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On $\mathcal{I}-$Statistical $\phi-$Limit Superior and $\mathcal{I}-$Statistical $\phi-$Limit Inferior

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Abstract. In this paper, we have extended the concepts of $\mathcal{I}-$statistical limit superior and $\mathcal{I}-$statistical limit inferior to $\mathcal{I}-$statistical ϕ-limit superior and $\mathcal{I}-$statistical ϕ-limit inferior and studied some of their properties for sequence of real numbers.

Keywords: $\mathcal{I}-$statistical limit superior; $\mathcal{I}-$statistical limit inferior; $\mathcal{I}-$statistical ϕ-convergence.

1. Introduction

The notion of statistical convergence was first introduced in the year 1951 independently by Fast [10] and Steinhaus [29] in connection with summability theory. Following them, the concept was investigated by Fridy [11, 12], Salat [25] and many others from the sequence space point of view. Various applications of statistical convergence can be found in [2, 3, 19, 20, 30].

Generalizing the concept of statistical convergence in 2001, Kostyrko et al. introduced the idea of $\mathcal{I}-$convergence in [17] where the ideal $\mathcal{I}$ basically represents a set of subsets of $\mathbb{N}$ which satisfy some specific conditions. As this concept was the generalization of so many known convergence methods, eventually it becomes a very interesting area of research. Several works related to $\mathcal{I}-$convergence can be found in [1, 8, 14, 18, 22, 24, 26].

In [27], Savas and Das generalized the idea of $\mathcal{I}-$convergence to $\mathcal{I}-$statistical convergence. Several investigation in this direction was done by Das and Savas [4], Debnath and Debnath [6], Mursaleen et al. [7], Et et al. [9] and many others [13, 15, 32, 31].

An Orlicz function (see [23]) is defined as a map $\phi : \mathbb{R} \to \mathbb{R}$ which satisfies the following criterion:

(i) $\phi(-x) = \phi(x) \, \forall x \in \mathbb{R}$, i.e. ϕ is an even function.

(ii) $\phi^{'}(x) \geq 0 \, \forall x \in \mathbb{R}^+$, i.e. ϕ is non-decreasing on $\mathbb{R}^+$.

(iii) $\phi(x) = 0$ if and only if $x = 0$.

(iv) ϕ is continuous on entire $\mathbb{R}$.

(v) $x \to \infty$ implies $\phi(x) \to \infty$.

Further, an Orlicz function ϕ is said to fulfil the condition $\triangle_2$, if there exists an positive real number M satisfying the condition $\phi(2x) \leq M \cdot \phi(x) \, \forall x \in \mathbb{R}^+$.

In [23], Rao and Ren describes some important applications of Orlicz functions in many areas such as economics, stochastic problems etc. Few examples of Orlicz functions are given below:

Example 1.1.

(i) For a fixed $q \in \mathbb{N}$, the function $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = |x|^q$ is an Orlicz function.

(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = x^5$ is not an Orlicz function.

(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = e^{|x|} - \mid x \mid -1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

The idea of $\phi-$convergence was introduced by Khusnussaadah and Supama [16]. Recently, it was generalized to $\mathcal{I}-$statistically $\phi-$convergence by Debnath and Choudhury [5] and Lacunary statistically $\phi-$convergence by Savas and Debnath [28].

Mursaleen, Debnath and Rakshit [21] introduced the concepts of $\mathcal{I}-$statistical limit superior and $\mathcal{I}-$statistical limit inferior. In the present paper we generalized it and introduced $\mathcal{I}-$statistical ϕ-limit superior and $\mathcal{I}-$statistical ϕ-limit inferior and studied some of their properties for sequence of real numbers.

2. Definitions and Preliminaries

Definition 2.1. [18] *Let X is a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if*

(i) $\emptyset \in \mathcal{I}$;

(ii) *for each* $A, B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$;
(iii) *for each* $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$.

Definition 2.2. [18] *Let* X *is a non-empty set. A family of subsets* $\mathcal{F} \subset P(X)$ *is called a filter on* X *if and only if*

(i) $\emptyset \notin \mathcal{F}$;
(ii) *for each* $A, B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$;
(iii) *for each* $A \in \mathcal{F}$ *and* $B \supset A$ *implies* $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.3. [27] *A sequence* $x = (x_n)$ *is said to be* $\mathcal{I}-$*statistically convergent to* x_0 *if for every* $\varepsilon > 0$ *and every* $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : | x_k - x_0 | \geq \varepsilon\} | \geq \delta\right\} \in \mathcal{I}.$$

x_0 *is called* $\mathcal{I}-$*statistical limit of the sequence* (x_n) *and we write*, $\mathcal{I} - st\, lim\, x_n = x_0$.

Definition 2.4. [21] *Let* $\mathcal{I}$ *be an admissible ideal of* $\mathbb{N}$ *and let* $x = (x_n)$ *be a real sequence. Let* B_x *denote the set*

$$B_x = \left\{b \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : x_k > b\} | > \delta\right\} \notin \mathcal{I}\right\}.$$

Similarly,

$$A_x = \left\{a \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : x_k < a\} | > \delta\right\} \notin \mathcal{I}\right\}.$$

Then $\mathcal{I}-$*statistical limit superior of* x *is given by,*

$$\mathcal{I} - st \limsup x = \begin{cases} sup\, B_x & if \;\; B_x \neq \emptyset, \\ -\infty & if \;\; B_x = \emptyset. \end{cases}.$$

Also, $\mathcal{I}-$*statistical limit inferior of* x *is given by,*

$$\mathcal{I} - st \liminf x = \begin{cases} inf\, A_x & if \;\; A_x \neq \emptyset, \\ \infty & if \;\; A_x = \emptyset. \end{cases}.$$

Definition 2.5. [16] *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be $\phi-$convergent to x_0 if $\lim_n \phi(x_n - x_0) = 0$.*

In this case, x_0 is called the $\phi-$limit of (x_n) and denoted by $\phi - \lim x = x_0$.

Remark 2.6. If we take $\phi(x) =| x |$, then $\phi-$convergent concepts coincide with usual convergence. Also it is easy to check, if $x = (x_n)$ is $\phi-$convergent to x_0, then any of its subsequence is $\phi-$convergent to x_0 as well.

Definition 2.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be statistically $\phi-$convergent to x_0 if for each $\varepsilon > 0$,*

$$\lim_n \frac{1}{n} \mid \{k \leq n : \phi(x_k - x_0) \geq \varepsilon\} \mid = 0.$$

x_0 is called the statistical $\phi-$ limit of the sequence (x_n) and we write $S - \phi \lim x = x_0$ or $x_k \to x_0 (S - \phi)$. We shall also use $S - \phi$ to denote the set of all statistically $\phi-$convergent sequences.

Remark 2.8. If we take $\phi(x) =| x |$, then $S - \phi$ convergence concepts coincide with statistically convergence.

We introduce the following definition as a generalization of $\mathcal{I}-$statistical convergence. Throughout the paper we consider $\mathcal{I}$ as an admissible ideal.

Definition 2.9. [5] *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically $\phi-$ convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi\,(x_k - x_0) \geq \varepsilon\} | \geq \delta \right\} \in \mathcal{I}.$$

x_0 is called $\mathcal{I}-$statistical $\phi-$ limit of the sequence (x_n) and we write, $\mathcal{I}_S - \phi \lim x_n = x_0$.

Remark 2.10. If we take $\phi(x) =| x |$, then $\mathcal{I}_S - \phi$ convergent coincide with $\mathcal{I}_S-$convergent. Thus $\mathcal{I}_S - \phi$ convergence is a generalization of $\mathcal{I}_S$ convergence.

3. Main Results

Definition 3.1. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. For a real sequence $x = (x_n)$, let*

$$B_x = \left\{ b \in \mathbb{R} : \left\{ n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) > b\} | > \delta \right\} \notin \mathcal{I} \right\}$$

and

$$A_x = \left\{ a \in \mathbb{R} : \left\{ n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) < a\} | > \delta \right\} \notin \mathcal{I} \right\}.$$

We define $\mathcal{I}-$statistical $\phi-$limit superior of x by

$$\mathcal{I} - st\ \phi - \limsup x = \begin{cases} sup\ B_x & if\ \ B_x \neq \emptyset, \\ -\infty & if\ \ B_x = \emptyset. \end{cases}.$$

Also $\mathcal{I}-$statistical $\phi-$limit inferior of x by

$$\mathcal{I} - st\ \phi - \liminf x = \begin{cases} inf\ A_x & if\ \ A_x \neq \emptyset, \\ \infty & if\ \ A_x = \emptyset. \end{cases}.$$

Theorem 3.2. *If $\beta = \mathcal{I} - st\,\phi - \limsup x$ is finite, then for every positive number ε,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) > \beta - \varepsilon\}\,| > \delta \right\} \notin \mathcal{I}$$

and

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) > \beta + \varepsilon\}\,| > \delta \right\} \in \mathcal{I}.$$

Similarly, if $\alpha = \mathcal{I} - st\ \phi - \liminf x$ is finite, then for every positive number ε,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) < \alpha + \varepsilon\}\,| > \delta \right\} \notin \mathcal{I}$$

and

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) < \alpha - \varepsilon\}\,| > \delta \right\} \in \mathcal{I}.$$

Proof. It follows from the definition. ■

Theorem 3.3. *Let ϕ be an Orlicz function. Then for any real number sequence (x_n), $\mathcal{I} - st\ \phi - \liminf x_n \leq \mathcal{I} - st\ \phi - \limsup x_n$.*

Proof. If $\mathcal{I} - st\,\phi - \limsup x_n = -\infty$, then we have $B_x = \emptyset$. So for every $b \in \mathbb{R}$, $\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) > b\}\,| > \delta \right\} \in \mathcal{I}$, which implies $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) > b\}| < \delta\} \in \mathcal{F}(\mathcal{I})$ i.e, $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) < b\}| > \delta\} \in \mathcal{F}(\mathcal{I})$.

So for every $a \in \mathbb{R}$, $\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) < a\}\,| > \delta \right\} \notin \mathcal{I}$. Hence, $\mathcal{I} - st\ \phi - \liminf x_n = -\infty$ (since $A_x = \mathbb{R}$).

If $\mathcal{I} - st\ \phi - \limsup x_n = \infty$, then it is obvious.

Let $\beta = \mathcal{I} - st\ \phi - \limsup x_n$ is finite and $\alpha = \mathcal{I} - st\ \phi - \liminf x_n$. So for $\varepsilon > 0$, $\delta > 0$, $\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) > \beta + \varepsilon\}\,| > \delta \right\} \in \mathcal{I}$ which implies $\left\{ n \in \mathbb{N} : \frac{1}{n} |\,\{k \leq n : \phi(x_k) < \beta + \varepsilon\}\,| > \delta \right\} \in \mathcal{F}(\mathcal{I})$ i.e., $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) < \beta + \varepsilon\}| > \delta\} \notin \mathcal{I}$. So $\beta + \varepsilon \in A_x$. Since ε is arbitrary and by definition $\alpha =$inf A_x. Therefore $\alpha < \beta + \varepsilon$. This proves that $\alpha \leq \beta$. ■

Definition 3.4. *Let ϕ be an Orlicz function. The real number sequence $x = (x_n)$ is said to be $\mathcal{I}-st\ \phi-$bounded if there is a positive number G such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) > G\} | > \delta\right\} \in \mathcal{I}.$$

Remark 3.5. If a sequence is $\mathcal{I}-$st $\phi-$bounded then $\mathcal{I} - st\ \phi - \limsup$ and $\mathcal{I} - st\ \phi - \liminf$ of the sequence are finite.

Definition 3.6. *Let ϕ be an Orlicz function. An element ξ is said to be an $\mathcal{I}-$statistical $\phi-$cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$ the set*

$$\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k - \xi) \geq \varepsilon\} | < \delta\right\} \notin \mathcal{I}.$$

Theorem 3.7. *Let ϕ be an Orlicz function. If a $\mathcal{I}-$statistically $\phi-$ bounded sequence has one cluster point then it is $\mathcal{I}-$statistically $\phi-$convergent.*

Proof. Let (x_n) be a $\mathcal{I}-$statistically $\phi-$bounded sequence which has one cluster point. i.e., $M = \left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) > G\} | > \delta\right\} \in \mathcal{I}$. So there exist a set $M' = \{n_1 < n_2 < ....\} \subset \mathbb{N}$ such that $M' \notin \mathcal{I}$ and (x_{n_k}) is a statistically bounded sequence.

Now since (x_n) has only one cluster point and (x_{n_k}) is a statistically bounded subsequence of (x_n), so (x_{n_k}) also has only one cluster point. Hence (x_{n_k}) is statistically ϕ convergent.

Let st-ϕ $\lim x_{n_k} = \xi$. Then for any $\varepsilon > 0$ and $\delta > 0$ we have the inclusion, $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k - \xi) \geq \varepsilon\} | \geq \delta\right\} \subseteq M \cup A \in \mathcal{I}$ where A is a finite set. That is, (x_n) is $\mathcal{I}-$statistically $\phi-$ convergent to ξ. ■

Theorem 3.8. *Let ϕ be an Orlicz function. For a $\mathcal{I}-st\ \phi-$bounded sequence (x_n), $(\phi(x_n))$ is $\mathcal{I}-st$ convergent if and only if $\mathcal{I}-st\phi-\liminf x_n = \mathcal{I}-st\phi-\limsup x_n$.*

Proof. Let $(\phi(x_n))$ be $\mathcal{I}-$statistically convergent, say to x. Then for any $\varepsilon > 0$, $\delta > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n :| \phi(x_k) - x |> \varepsilon\} | > \delta\right\} \in \mathcal{I}$. So, $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) > x + \varepsilon\} | > \delta\right\} \in \mathcal{I}$, which means that $\mathcal{I} - st\ \phi - \liminf x_n \leq x + \varepsilon$.

We have also $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) < x - \varepsilon\} | > \delta\right\} \in \mathcal{I}$, i.e. $\mathcal{I} - st\ \phi - \limsup x_n \geq x - \varepsilon$.

For the converse, let us assume $\mathcal{I}-st\phi-\liminf x_n = \mathcal{I}-st\phi-\limsup x_n$. Now for any $\varepsilon > 0$, $\delta > 0$, we have $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) > x + \varepsilon\} | > \delta\right\} \in \mathcal{I}$ and $\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : \phi(x_k) < x - \varepsilon\} | > \delta\right\} \in \mathcal{I}$. These implies that $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n :| \phi(x_k) - x |> \varepsilon\}| > \delta\} = A \cup B \in \mathcal{I}$ where $A = \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) > x + \varepsilon\}| > \delta\}$ and $B = \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) < x - \varepsilon\}| > \delta\}$.

This means $(\phi(x_n))$ is $\mathcal{I}-$statistically convergent to x. ■

Theorem 3.9. *Let ϕ be an Orlicz function. If $(x_n), (y_n)$ are two $\mathcal{I}-st\ \phi-$bounded sequences, then*

(i) $\mathcal{I} - st\,\phi - \limsup\,(x_n + y_n) \leq \mathcal{I} - st\,\phi - \limsup x_n + \mathcal{I} - st\,\phi - \limsup y_n$.
(ii) $\mathcal{I} - st\,\phi - \liminf\,(x_n + y_n) \geq \mathcal{I} - st\,\phi - \liminf x_n + \mathcal{I} - st\,\phi - \liminf y_n$.

Proof. (i) Let $l_1 = \mathcal{I} - st\,\phi - \limsup x_n$ and $l_2 = \mathcal{I} - st\,\phi - \limsup y_n$. So $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) > l_1 + \frac{\varepsilon}{2}\right\}\right| > \delta\right\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(y_k) > l_2 + \frac{\varepsilon}{2}\}| > \delta\} \in \mathcal{I}$. Now $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k + y_k) > l_1 + l_2 + \varepsilon\}| > \delta\} \subset \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) > l_1 + \frac{\varepsilon}{2}\}| > \delta\} \cup \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(y_k) > l_2 + \frac{\varepsilon}{2}\}| > \delta\}$. So, $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k + y_k) > l_1 + l_2 + \varepsilon\}| > \delta\} \in \mathcal{I}$.

If $c \in B_{(x+y)}$, then by definition $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k + y_k) > c\}| > \delta\} \notin \mathcal{I}$.

We show that $c < l_1 + l_2 + \varepsilon$. If $c \geq l_1 + l_2 + \varepsilon$ then $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k + y_k) > c\right\}\right| > \delta\right\} \subseteq \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k + y_k) > l_1 + l_2 + \varepsilon\}| > \delta\}$. Therefore $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k + y_k) > c\right\}\right| > \delta\right\} \in \mathcal{I}$ which is a contradiction. Hence, $c < l_1 + l_2 + \varepsilon$. As this is true for all $c \in B_{(x+y)}$, so, $\mathcal{I}-$st$\phi-$ lim sup $(x + y)$ =sup $B_{(x+y)} < l_1 + l_2 + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, so $\mathcal{I}-$st $\phi-$lim sup $(x_n + y_n) \leq \mathcal{I}-$st $\phi-$lim sup $x_n + \mathcal{I}-$st $\phi-$lim sup y_n.

(ii) Similar to above technique of proof. ■

Definition 3.10. *Let ϕ be an Orlicz function. A sequence (x_n) is said to be $\mathcal{I}-st$ $\phi-$convergent to $+\infty$ (or$-\infty$) if for every real number $G > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) \leq G\right\}\right| > \delta\right\} \in \mathcal{I}$(or, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) \geq -G\right\}\right| > \delta\right\} \in \mathcal{I}$).*

Theorem 3.11. *Let ϕ be an Orlicz function. If $\mathcal{I} - st\,\phi - \limsup x = l$, then there exists a subsequence of x which is $\mathcal{I}-st$ $\phi-$ convergent to l.*

Proof. *Case-I*: If $l = -\infty$ then $B_x = \emptyset$.

So for any real number $G > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) \geq -G\right\}\right| > \delta\right\} \in \mathcal{I}$ i.e., $\mathcal{I} - st\,\phi - \lim x = -\infty$.

Case-II: If $l = +\infty$, then $B_x = \mathbb{R}$. So for any $b \in \mathbb{R}$, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) > b\right\}\right| > \delta\right\} \notin \mathcal{I}$. Let x_{n_1} be arbitrary member of x. Then, $A_{n_1} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) > x_{n_1} + 1\right\}\right| > \delta\right\} \notin \mathcal{I}$. Since $\mathcal{I}$ is an admissible ideal, so A_{n_1} must be an infinite set. That is, $d(\{k \leq n : \phi(x_k) > x_{n_1}+1\}) \neq 0$. We claim that there is at least $k \in \{k \leq n : \phi(x_k) > x_{n_1} + 1\}$ such that $k > n_1 + 1$, for otherwise $\{k \leq n : \phi(x_k) > x_{n_1} + 1\} \subseteq \{1, 2, ...n_1, n_1 + 1\}$, i.e., $d\left(\{k \leq n : \phi(x_k) > x_{n_1} + 1\}\right) \leq d\left(\{1, 2, ...n_1, n_1 + 1\}\right) = 0$, which is a contradiction.

We call this k as n_2, thus $x_{n_2} > x_{n_1} + 1$. Proceeding in this way we obtain a subsequence $\{x_{n_k}\}$ of x with $x_{n_k} > x_{n_{k-1}} + 1$. Since for any $G > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) \leq G\right\}\right| > \delta\right\} \in \mathcal{I}$, so $\mathcal{I}-$st $\phi-$ lim $x_{n_k} = +\infty$.

Case-III: $-\infty < l < +\infty$.

So, $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \phi(x_k) > l + \frac{1}{2}\right\}\right| > \delta\right\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) > l - 1\}| > \delta\} \notin \mathcal{I}$. So there must be a m in this set for which

$\frac{1}{m}|\left\{k \leq m : \phi(x_k) > l-1\right\}| > \delta$ and $\frac{1}{m}|\left\{k \leq m : \phi(x_k) \leq l+\frac{1}{2}\right\}| > \delta$.

For otherwise $\left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \phi(x_k) > l-1\right\}| > \delta\right\} \subset \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \phi(x_k) > l+\frac{1}{2}\}| > \delta\} \in \mathcal{I}$, which is a contradiction.

Now for maximum $k \leq m$ will satisfy $\phi(x_k) > l-1$ and $\phi(x_k) \leq l+\frac{1}{2}$ so we must have a n_1 for which $l-1 < \phi(x_{n_1}) \leq l+\frac{1}{2} < l+1$.

Next we proceed to choose an element $\phi(x_{n_2})$ from $\phi(x)$, $n_2 > n_1$ such that $l-\frac{1}{2} < \phi(x_{n_2}) < l+\frac{1}{2}$. Now $\left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \phi(x_k) > l-\frac{1}{2}\right\}| > \delta\right\}$ is an infinite set. So, $d\left(\left\{k \leq n : \phi(x_k) > l-\frac{1}{2}\right\}\right) \neq 0$. We observe that there is at least one $k > n_1$ for which $\phi(x_k) > l-\frac{1}{2}$, for otherwise $d\left(\left\{k \leq n : \phi(x_k) > l-\frac{1}{2}\right\}\right) \leq d\left(\{1,2,...n_1\}\right) = 0$ which is a contradiction.

Let $E_{n_1} = \left\{k \leq n : k > n_1, \phi(x_k) > l-\frac{1}{2}\right\} \neq \emptyset$. If $k \in E_{n_1}$ always implies $x_k \geq l+\frac{1}{2}$ then, $E_{n_1} \subseteq \left\{k \leq n : \phi(x_k) > l+\frac{1}{2}\right\}$, i.e., $d\left(E_{n_1}\right) \leq d\left(\left\{k \leq n : \phi(x_k) > l+\frac{1}{2}\right\}\right) = 0$.

Since $\left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \phi(x_k) > l+\frac{1}{2}\right\}| < \delta\right\} \in \mathcal{F}(\mathcal{I})$, thus $\{k \leq n : \phi(x_k) > l-\frac{1}{2}\} \subseteq \{1,2,...n_1\} \cup E_{n_1}$. So, $d\left(\left\{k \leq n : \phi(x_k) > l-\frac{1}{2}\right\}\right) \leq d\left(\{1,2,...n_1\}\right) + d\left(E_{n_1}\right) \leq 0$, which is a contradiction.

This shows that there is a $n_2 > n_1$ such that $l-\frac{1}{2} < \phi(x_{n_2}) < l+\frac{1}{2}$. Proceeding in this way we obtain a subsequence $\phi(x_{n_k})$ of $\phi(x)$, $n_k > n_{k-1}$ such that $l-\frac{1}{k} < \phi(x_{n_k}) < l+\frac{1}{k}$ for each k. This subsequence $\{\phi(x_{n_k})\}$ ordinarily converges to l and thus $\mathcal{I}-$st $\phi-$ convergent to l. ∎

Theorem 3.12. *If $\mathcal{I}-st$ $\phi-$ lim inf $x = l$, then there exists a subsequence of x which is $\mathcal{I}-st$ $\phi-$ convergent to l.*

Proof. The proof is similar to Thm. 3.11 and so omitted. ∎

Theorem 3.13. *Let ϕ be an Orlicz function. Every $\mathcal{I}-st$ $\phi-$ bounded sequence x has a subsequence which is $\mathcal{I}-st$ $\phi-$convergent to a finite real number.*

Proof. The proof follows from Remark 3.5 and Thm. 3.11. ∎

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