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Functions Which Are Related to Dirichlet Series Analytic in the Half Plane

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Abstract. In the present paper, let X be a class of functions represented by Dirichlet series analytic in the half plane. Various results on the topological structure in terms of continuous linear functionals on X and continuous linear operator T' from X to X have been established. Further the conditions under which a base in X becomes a proper base have been characterised. Also the results on Frechet space and Total set are discussed.

Keywords: Dirichlet series; Analytic functions; Proper bases; Total set.

1. Introduction

Let X be a class of functions represented by Dirichlet series

$$f(z) = \sum_{k=1}^{\infty} a_k \, e^{<\lambda^k, z>}, \qquad z \in \mathbb{C}^n \tag{1}$$

where $\{\lambda^k\}$; $\lambda^k = (\lambda_1^k, \lambda_2^k, ..., \lambda_n^k),\ k = 1, 2, ...$ be a sequence of complex vectors in $\mathbb{C}^n$ and $<\lambda^k, z>= \lambda_1^k z_1 + \lambda_2^k z_2 + ... + \lambda_n^k z_n$. Also $\{\lambda^k\}'s$ satisfy the condition $|\lambda^k| \to \infty$ as $k \to \infty$. If $a_k{}'s \in \mathbb{C}$ satisfy

$$\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|} \leq -B \tag{2}$$

where B is assumed to be a given positive number and

$$\limsup_{k\to\infty} \frac{\log\ k}{|\lambda^k|} = 0 \tag{3}$$

then for each $f(z) \in X$,

$$\sigma_c(f) = \sigma_a(f) = -\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|},$$

where $\sigma_c(f)$ and $\sigma_a(f)$ denote the abscissa of convergence and the abscissa of absolute convergence of f respectively. Suppose that (1) converges absolutely in the left half plane $r < B$, where r is the real part of z. Then X includes all the analytic functions represented by series (1). If

$$f(z) = \sum_{k=1}^{\infty} a_k e^{<\lambda^k, z>} \qquad \text{and} \qquad g(z) = \sum_{k=1}^{\infty} b_k e^{<\lambda^k, z>}$$

Define the binary operations defined in X as

$$\begin{aligned} f(z) + g(z) &= \sum_{k=1}^{\infty} (a_k + b_k) e^{<\lambda^k, z>}, \\ \xi.f(z) &= \sum_{k=1}^{\infty} (\xi.a_k) e^{<\lambda^k, z>}. \end{aligned}$$

Obviously X forms a vector space with usual pointwise addition and scalar multiplication. Now for each $f(z) \in X$, define

$$\|f\|_r = \sum_{k=1}^{\infty} |a_k| e^{r \sum_{s=1}^{n} |\lambda_s^k|} \tag{4}$$

for every $r < B$. Clearly $\|f\|_r$ exists on account of (2) and defines a norm on X, for each $r < B$. We denote by $X(r)$, the space X equipped with the norm $\|.\|_r$.

Let D be the topology generated by the family of norms $\{\|f\|_r : r < B\}$. This topology is equivalent to the topology generated by the invariant metric d, where

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g\|_{r_k}}{1 + \|f - g\|_{r_k}}, \quad (Frechet\ \ Combination),$$

where, $\{r_k\}$ is a sequence such that $r_1 < r_2 < ... < r_k \ldots;\ r_k \to B$ as $k \to \infty$.

In the previous years immense research has been carried out in the field of Dirichlet series of one and several complex variables. Recently Kumar and Manocha in [5] generalised the condition of weighted norm for a Dirichlet series of one variable given by $\sum_{n=1}^{\infty} a_n e^{\lambda_n s}$ and thus proved some results. For the same series, Kumar and Manocha in [6] considered a class of entire functions, established it to be a locally convex topological linear space and thus proved it to be a complex FK-space and a Frechet space. Very recently Akanksha and Srivastava in [1] studied the spaces of vector-valued Dirichlet series in a half-plane and thereby proved many important results. Kong and Gan in [4] and Kong in [3] studied various results on order and type of Dirichlet series.

The origin of Dirichlet series with complex frequencies $f(z) = \sum_{k=1}^{\infty} a_k \times e^{<\lambda^k, z>}$ can be traced from the fact that every entire function in $\mathbb{C}^n$ as well as each holomorphic function in a convex domain of $\mathbb{C}^n$ can always be represented in the form of Dirichlet series with complex frequencies. It finds its numerous applications in the theory of functional equations.

Khoi in [2] studied the coefficient multipliers for some classes of Dirichlet series with complex frequencies including those that define entire functions in $\mathbb{C}^n$.

In the present paper, the space X of functions defined by (1) and analytic in the half-plane are considered. Initially it is shown that the space X equipped with certain locally convex topology D becomes a Frechet space. The form of continuous linear functionals on X and continuous linear operator T' from X to X is then characterized. Further the conditions under which a base in X becomes a proper base, in terms of semi-norms which generate the topology D on X, have been discussed. Lastly the result on total set is also studied. The purpose of this paper is to widen the scope of the study of Dirichlet series with complex frequencies.

2. Basic Definitions

The following definitions are required to prove the main results. For the definitions of terms used refer [9]–[8].

Definition 2.1. *A sequence* $\{c_k\} \subset X$ *will be linearly independent if* $\sum_{k=1}^{\infty} a_k c_k = 0$ *implies* $a_k = 0$*, for all* $k \geq 1$ *that is for all sequences* $\{a_k\}$ *of complex numbers for which* $\sum_{k=1}^{\infty} a_k c_k$ *converges in* X*.*

The sequence $\{c_k\} \subset X$ *spans a subspace* X_0 *of* X*, if* X_0 *consists of all linear combinations* $\sum_{k=1}^{\infty} a_k c_k$*, such that* $\sum_{k=1}^{\infty} a_k c_k$ *converges in* X*.*

A sequence $\{c_k\} \subset X$*, which is linearly independent and spans a closed subspace* X_0 *of* X*, will be a base in* X_0*. If* $e_k \subset X$*,* $e_k(z) = e^{<\lambda^k, z>}$*,* $k \geq 1$ *then clearly* $\{e_k\}$ *is a base in* X*.*

A sequence $\{c_k\} \subset X$ *will be a proper base if it is a base and it satisfies the condition that for all sequences* $\{a_k\}$ *of complex numbers,* $\sum_{k=1}^{\infty} a_k c_k$ *converges* *in* X *if and only if* $\sum_{k=1}^{\infty} a_k e_k$ *converges in* X.

Definition 2.2. *Let* X *be a locally convex topological vector space. A set* $E' \subset X$ *is said to be total if and only if for any* $\psi \in X'$ *where* X' *denotes the dual of* X *with* $\psi(E') = 0$ *we have* $\psi = 0$.

3. Main Results

In this section main results are proved.

Theorem 3.1. *The space* X *is complete with respect to the metric* d *and hence is a Frechet space.*

Proof. Let $f_p(z) = \sum_{k=1}^{\infty} a_k^{(p)} e^{<\lambda^k, z>}$; $p = 1, 2, ...$ be a cauchy sequence in X. Let $\epsilon > 0$ be given such that

$$\|f_p - f_q\| < \epsilon \text{ where } p, q \geq N.$$

This implies that

$$\sum_{k=1}^{\infty} |a_k^{(p)} - a_k^{(q)}| e^{r \sum_{s=1}^{n} |\lambda_s^k|} < \epsilon \text{ for all } p, q \geq N.$$

Clearly $\{a_k^{(p)}\}$ being a cauchy sequence in the set of complex numbers converges to some element say a_k for every value of $k \geq 1$. This implies

$$\sum_{k=1}^{\infty} |a_k^{(p)} - a_k| e^{r \sum_{s=1}^{n} |\lambda_s^k|} < \epsilon \text{ for } p \geq N.$$

Also

$$\sum_{k=1}^{\infty} |a_k| e^{r \sum_{s=1}^{n} |\lambda_s^k|} \leq \sum_{k=1}^{\infty} |a_k^{(p)} - a_k| e^{r \sum_{s=1}^{n} |\lambda_s^k|} + \sum_{k=1}^{\infty} |a_k^{(p)}| e^{r \sum_{s=1}^{n} |\lambda_s^k|}.$$

Hence $\sum_{k=1}^{\infty} a_k e_k \in X$. Thus $f_p \to f$ in X, where $f(z) \in X$ and this proves the theorem. ∎

Lemma 3.2. *A continuous linear functional* η *on* $X(r)$, *is of the form*

$$\eta(f) = \sum_{k=1}^{\infty} a_k \, p_k \, e^{r \sum_{s=1}^{n} |\lambda_s^k|}$$

if and only if $\{p_k\}$ *is a bounded sequence in* $\mathbb{C}$, *where* $f(z)$ *is as defined in* (1).

Proof. Let η be a continuous linear functional on $X(r)$. Then

$$\eta(f) = \eta\left(\lim_{N\to\infty} f^{(N)}\right)$$

where

$$f^{(N)}(z) = \sum_{k=1}^{N} a_k\, e^{<\lambda^k, z>}.$$

Define a sequence $\{f_k\}$ as $f_k = e^{-r\sum_{s=1}^{n}|\lambda_s^k|}\, e^{<\lambda^k,z>} \subseteq X$. Then

$$\begin{aligned}\eta(f) &= \eta\left(\lim_{N\to\infty}\sum_{k=1}^{N} a_k\ e^{r\sum_{s=1}^{n}|\lambda_s^k|}\, f_k\right)\\ &= \lim_{N\to\infty}\sum_{k=1}^{N} a_k\ e^{r\sum_{s=1}^{n}|\lambda_s^k|}\,\eta(f_k).\end{aligned}$$

Since η is a linear functional, therefore $\eta(f_k) = p_k$, which implies

$$\eta(f) = \sum_{k=1}^{\infty} a_k\, p_k\, e^{r\sum_{s=1}^{n}|\lambda_s^k|}.$$

Now $|p_k| = |\eta(f_k)| \le P'\|f_k\|$ and $\|f_k\| = 1$. This implies $|p_k| \le P'$. Thus $\{p_k\}$ is a bounded sequence in $\mathbb{C}$.

Conversely, let $\{p_k\}$ be a bounded sequence in $\mathbb{C}$, satisfying

$$\eta(f) = \sum_{k=1}^{\infty} a_k\, p_k\, e^{r\sum_{s=1}^{n}|\lambda_s^k|}.$$

Then η is well defined and linear. Also

$$|\eta(f)| \le P'\sum_{k=1}^{\infty}|a_k|\ e^{r\sum_{s=1}^{n}|\lambda_s^k|} = P'\|f\|.$$

Thus, η is a continuous linear functional. ∎

Remark 3.3. A continuous linear functional η on $X(r)$, is of the form

$$\eta(f) = \sum_{k=1}^{\infty} a_k w_k,\ \text{where } f(z) = \sum_{k=1}^{\infty} a_k e_k,\ \ w_k = \eta(e_k)$$

if and only if $\left\{|w_k|/e^{r\sum_{s=1}^{n}|\lambda_s^k|}\right\}$ is bounded, for all $k \ge 1$.

Theorem 3.4. *A sufficient condition that there exists a continuous linear transformation $T' : X \to X$ with $T'(e_k) = c_k,\ k = 1, 2, ...$ is that for each $r < B$,*

$$\limsup_{k\to\infty}\frac{\log\|c_k\|_r}{|\lambda^k|} < B. \tag{5}$$

Proof. Let (5) holds and let $f(z)=\sum_{k=1}^{\infty} a_k e_k \in X$. Then there exists an $\epsilon>0$, such that

$$\frac{\log\ \|c_k\|_r}{|\lambda^k|} \leq B-\epsilon,\ \ \forall\ \ k \geq K_1(\epsilon)$$

which further implies

$$\|c_k\|_r \leq e^{(B-\epsilon)|\lambda^k|}.$$

Choose $\rho>0$, such that $\rho<\epsilon$. Then

$$|a_k| \leq e^{(-B+\rho)|\lambda^k|},\ \ \forall\ \ k \geq K_2(\rho).$$

Thus

$$\begin{aligned} |a_k|\|c_k\|_r &\leq e^{(-B+\rho)|\lambda^k|}\ e^{(B-\epsilon)|\lambda^k|} \\ &= e^{(\rho-\epsilon)|\lambda^k|} \end{aligned}$$

for all $k \geq \max(K_1, K_2)$. This implies $\sum_{k=1}^{\infty}|a_k|\|c_k\|_r$ is convergent and as r is arbitrarily less than B, one obtains that $\sum_{k=1}^{\infty} a_k c_k$ is convergent in X. Thus there exists a transformation $T': X \to X$ such that

$$T'(f)=\sum_{k=1}^{\infty} a_k c_k, \quad \text{for each} \quad f(z) \in X.$$

Then T' is linear, $T'(e_k)=c_k,\ k=1,2,...$ and for given $r<B$, there exists $\rho>0$, such that

$$\frac{\log\ \|c_k\|_r}{|\lambda^k|} \leq B-\rho,\ \ \forall\ \ k \geq K$$

which implies

$$\|c_k\|_r \leq e^{(B-\rho)|\lambda^k|},\ \forall\ k \geq K$$

which further implies

$$\|c_k\|_r \leq N'\, e^{(B-\rho)|\lambda^k|},\ \ \forall\ k \geq 1.$$

Thus

$$\begin{aligned} \|T'(f)\| &\leq N' \sum_{k=1}^{\infty} |a_k| e^{(B-\rho)\sum_{s=1}^{n}|\lambda_s^k|} \\ &= N'\|f\|_{(B-\rho)}. \end{aligned}$$

Hence T' is continuous. This completes the proof. ∎

Lemma 3.5. *Let $\{c_k\} \subset X$. Then the following three properties are equivalent:*

(A') $\limsup_{k\to\infty} \frac{\log \|c_k\|_r}{|\lambda^k|} < B$, *for all* $r<B$.

(B') *For all sequences $\{a_k\}$ of complex numbers, convergence of $\sum_{k=1}^{\infty} a_k e_k$ implies the convergence of $\sum_{k=1}^{\infty} a_k c_k$ in X.*

(C') *For all sequences* $\{a_k\}$ *of complex numbers, convergence of* $\sum_{k=1}^{\infty} a_k e_k$ *implies that* $\{a_k c_k\}$ *tends to zero in* X*.*

Proof. We have already proved (A') $\Rightarrow$ (B') in the proof of sufficiency part of Theorem 3.4. Also it is clear that (B') $\Rightarrow$ (C'). We need to prove only (C') $\Rightarrow$ (A').

Assume that (C') is true and (A') is false. This implies that, for some $r' < B$,

$$\limsup_{k\to\infty} \frac{\log \|c_k\|_{r'}}{|\lambda^k|} \geq B.$$

Hence there exists a sequence $\{k_n\}$ of positive integers, such that

$$\frac{\log \|c_{k_n}\|_{r'}}{|\lambda^{k_n}|} \geq B - \frac{1}{n}, \quad \text{for all} \quad n = 1, 2, \ldots.$$

Define $\{a_k\}$ by-

$$a_k = \begin{cases} e^{-(B-\frac{1}{n})\sum_{s=1}^{n} |\lambda_s^{k_n}|}, \ n = 1, 2, ... \\ 0, \ k \neq k_n \end{cases}$$

So we have

$$\begin{aligned} |a_{k_n}| e^{r\sum_{s=1}^{n} |\lambda_s^{k_n}|} &= e^{-(B-\frac{1}{n})\sum_{s=1}^{n} |\lambda_s^{k_n}|} \ e^{r\sum_{s=1}^{n} |\lambda_s^{k_n}|} \\ &= e^{-(B-(r+\frac{1}{n}))\sum_{s=1}^{n} |\lambda_s^{k_n}|}. \end{aligned}$$

There exists a n, large enough such that $B - (r + \frac{1}{n}) > 0$. This implies

$$\sum_{n=1}^{\infty} |a_{k_n}| e^{r\sum_{s=1}^{n} |\lambda_s^{k_n}|}$$

converges in X, for all $r < B$.

But

$$|a_{k_n}| \|c_{k_n}\|_{r'} \geq e^{-(B-\frac{1}{n})\sum_{s=1}^{n} |\lambda_s^{k_n}|} \ e^{(B-\frac{1}{n})|\lambda^{k_n}|} \nrightarrow 0.$$

This implies $a_{k_n} c_{k_n}$ does not tend to zero in X, a contradiction and this contradicts (C'). So (C') $\Rightarrow$ (A'). ■

Lemma 3.6. *The following three conditions are equivalent for any sequence* $\{c_k\} \subset X$*:*

(a) $\lim_{r\to B} \left\{ \liminf_{k\to\infty} \frac{\log \ \|c_k\|_r}{|\lambda^k|} \right\} \geq B$.

(b) *For all sequences* $\{a_k\}$ *of complex numbers, convergence of* $\sum_{k=1}^{\infty} a_k c_k$ *in* X *implies the convergence of* $\sum_{k=1}^{\infty} a_k e_k$ *in* X*.*

(c) *For all sequences* $\{a_k\}$ *of complex numbers,* $\{a_k c_k\}$ *tends to zero in* X *implies the convergence of* $\sum_{k=1}^{\infty} a_k e_k$ *in* X*.*

Proof. Clearly $(c) \Rightarrow (b)$. Now one needs to prove that $(b) \Rightarrow (a)$ and $(a) \Rightarrow (c)$.

First let us suppose that (b) holds but (a) doesnot hold. Therefore

$$\lim_{r\to B}\left\{\liminf_{k\to\infty}\frac{\log\ \|c_k\|_r}{|\lambda^k|}\right\} < B.$$

Since, $\|...\|_r$ increases as r increases, this implies that for each $r < B$,

$$\liminf_{k\to\infty}\frac{\log\ \|c_k\|_r}{|\lambda^k|} < B, \quad \forall \quad r < B.$$

If α be a small positive number, then there exists an increasing sequence $\{k_p\}$, such that

$$\frac{\log\ \|c_{k_p}\|_r}{|\lambda^{k_p}|} \leq B - \alpha$$

which implies

$$\|c_{k_p}\|_r \leq e^{(B-\alpha)|\lambda^{k_p}|}.$$

Choose $\rho < \alpha$ and define $\{a_k\}$ by-

$$a_k = \begin{cases} e^{-(B-\rho)|\lambda^{k_p}|} & \text{if} \quad p = 1, 2, ..., \\ 0 & \text{if} \quad k \neq k_p. \end{cases}$$

Then for every $r < B$,

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k|\|c_k\|_r &= \sum_{p=1}^{\infty}|a_{k_p}|\|c_{k_p}\|_r \\ &\leq \sum_{p=1}^{\infty}e^{-(B-\rho)|\lambda^{k_p}|}\,e^{(B-\alpha)|\lambda^{k_p}|} \\ &= \sum_{p=1}^{\infty}e^{(\rho-\alpha)|\lambda^{k_p}|}\end{aligned}$$

and the last series is convergent since $\rho < \alpha$. Hence for this sequence $\{a_k\}$, $\sum_{k=1}^{\infty} a_k c_k$ converges in $X(r)$, for each $r < B$, and hence converges in X. But,

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k|e^{r\sum_{s=1}^{n}|\lambda_s^k|} &= \sum_{p=1}^{\infty}|a_{k_p}|e^{r\sum_{s=1}^{n}|\lambda_s^{k_p}|} \\ &\geq \sum_{p=1}^{\infty}|a_{k_p}|\,e^{r|\lambda^{k_p}|} \\ &= \sum_{p=1}^{\infty}e^{-(B-\rho)|\lambda^{k_p}|}\,e^{r|\lambda^{k_p}|} \\ &= \sum_{p=1}^{\infty}e^{(r+\rho-B)|\lambda^{k_p}|}.\end{aligned}$$

Given ρ choose $r < B$ such that $r + \rho > B$, and thus the last series is divergent for this r. Hence $\sum_{k=1}^{\infty} a_k e_k$ does not converge in X and this contradicts (b). Thus (b) implies (a).

Now to prove $(a) \Rightarrow (c)$, assume that (a) is true but (c) is not true. Thus there exists a sequence $\{a_k\}$ of complex numbers for which $\{a_k c_k\}$ tends to zero in X, but $\sum_{k=1}^{\infty} a_k e_k$ doesnot converge in X. This implies

$$\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|} > -B.$$

Hence given $\delta > 0$, there exists a sequence $\{k_n\}$ of positive integers, such that

$$|a_{k_n}| \geq e^{(-B+\delta)|\lambda^{k_n}|}.$$

Now choose a positive number r, such that $\delta > 2\beta$. (a) being true, one can find a number $r = r(\beta)$, such that

$$\liminf_{k\to\infty} \frac{\log \; \|c_k\|}{|\lambda^k|} \geq B - \beta.$$

Hence there exists $N = N(\beta)$, such that

$$\frac{\log \; \|c_k\|_r}{|\lambda^k|} \geq B - 2\beta, \;\text{ for all }\; k \geq N.$$

Therefore,

$$\begin{aligned} |a_{k_n}| \|c_{k_n}\|_r &\geq e^{(-B+\delta)|\lambda^{k_n}|}\, e^{(B-2\beta)|\lambda^{k_n}|} \\ &= e^{(\delta-2\beta)|\lambda^{k_n}|} \to +\infty \end{aligned}$$

as $n \to \infty$, since $\delta > 2\beta$. This shows that $\{a_k c_k\}$ does not tend to zero in X which is a contradiction. Thus we conclude that $(a) \Rightarrow (c)$. ∎

Theorem 3.7. *A base in a closed subspace X_0 of X is proper if and only if conditions (A') and (a) are satisfied.*

Theorem 3.8. *Let $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>} \in X,\; a_k \neq 0,\; k \geq 1$. Let $D \subset \mathbb{C}^n$ be a region having at least one limit point. Define*

$$f_\tau(z) = \sum_{k=1}^{\infty} a_k\, e^{-r\sum_{s=1}^{n}|\lambda_s^k|} e^{<\lambda^k, \tau+z>}. \tag{6}$$

Then the set $A_f = \{f_\tau : \tau \in D\}$ is a total set in X.

Proof. Note first that for all $\tau \in \mathbb{C}^n$,

$$\|f_\tau\| = \sum_{k=1}^{\infty} |a_k|\, e^{Re<\lambda^k, \tau>}$$

which is clearly convergent.

Let ψ^* be a continuous linear transformation such that $\psi^*(A_f) \equiv 0$, that is $\psi^*(f_\tau) = 0$. Thus by Lemma 3.2,

$$\sum_{k=1}^{\infty}\{a_k\, e^{-r\sum_{s=1}^{n}|\lambda_s^k|}\, e^{<\lambda^k,\tau>}\, p_k\}e^{r\sum_{s=1}^{n}|\lambda_s^k|} = 0$$

which implies

$$\sum_{k=1}^{\infty} a_k\, p_k\, e^{<\lambda^k,\tau>} = 0, \text{ for all } \tau \in \mathbb{C}^n. \tag{7}$$

Define $h(z) = \sum_{k=1}^{\infty} a_k\, p_k\, e^{<\lambda^k,z>}$. Since $\{p_k\}$ is a bounded sequence in $\mathbb{C}$ and $f(z) = \sum_{k=1}^{\infty} a_k e^{<\lambda^k,z>}$, we have $h(z)$ also belongs to X. But by (7),

$$h(\tau) = \sum_{k=1}^{\infty} a_k\, p_k\, e^{<\lambda^k,\tau>} = 0 \text{ for all } \tau \in \mathbb{C}^n.$$

Since $\mathbb{C}^n$ has a finite limit point, we have $h \equiv 0$. This further implies $a_k\, p_k = 0$, for all $k \geq 1$ and as $a_k \neq 0$ implies $p_k = 0$, for all $k \geq 1$. Thus $\psi^* = 0$. This completes the proof. ■

References

[1] Akanksha and G.S. Srivastava, Spaces of vector-valued Dirichlet series in a half-plane, *Front. Math. China* **9** (6) (2014) 1239–1252.

[2] L.H. Khoi, Coefficient multipliers for some classes of Dirichlet series in several complex variables, *Acta Math. Vietnamica* **24** (2) (1999) 169–182.

[3] Y.Y. Kong, On some q-orders and q-types of Dirichlet-Hadamard product function, *Acta Mathematica Sinica, Chinese Series* **52** (2009) 1165–1172.

[4] Y. Kong and H. Gan, On orders and types of Dirichlet series of slow growth, *Turk. J. Math.* **34** (2010) 1–11.

[5] N. Kumar and G. Manocha, On a class of entire functions represented by Dirichlet series, *J. Egypt. Math. Soc.* **21** (2013) 21–24.

[6] N. Kumar and G. Manocha, A class of entire Dirichlet series as an FK-space and a Frechet space, *Acta Math. Scientia* **33B** (6) (2013) 1571–1578.

[7] R. Larsen, *Banach Algebras- An Introduction*, Marcel Dekker Inc., New York, 1973.

[8] R. Larsen, *Functional Analysis- An Introduction*, Marcel Dekker Inc., New York, 1973.

[9] J.R. Yu, *Dirichlet Series and Random Dirichlet Series*, Science Press, Beijing, 1997.