+|l-m|\rho e^{i\alpha}+(1+\beta+\rho e^{i\alpha})\right]C_{lm}\overline{b_l z^l}| \\
\geq\;& (2-\beta)|z|-\sum_{l=2}^{\infty}\left[|l-m|+|l-m|\rho+(1-\beta-\rho)\right]C_{lm}|a_l||z|^l \\
&-\sum_{l=1}^{\infty}\left[|l-m|+|l-m|\rho-(1-\beta-\rho)\right]C_{lm}|b_l||z|^l \\
&-\beta|z|-\sum_{l=2}^{\infty}\left[|l-m|+|l-m|\rho-(1+\beta+\rho)\right]C_{lm}|a_l||z|^l \\
&-\sum_{l=1}^{\infty}\left[|l-m|+|l-m|\rho+(1+\beta+\rho)\right]C_{lm}|b_l||z|^l \\
=\;& (2-\beta)|z|-2\sum_{l=2}^{\infty}\left[|l-m|-\beta+(|l-m|-1)\rho\right]C_{lm}|a_l||z|^l \\
&-2\sum_{l=1}^{\infty}\left[|l-m|+\beta+(|l-m|+1)\rho\right]C_{lm}|b_l||z|^l \\
=\;& 2(1-\beta)|z|\left[1-\sum_{l=2}^{\infty}\frac{\left[|l-m|-\beta+(|l-m|-1)\rho\right]C_{lm}|a_l||z|^{l-1}}{1-\beta}\right. \\
&\left.-\sum_{l=1}^{\infty}\frac{\left[|l-m|+\beta+(|l-m|+1)\rho\right]C_{lm}|b_l||z|^{l-1}}{1-\beta}\right] \\
&\qquad\text{(since } z\in\mathbb{U}, |z|<1) \\
>\;& 2(1-\beta)\left[1-\sum_{l=2}^{\infty}\frac{\left[|l-m|-\beta+(|l-m|-1)\rho\right]C_{lm}|a_l|}{1-\beta}\right. \\
&\left.-\sum_{l=1}^{\infty}\frac{\left[|l-m|+\beta+(|l-m|+1)\rho\right]C_{lm}|b_l|}{1-\beta}\right].
\end{aligned}
$$

Last expression is non-negative by (6), therefore the proof is complete. ■

If we plug $\rho = 0$ in Theorem 2.1, then Corollary 2.2 is obtained.

Corollary 2.2. [5] *Let* $f(z) = h(z) + \overline{g(z)}$ *be given by* (2). *If*

$$\sum_{l=2}^{\infty} [|l-m| - \beta]\, C_{lm}|a_l| + \sum_{l=1}^{\infty} [|l-m| + \beta]\, C_{lm}|b_l| \leq 1 - \beta$$

where $0 \leq \beta < 1, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $C_{lm} = \frac{l!}{|(l-m)|!}$. *Then* f *is sense-preserving, harmonic univalent in* $\mathbb{U}$ *and* $f \in \mathcal{B}_H(m, \beta)$.

The harmonic function given below shows that the coefficient bound given by (6) is sharp.

$$\begin{aligned} f(z) =& z + \sum_{l=2}^{\infty} \frac{1-\beta}{[|l-m| - \beta + (|l-m|-1)\rho]\, C_{lm}} u_l z^l \\ &+ \sum_{l=1}^{\infty} \frac{1-\beta}{[|l-m| + \beta + (|l-m|+1)\rho]\, C_{lm}} \overline{v_l z^l}, \end{aligned}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \rho \geq 0$ and $\sum_{l=2} |u_l| + \sum_{l=1} |v_l| = 1$.

The above defined harmonic function is in $\mathcal{G}_H(m, \rho, \beta)$. We have

$$\sum_{l=i}^{\infty} \left[\begin{array}{l} \frac{[|l-m| - \beta + (|l-m|-1)\rho]\, C_{lm}}{1-\beta} |a_l| \\ + \frac{[|l-m| + \beta + (|l-m|+1)\rho]\, C_{lm}}{1-\beta} |b_l| \end{array} \right] = 1 + \sum_{l=2} |u_l| + \sum_{l=1} |v_l| = 2.$$

The following theorem shows that, the necessary condition for the function $f_m(z) = h(z) + \overline{g_m(z)}$ of the form (5) is the condition (6).

Theorem 2.3. *Let function* $f_m(z) = h(z) + \overline{g_m(z)}$ *be given by* (5). *Then* $f_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta)$ *if and only if*

$$\begin{aligned} &\sum_{l=2}^{\infty} [|l-m| - \beta + (|l-m|-1)\rho]\, C_{lm}|a_l| \\ &+ \sum_{l=1}^{\infty} [|l-m| + \beta + (|l-m|+1)\rho]\, C_{lm}|b_l| \leq 1 - \beta. \end{aligned} \tag{7}$$

Proof. It is easy to prove the 'if part', since $f_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta) \subset \mathcal{G}_H(m, \rho, \beta)$. Now, we prove the 'only if' part of Theorem 2.3. Let $f_m(z) = h(z) + \overline{g_m(z)} \in \mathcal{G}_{\overline{H}}(m, \rho, \beta)$. Then the condition (4) is equivalent to

$$Re\left\{ (1 + \rho e^{i\alpha}) \frac{F^{m+1} f(z)}{F^m f(z)} - (\rho e^{i\alpha} + \beta) \right\} \geq 0,$$

implies that,

$$Re\left\{\frac{(1+\rho e^{i\alpha})F^{m+1}f(z)-(\rho e^{i\alpha}+\beta)F^{m}f(z)}{F^{m}f(z)}\right\}\geq 0.$$

Therefore,

$$Re\left\{\begin{array}{c}\dfrac{(1+\rho e^{i\alpha})\left[z-\sum\limits_{l=2}^{\infty}C_{l(m+1)}|a_l|z^l+(-1)^{2m+1}\sum\limits_{l=1}^{\infty}C_{l(m+1)}|b_l|\overline{z^l}\right]}{z-\sum\limits_{l=2}^{\infty}C_{lm}|a_l|z^l+\sum\limits_{l=1}^{\infty}C_{lm}|b_l|\overline{z^l}}\\ -\dfrac{(\rho e^{i\alpha}+\beta))\left[z-\sum\limits_{l=2}^{\infty}C_{lm}|a_l|z^l+(-1)^{2m}\sum\limits_{l=1}^{\infty}C_{lm}|b_l|\overline{z^l}\right]}{z-\sum\limits_{l=2}^{\infty}C_{lm}|a_l|z^l+\sum\limits_{l=1}^{\infty}C_{lm}|b_l|\overline{z^l}}\end{array}\right\}\geq 0$$

Let $H=(1-\beta)-\sum_{l=2}^{\infty}\left[|l-m|-\beta+(|l-m|-1)\rho e^{i\alpha}\right]C_{lm}|a_l|z^{l-1}$. After simplification, we get

$$Re\left\{\frac{H-\frac{\overline{z}}{z}(-1)^{2m}\sum\limits_{l=1}^{\infty}\left[|l-m|+\beta+(|l-m|+1)\rho e^{i\alpha}\right]C_{lm}|b_l|\overline{z^{l-1}}}{1-\sum\limits_{l=2}^{\infty}C_{lm}|a_l|z^{l-1}+\frac{\overline{z}}{z}\sum\limits_{l=1}^{\infty}C_{lm}|b_l|\overline{z^{l-1}}}\right\}\geq 0. \quad (8)$$

The condition (8) must hold for all values of z on the positive real axis, $0\leq |z|=r<1$. Choose the values of z on positive real axis, where $0\leq|z|=r<1$. Now we have

$$Re\frac{\left\{H-e^{i\alpha}\left[\sum\limits_{l=2}^{\infty}\left[|l-m|-1\right]\rho C_{lm}|a_l|r^{l-1}+\sum\limits_{l=1}^{\infty}\left[|l-m|+1\right]\rho C_{lm}|b_l|r^{l-1}\right]\right\}}{1-\sum\limits_{l=2}^{\infty}C_{lm}|a_l|r^{l-1}+\sum\limits_{l=1}^{\infty}C_{lm}|b_l|r^{l-1}}\geq 0.$$

Since $Re(-e^{i\alpha})\geq -|e^{i\alpha}|=-1$, the above inequality become

$$Re\frac{\left[\begin{array}{l}(1-\beta)-\sum\limits_{l=2}^{\infty}\left[|l-m|-\beta+(|l-m|-1)\rho\right]C_{lm}|a_l|r^{l-1}\\ -\sum\limits_{l=1}^{\infty}\left[|l-m|+\beta+(|l-m|+1)\rho\right]C_{lm}|b_l|r^{l-1}\end{array}\right]}{1-\sum_{l=2}^{\infty}C_{lm}|a_l|r^{l-1}+\sum_{l=1}^{\infty}C_{lm}|b_l|r^{l-1}}\geq 0. \quad (9)$$

If condition (7) is not satisfied, then numerator in (9) is negative for r sufficient close to 1. This contradicts the condition for $f_m(z)\in\mathcal{G}_{\overline{H}}(m,\rho,\beta)$. Therefore proof is complete. ∎

3. Convolution

For harmonic functions

$$f_m(z) = z - \sum_{l=2}^{\infty} |a_l| z^l + (-1)^m \sum_{l=1}^{\infty} |b_l| \overline{z^l}$$

and

$$F_m(z) = z - \sum_{l=2}^{\infty} |A_l| z^l + (-1)^m \sum_{l=1}^{\infty} |B_l| \overline{z^l}.$$

We define the convolution of two harmonic function $f_m(z)$ and $F_m(z)$ as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{l=2}^{\infty} |a_l A_l| z^l + (-1)^m \sum_{l=1}^{\infty} |b_l B_l| \overline{z^l}. \quad (10)$$

Theorem 3.1. *For $0 \le \beta_1 \le \beta_2 < 1$, let $f_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_2)$ and $F_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_1)$. Then $f_m(z) * F_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_2) \subset \mathcal{G}_{\bar{H}}(m, \rho, \beta_1)$.*

Proof. Let $f_m(z) = z - \sum_{l=2}^{\infty} |a_l| z^l + (-1)^m \sum_{l=1}^{\infty} |b_l| \overline{z^l} \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_2)$ and $F_m(z) = z - \sum_{l=2}^{\infty} |A_l| z^l + (-1)^m \sum_{l=1}^{\infty} |B_l| \overline{z^l} \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_1)$, then the convolution $(f_m * F_m)(z)$ is given by (10). We wish to show that the coefficient of $(f_m * F_m)(z)$ satisfies the required condition given in Theorem 2.3. For $F_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_1)$, we note that $|C_l| \le 1$ and $|D_l| \le 1$. Now, for the coefficient of $(f_m * F_m)(z)$, we have

$$\begin{aligned}
&\sum_{l=2}^{\infty} \left[\frac{[|l-m| - \beta_1 + (|l-m|-1)\rho]\, C_{lm}}{1-\beta_1} |a_l A_l| \right] \\
&\quad + \sum_{l=1}^{\infty} \left[\frac{[|l-m| + \beta_1 + (|l-m|+1)\rho]\, C_{lm}}{1-\beta_1} |b_l B_l| \right] \\
&\le \sum_{l=2}^{\infty} \left[\frac{[|l-m| - \beta_1 + (|l-m|-1)\rho]\, C_{lm}}{1-\beta_1} |a_l| \right] \\
&\quad + \sum_{l=1}^{\infty} \left[\frac{[|l-m| + \beta_1 + (|l-m|+1)\rho]\, C_{lm}}{1-\beta_1} |b_l| \right] \\
&\le \sum_{l=2}^{\infty} \left[\frac{[|l-m| - \beta_2 + (|l-m|-1)\rho]\, C_{lm}}{1-\beta_2} |a_l| \right] \\
&\quad + \sum_{l=1}^{\infty} \left[\frac{[|l-m| + \beta_2 + (|l-m|+1)\rho]\, C_{lm}}{1-\beta_2} |b_l| \right] \le 1.
\end{aligned}$$

Since $0 \le \beta_1 \le \beta_2 < 1$ and $f_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_2)$. Thus $f_m(z) * F_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta_2) \subset \mathcal{G}_{\overline{H}}(m, \rho, \beta_1)$. Therefore proof of theorem is complete. ■

4. Convex Combination

Let the functions $f_{m_i}(z)$ be defined, for $i = 1, 2, 3, \cdots, j, \cdots$ by

$$f_{m_i}(z) = z - \sum_{l=2}^{\infty} |a_{l,i}| z^l + (-1)^m \sum_{l=1}^{\infty} |b_{l,i}| \overline{z^l}. \tag{11}$$

Theorem 4.1. *Let the functions $f_{m_i}(z)$ of the forms (11) belong to the class $\mathcal{G}_{\overline{H}}(m, \rho, \beta)$ for every $i = 1, 2, 3, \cdots, j$. Then the functions $t_i(z)$ defined by $t_i(z) = \sum_{i=1}^{j} d_i f_{m_i}(z), 0 \leq d_i \leq 1$ are also in the class $\mathcal{G}_{\overline{H}}(m, \rho, \beta)$, where $\sum_{i=1}^{j} d_i = 1$.*

Proof. By the definition of $t_i(z)$, we can write

$$t_i(z) = z - \sum_{l=2}^{\infty} \left(\sum_{i=1}^{j} d_i |a_{l,i}| \right) z^l + (-1)^m \sum_{l=1}^{\infty} \left(\sum_{i=1}^{j} d_i |b_{l,i}| \right) \bar{z}^l.$$

On comparing above equation with (5), we obtain $|a_l| = \left(\sum_{i=1}^{j} d_i |a_{l,i}| \right)$ and $|b_l| = \left(\sum_{i=1}^{j} d_i |b_{l,i}| \right)$. In order to prove $t_i(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta)$, we show that the condition (7) satisfies. Consider

$$\begin{aligned}
&\sum_{l=2}^{\infty} \left[|l-m| - \beta + (|l-m|-1)\rho \right] C_{lm} |a_l| \\
&\quad + \sum_{l=1}^{\infty} \left[|l-m| + \beta + (|l-m|+1)\rho \right] C_{lm} |b_l| \\
&= \sum_{l=2}^{\infty} \left[|l-m| - \beta + (|l-m|-1)\rho \right] C_{lm} \left(\sum_{i=1}^{j} d_i |a_{l,i}| \right) \\
&\quad + \sum_{l=1}^{\infty} \left[|l-m| + \beta + (|l-m|+1)\rho \right] C_{lm} \left(\sum_{i=1}^{j} d_i |b_{l,i}| \right) \\
&= \sum_{i=1}^{j} d_i \left(\sum_{l=2}^{\infty} \left[|l-m| - \beta + (|l-m|-1)\rho \right] C_{lm} |a_{l,i}| \right. \\
&\quad \left. + \sum_{l=1}^{\infty} \left[|l-m| + \beta + (|l-m|+1)\rho \right] C_{lm} |b_{l,i}| \right) \\
&\leq \sum_{i=1}^{j} d_i 2(1-\beta) \leq 2(1-\beta).
\end{aligned}$$

Since $f_{m_i}(z)$ are in $\mathcal{G}_{\overline{H}}(m, \rho, \beta)$ for every $i = 1, 2, 3, \cdots$.

Therefore by Theorem 2.3, $t_i(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$ and so the proof is complete. ■

5. Extreme Points

Theorem 5.1. *Let $f_m(z)$ be given by (5). Then $f_m(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$ if and only if*

$$f_m(z) = \sum_{l=1}^{\infty} \left[\mu_l h_l(z) + \eta_l g_{m_l}(z)\right], \tag{12}$$

where

$$\begin{aligned} h_1(z) =&\ z, h_l(z) = z - \left(\tfrac{1-\beta}{[|l-m|-\beta+(|l-m|-1)\rho]C_{lm}}\right) z^l, l = 2,3,\cdots, \\ g_{m_l}(z) =&\ \ z + (-1)^m \left(\tfrac{1-\beta}{[|l-m|+\beta+(|l-m|+1)\rho]C_{lm}}\right) \bar{z}^l, l = 1,2,3,\cdots, \end{aligned}$$

and $\sum_{l=1}^{\infty}(\mu_l + \eta_l) = 1, \mu_l \geq 0, \eta_l \geq 0$. In particular, the extreme points of $\mathcal{G}_{\overline{H}}(m,\rho,\beta)$ are $\{h_l(z)\}$ and $\{g_{m_l}(z)\}$.

Proof. For functions $f_m(z)$ of the form (12), we have

$$\begin{aligned} f_m(z) &= \sum_{l=1}^{\infty}\left[\mu_l h_l(z) + \eta_l g_{m_l}(z)\right] \\ &= \sum_{l=1}^{\infty}(\mu_l + \eta_l) z - \sum_{l=2}^{\infty} \frac{1-\beta}{[|l-m|-\beta+(|l-m|-1)\rho]\, C_{lm}} \mu_l z^l \\ &\quad +(-1)^m \sum_{l=1}^{\infty} \frac{1-\beta}{[|l-m|+\beta+(|l-m|+1)\rho]\, C_{lm}} \eta_l \overline{z^l}. \end{aligned}$$

Now, on comparing above equation with (5), we obtain $|a_l| = \frac{1-\beta}{[|l-m|-\beta+(|l-m|-1)\rho]C_{lm}}\mu_l$ and $|b_l| = \frac{1-\beta}{[|l-m|+\beta+(|l-m|+1)\rho]C_{lm}}\eta_l$. For proving $f_m(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$ we show that condition (7) satisfies. Consider,

$$\begin{aligned} &\sum_{l=2}^{\infty}\left[\frac{[|l-m|-\beta+(|l-m|-1)\rho]\, C_{lm}}{1-\beta}\right]|a_l| \\ &+\sum_{l=1}^{\infty}\left[\frac{[|l-m|+\beta+(|l-m|+1)\rho]\, C_{lm}}{1-\beta}\right]|b_l| \\ =& \sum_{l=2}^{\infty}\mu_l + \sum_{l=1}^{\infty}\eta_l = 1-\mu_1 \leq 1. \end{aligned}$$

Therefore by Theorem 2.3, $f_m(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$.

Conversely, suppose that $f_m(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$. Set

$$\mu_l = \frac{[|l-m|-\beta+(|l-m|-1)\rho]\, C_{lm}}{1-\beta}|a_l|, 0 \leq \mu_l \leq 1, l = 2,3,\cdots,$$

and

$$\eta_l = \frac{[|l-m| + \beta + (|l-m|+1)\rho]\, C_{lm}}{1-\beta}|b_l|, 0 \le \eta_l \le 1, l = 1, 2, 3, \cdots ,$$

and $\mu_1 = 1 - \sum_{l=2}^{\infty} \mu_l - \sum_{l=1}^{\infty} \eta_l$. Then $f_m(z)$ can be written as

$$\begin{aligned}
f_m(z) &= z - \sum_{l=2}^{\infty} |a_l| z^l + (-1)^m \sum_{l=1}^{\infty} |b_l| \overline{z^l} \\
&= z - \sum_{l=2}^{\infty} \frac{(1-\beta)}{[|l-m| - \beta + (|l-m|-1)\rho]\, C_{lm}} \mu_l z^l \\
&\quad + (-1)^m \sum_{l=2}^{\infty} \frac{(1-\beta)}{[|l-m| + \beta + (|l-m|+1)\rho]\, C_{lm}} \eta_l \bar{z}^l \\
&= z + \sum_{l=2}^{\infty} (h_l(z) - z)\mu_l + \sum_{l=1}^{\infty} (g_{m_l}(z) - z)\eta_l \\
&= \sum_{l=2}^{\infty} h_l(z)\mu_l + \sum_{l=1}^{\infty} g_{m_l}(z)\eta_l + z\left(1 - \sum_{l=2}^{\infty} \mu_l - \sum_{l=1}^{\infty} \eta_l\right) \\
&= \sum_{l=1}^{\infty} [\, h_l(z)\mu_l + g_{m_l}(z)\eta_l] \,,
\end{aligned}$$

so the proof is complete. ∎

Theorem 5.2. *Each member of* $\mathcal{G}_{\overline{H}}(m, \rho, \beta)\, , (0 \le \beta < 1)$ *maps* $\mathbb{U}$ *onto a starlike domain.*

Proof. We only need to show that, if $f_m(z) \in \mathcal{G}_{\overline{H}}(m, \rho, \beta)$, then

$$Re\left\{\frac{zh'(z) - \overline{zg'_m(z)}}{h(z) + \overline{g_m(z)}}\right\} > 0.$$

Using the fact that $Re(\alpha) > 0$ if and only if $|1+\alpha| \ge |1-\alpha|$. It is suffices to show that

$$\begin{aligned}
&\left|h(z) + \overline{g_m(z)} + zh'(z) - \overline{zg'_m(z)}\right| \\
&\quad - \left|h(z) + \overline{g_m(z)} - zh'(z) + \overline{zg'_m(z)}\right| \\
&= \left|2z - \sum_{l=2}^{\infty}(l+1)|a_l|z^l - (-1)^m \sum_{l=1}^{\infty}(l-1)|b_l|\overline{z^l}\right| \\
&\quad - \left|\sum_{l=2}^{\infty}(l-1)|a_l|z^l + (-1)^m \sum_{l=1}^{\infty}(l+1)|b_l|\overline{z^l}\right|
\end{aligned}$$

$$\begin{aligned}
&\geq \left(2|z| - \sum_{l=2}^{\infty}(l+1)|a_l||z|^l - \sum_{l=1}^{\infty}(l-1)|b_l||z|^l\right) \\
&\quad - \left(\sum_{l=2}^{\infty}(l-1)|a_l||z|^l + \sum_{l=1}^{\infty}(l+1)|b_l||z|^l\right) \\
&= 2|z|\left[1 - \left(\sum_{l=2}^{\infty} l|a_l||z|^{l-1} + \sum_{l=1}^{\infty} l|b_l||z|^{l-1}\right)\right] \\
&\geq 2|z|\left\{1 - \sum_{l=2}^{\infty}\left[|l-m| - \beta + (|l-m|-1)\rho\right] C_{lm}|a_l|\right. \\
&\quad \left. - \sum_{l=1}^{\infty}\left[|l-m| + \beta + (|l-m|+1)\rho\right] C_{lm}|b_l|\right\} \\
&\geq 2|z|[1-(1-\beta)] = 2|z|\beta \geq 0.
\end{aligned}$$

Hence proof is complete. ■

Theorem 5.3. *Let $f_m(z)$ be given by (5) belong to the class $\mathcal{G}_{\overline{H}}(m,\rho,\beta)$ and c is any real number with $c > -1$. Then the function $L_c(f_m(z))$ defined as $L_c(f_m(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f_m(t)dt, c > -1$, is also belong to the class $\mathcal{G}_{\overline{H}}(m,\rho,\beta)$.*

Proof. From definition of $L_c(f_m(z))$, it follows that

$$\begin{aligned}
L_c(f_m(z)) &= \frac{c+1}{z^c}\int_0^z t^{c-1} f_m(t)dt \\
&= \frac{c+1}{z^c}\int_0^z t^{c-1}\left(t - \sum_{l=2}^{\infty}|a_l|t^l + (-1)^m \sum_{l=1}^{\infty}|b_l|\overline{t^l}\right) dt \\
&= z - \sum_{l=2}^{\infty}\frac{c+1}{c+l}|a_l|z^l + (-1)^m \sum_{l=1}^{\infty}\frac{c+1}{c+l}|b_l|\overline{z^l} \\
&= z - \sum_{l=2}^{\infty}|A_l|z^l + (-1)^m \sum_{l=1}^{\infty}|B_l|\overline{z^l}
\end{aligned}$$

Now,on comparing the above equation with (5), we obtain $|a_l| = |A_l| = \frac{c+1}{c+l}|a_l|$ and $|b_l| = |B_l| = \frac{c+1}{c+l}|b_l|$. In order to prove $L_c(f_m(z)) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$, we need to show that condition (7) satisfies. Consider,

$$\begin{aligned}
&\sum_{l=2}^{\infty}\frac{\left[|l-m| - \beta + (|l-m|-1)\rho\right] C_{lm}}{1-\beta}\left(\frac{c+1}{c+l}|a_l|\right) \\
&\quad + \sum_{l=1}^{\infty}\frac{\left[|l-m| + \beta + (|l-m|+1)\rho\right] C_{lm}}{1-\beta}\left(\frac{c+1}{c+l}|b_l|\right) \\
&\leq \sum_{l=2}^{\infty}\left[\frac{\left[|l-m| - \beta + (|l-m|-1)\rho\right] C_{lm}}{1-\beta}\right]|a_l|
\end{aligned}$$

$$+\sum_{l=1}^{\infty}\left[\frac{[|l-m|+\beta+(|l-m|+1)\rho]\,C_{lm}}{1-\beta}\right]|b_l|\leq 1.$$

Since $f_m(z) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$. Therefore by Theorem 2.3, $L_c(f_m(z)) \in \mathcal{G}_{\overline{H}}(m,\rho,\beta)$. Proof is complete. ■

References

[1] O.P. Ahuja, Planer Harmonic Univalent and related mappings, *J. Inequal. Pure Appl. Math.* **6** (4) (2005) 1–18.

[2] A.K. AL-khafaji, W.G. Atshan, S.S. Abed, On the generalization of a class of harmonic univalent functions defined by differential operator, *Mathematics* **6** (2018), 9 pages.

[3] K. Al-Shaqsi and M. Darus, On harmonic functions defined by derivative operator, *Journal of Inequalities and Applications* (2008) Art. ID 263413, 10 pages.

[4] H. Bayram and S. Yalin, On a new subclass of harmonic univalent functions, *Malaysian Journal of Mathematical Sciences* **14** (1) (2020) 63–75.

[5] R. Bharavi Sharma and B. Ravinder, On a Subclass of harmonic univalent functions, *J. Phys. Conf. Ser* **1000**, (2018), 012115.

[6] D.D. Bobalade and N.D. Sangle, Certain subclass of harmonic univalent functions defined by generalized differential operator, *Design Engineering* **2020** (8) (2021) 7410–7422.

[7] D.D. Bobalade and N.D. Sangle, Subclass of harmonic univalent functions defined by differential operator, *Jñānābha* **51** (2) (2021) 130–136.

[8] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fen. Series Al. Math.* **9** (3) (1984) 3–25.

[9] P. Duren, *Harmonic Mapping in the Plane*, Cambridge Tracts in Mathematics **156**, Cambridge University Press, Cambridge, 2004.

[10] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, *Southwest J. Pure Appl. Math.* **2** (2002) 77–82.

[11] F.F. Aubdulnabi and K.A. Jassim, A class of harmonic univalent functions defined by differential operator and the generalization, *Iraqi Journal of Science* **61** (6) (2020) 1440–1445.

[12] D.O. Makinde, On a new differential operator, *Theoretical Mathematics and Applications* **6** (4) (2016) 71–74.

[13] S. Ponnusamy and A. Rasila, Planar harmonic mapping, *RMS Mathematics Newsletter* **17** (2) (2007) 40–57.

[14] A.L. Pathak, S.B. Joshi, D. Preeti, R. Agarwal, Subclass of harmonic univalent functions associated with the derivative operator, *Hacettepe Journal of Mathematics and Statistics* **4** (1) (2012) 47–58.

[15] T. Rosy, B.A. Stephen, K.G. Subramanian, J.M. Jahangiri, Goodman-Ronning-type harmonic univalent functions, *Kyungpook Math. J.* **41** (2001) 45–54.

[16] H. Silverman, Harmonic univalent function with negative coefficients, *J. Math Anal. Appl.* **220** (3) (1998) 233–289.

[17] H. Silverman and E.M. Silvia, Subclasses of harmonic univalent functions, *New Zeal. J. Math.* **28** (1999) 275–284.