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Positive Solutions to Conformable Fractional Differential Equation with Integral Boundary Condition with p-Laplacian Operator

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Abstract. The aim of this work is to investigate the existence and multiplicity of positive solutions of a kind of conformable fractional differential equations with an integral boundary conditions with p-Laplacian operator. By using some fixed point theorems such as, Krasnoselskii, Schaefer and Leggett-Williams fixed point theorems, necessary and sufficient conditions are presented to obtain existence and multiplicity results. Two examples are given to illustrate our results.

Keywords: Fractional boundary value problem; Integral boundary condition; p-Laplacian operator.

1. Introduction

Fractional calculus as a generalization of ordinary order calculus from the early years has been of great interest of many mathematicians and scientists. With the development of fractional calculus, fractional differential equations have wide applications in modeling of different Physical phenomena and engineering, such as mechanics, chemistry, control system, etc. see [16, 27, 28, 29].

Since the early years, different definitions of fractional order derivatives have been proposed such as Riemann-Liouville, Grunwald-Letnikov, Caputo and etc. Each of these definitions had its advantages and disadvantages. Very recently in [19] Khalil et al. introduced a new well-behaved definition of fractional derivative termed the conformable fractional derivative. This definition has drawn much interest from many researchers [1, 17, 19]. Also different applications and generalizations of the conformable derivative were discussed in [5, 6, 32, 34, 35].

Beside to studying boundary and initial value problems including fractional differential equations with classical fractional operators, such as Riemann-Liouville and Caputo fractional derivative [2, 3, 4, 7, 8, 10, 11, 13, 15, 18, 20, 23, 24, 25, 26, 30, 31, 33, 36], investigating the existence of solutions to boundary and initial value problems involving this fractional operator has also attracted some researchers.

In 2015 Batarfi et al. [9] consider the following three-point boundary value problem for conformable fractional differential equation

$$\begin{aligned} D^{\alpha}(D+\lambda)x(t) &= f(t,x(t)), \qquad && t\in[0,1], \\ x(0) &= 0, x'(0)=0, && x(1)=\beta x(\eta) \end{aligned}$$

$1<\alpha\leq 2$ is a real number and D^{α} is the conformable derivative and D is the ordinary derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a known continuous function, λ and β are real constant numbers, $\lambda>0$, and $\eta\in(0,1)$. The existence results are obtained by means of Krasnoselskiis fixed point theorem and the classical Banach fixed point theorem.

In 2017, X. Dong et al. [12] studied the existence and multiplicity of positive solutions for the conformable fractional differential equation with p-Laplacian operator

$$\begin{aligned} &D^{\alpha}(\varphi_p(D^{\alpha}u(t)) = f(t,u(t)), \quad 0<t<1, \\ &u(0)=u(1)=D^{\alpha}u(0)=D^{\alpha}u(1)=0, \end{aligned}$$

where $1<\alpha\leq 2$ is a real number and D^{α} is the conformable derivative, $\varphi_p(s)=|s|^{p-2}s, p>1, \varphi_p^{-1}=\varphi_q, \frac{1}{p}+\frac{1}{q}=1$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. They used an approximation method and some fixed point theorems on cone and established some existence results for positive solutions.

In 2019, Zhou and Zhang studied the following fractional boundary value problem of conformable fractional derivative

$$\begin{aligned} &T_{\alpha}^{0+}(\varphi_p(T_{\alpha}^{0+}u(t)) = f(t,u(t),T_{\alpha}^{0+}u(t)), \quad 0\leq t\leq 1; \\ &u^{(i)}(0)=0, \quad [\varphi_p(T_{\alpha}^{0+}u(t)]^{(i)}=0, \quad i=0,1,2,\ldots,n-2; \\ &(T_{\beta}^{0+}u(t))_{t=1}=0, \quad m-1<\beta\leq m; \\ &(T_{\beta}^{0+}(\varphi_p(T_{\alpha}^{0+}u(t))))_{t=1}=0, \quad 1\leq m\leq n-1; \end{aligned}$$

where $n-1<\alpha\leq n$, φ_p is the p-Laplacian operator, $\varphi_p(s)=|s|^{p-2}s, p>1, \varphi_p^{-1}=\varphi_q$, $\frac{1}{p}+\frac{1}{q}=1$, and T_{α} is the conformable fractional derivative. Authors by using the Guo-Krasnoselskii fixed point theorem, were able to prove the existence of at least one positive solution.

Motivated by the aforementioned works, this paper discusses the existence of positive solutions for the fractional boundary value problem

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}u(t))) = f(t,u(t)), \quad 0<t<1 \\ &u'(0)=D^{\beta}u(0)=D^{\beta}u(1)=0, \quad u(1)=\textstyle\int_0^{\eta}u(t)dt \end{aligned} \tag{1}$$

2. Preliminaries

In this section, notations, definitions and preliminary facts which are used throughout this paper are introduced. At first, let us recall some basic definitions of fractional calculus which can be found in [1, 17, 19].

Definition 2.1. *Let α be in $(0,1]$. The conformable fractional derivative of a function $f:[0,\infty)\to\mathbb{R}$ of order α is defined by*

$$D^{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})}{\epsilon} \tag{2}$$

If $D^{\alpha}f(t)$ exists on $(0,b)$, then $D^{\alpha}f(0)=\lim_{t\to 0}D^{\alpha}f(t)$

Definition 2.2. *Let $\alpha\in(n,n+1]$. The conformable fractional derivative of a function $f:[0,\infty)\to\mathbb{R}$ defined by*

$$D^{\alpha}f(t)=D^{\beta}f^{(n)}(t)$$

where $\beta=\alpha-n$.

Definition 2.3. *Let $\alpha\in(n,n+1]$. The conformable fractional integral of a function $f:[0,\infty)\to\mathbb{R}$ of order α is defined by*

$$T^{\alpha}f(t)=\frac{1}{n!}\int_0^t(t-s)^n s^{\alpha-n-1}f(s)ds.$$

Lemma 2.4. *Let $\alpha\in(n,n+1]$. If f is a continuous function on $[0,\infty)$, then for all $t>0, D^{\alpha}I^{\alpha}f(t)=f(t)$.*

Lemma 2.5. *Let $\alpha\in(n,n+1]$. Then $D^{\alpha}t^k=0$ for $t\in[0,1]$ and $k=1,2,\dots,n$.*

Lemma 2.6. [29] *Let $\alpha\in(n,n+1]$. If $D^{\alpha}f(t)$ is continuous on $[0,\infty)$, then*

$$I^{\alpha}D^{\alpha}f(t)=f(t)+C_1+C_2t^2+\cdots+C_nt^n,$$

for some real numbers $c_k, k=1,2,\dots,n$.

In the rest of the section, we present some defintions, theorems and lemmas which are used in this paper.

Definition 2.7. *Let E be a real Banach space, A nonempty, closed, convex set $P\subset E$ is a cone if it satisfies the following conditions:*

(i) *If $x\in P$, $\lambda\geq 0$ then $\lambda x\in P$;*

(ii) *If $x \in P$ and $-x \in P$ then $x = 0$.*

Theorem 2.8. [21] *Let E be a Banach space and $P \subset E$, be a cone. Assume Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ and let*

$$T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$$

be a completely continuous operator such that

(i) *$\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or*

(ii) *$\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.9. [14] *Let E be a Banach space B a closed convex subset of E, U an open subset of B and $0 \in U$. Suppose that $T : \overline{U} \to B$ is a continuous, compact map. Then either*

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *There is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda A(u)$.*

Theorem 2.10. [22] *Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and ψ a nonnegative continuous concave functional on P such that $\psi(u) \leq \|u\|$ for all u in $\overline{P}_c$. Suppose that there exists constant $0 < a < b < d \leq c$ such that*

(i) *$\{u \in P(\psi, b, d) : \psi(u) > b\} \neq 0$ and $\psi(Tu) > b$ if $u \in P(\psi, b, d)$,*

(ii) *$\|Tu\| < a$ if $u \in P_a$,*

(iii) *$\psi(Tu) > b$ for $u \in P(\psi, b, c)$ with $\|Tu\| > d$.*

Then, T has at least three fixed points u_1, u_2 and u_3 such that $\|u_1\| < a$, $b < \psi(u_2)$ and $\|u_3\| > a$ with $\psi(u_3) < b$.

Lemma 2.11. [12] *Suppose E is a Banach space and $T_n : E \to E, n = 3, 4, \ldots$ are completely continuous operators, $T : E \to E$. If $\|T_n u - Tu\|$ uniformly converges to zero when $n \to \infty$ for all bounded set $\Omega \subset E$, then $T : E \to E$ is completely continuous.*

3. Green Function and Bounds

In order to apply the fixed point theorems, we need to calculate the Green function of the desired operator. In this section in addition to calculate Green function, we also outline some its properties which is used throughout this paper.

Lemma 3.1. [12] *Let $h \in C([0, 1])$ and $1 < \beta \leq 2$. Then the fractional boundary value problem*

$$\begin{aligned} &D^\beta x(t) = h(t), \\ &x'(0) = x(1) = 0, \end{aligned} \tag{3}$$

has a unique solution

$$x(t) = -\int_0^1 H(t,s)h(s)ds, \tag{4}$$

where

$$H(t,s) = \begin{cases} [(1-s)-(t-s)]s^{\beta-2} & \text{if } 0 \le s \le t \le 1, \\ s^{\beta-2}(1-s) & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{5}$$

Proof. Integrating of order β from Eq. (3) yields

$$x(t) = I^{\alpha}h(t) + c_0 + c_1 t = \int_0^t (t-s)s^{\beta-2}h(s)ds + c_0 + c_1 t \tag{6}$$

where c_0, c_1 are real constants. By condition $x'(0) = 0$ we get $c_1 = 0$ and from the other boundary condition we have

$$x(1) = \int_0^1 (1-s)s^{\beta-2}h(s)ds + c_0 = 0.$$

So

$$c_0 = -\int_0^1 (1-s)s^{\beta-2}h(s)ds$$

By replacing c_0 in (6) we have

$$\begin{aligned} x(t) &= \int_0^t (t-s)s^{\beta-2}h(s)ds - \int_0^1 (1-s)s^{\beta-2}h(s)ds \\ &= -\int_0^t [(1-s)-(t-s)]s^{\beta-2}h(s)ds - \int_t^1 (1-s)s^{\beta-2}h(s)ds \\ &= -\int_0^1 H(t,s)h(s)ds. \end{aligned}$$

∎

Lemma 3.2. *Let $y \in C([0,1])$, $1 \le \alpha \le 2$, $0 < \eta < 1$. Then boundary value problem*

$$\begin{aligned} &D^{\alpha}u(t) = y(t), \quad 0 \le t \le 1, \\ &u'(0) = 0, \quad u(1) = \textstyle\int_0^{\eta} u(t)dt, \end{aligned} \tag{7}$$

is equivalent to the integral equation

$$u(t) = -\int_0^t G(t,s)y(s)ds \tag{8}$$

where

$$G(t,s) = G_1(t,s) + \frac{1}{1-\eta}G_2(\eta,s), \tag{9}$$

$$G_1(t,s) = \begin{cases} (1-s)s^{\alpha-2} - (t-s)s^{\alpha-2} & \text{if } 0 \le s \le t \le 1, \\ (1-s)s^{\alpha-2} & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{10}$$

$$G_2(t,s) = \begin{cases} t(1-s)s^{\alpha-2} - \frac{1}{2}(t-s)^2 s^{\alpha-2} & \text{if } 0 \le s \le t \le 1, \\ t(1-s)s^{\alpha-2} & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{11}$$

Proof. Applying Lemma 2.3 we have

$$u(t) = I^{\alpha} y(t) + c_0 + c_1 t = \int_0^t (t-s)s^{\alpha-2} + c_0 + c_1 t \tag{12}$$

where c_0, c_1 are real constants. By the boundary condition $u'(0) = 0$, we obtain $c_1 = 0$ and from the other boundary condition we have

$$u(1) = \int_0^1 (1-s)s^{\alpha-2} y(s) + c_0 = \int_0^{\eta} u(t)dt.$$

Hence

$$c_0 = -\int_0^1 (1-s)s^{\alpha-2} y(s) + \int_0^{\eta} u(t)dt. \tag{13}$$

By replacing Eq. (13) in (12) we obtain

$$u(t) = \int_0^t (t-s)s^{\alpha-2} - \int_0^1 (1-s)s^{\alpha-2} y(s) + \int_0^{\eta} u(t)dt. \tag{14}$$

Now for determining value of $\int_0^{\eta} u(t)dt$, we integrating relation (14) from 0 to η,

$$\begin{aligned}
\int_0^{\eta} u(t)dt &= \int_0^{\eta}\int_0^t (t-s)s^{\alpha-2} y(s)dsdt - \int_0^{\eta}\int_0^t (1-s)s^{\alpha-2} y(s)ds \\
&\quad + \int_0^{\eta}\int_0^{\eta} u(s)dsdt \\
&= \int_0^{\eta} \frac{1}{2}(\eta-s)^2 s^{\alpha-2} y(s)ds - \eta \int_0^1 (1-s)s^{\alpha-2} y(s)ds \\
&\quad + \eta \int_0^{\eta} u(s)ds.
\end{aligned}$$

So

$$\int_0^{\eta} u(t)dt = \frac{1}{1-\eta}\int_0^{\eta} \frac{1}{2}(\eta-s)^2 s^{\alpha-2} y(s)ds - \frac{1}{1-\eta}\int_0^1 (1-s)s^{\alpha-2} y(s)ds \tag{15}$$

By replacing the right hand side of Eq. (15) in (14) we obtain

$$\begin{aligned}
u(t) &= \int_0^t (t-s)s^{\alpha-2} - \int_0^1 (1-s)s^{\alpha-2} y(s) \\
&\quad \frac{1}{1-\eta}\int_0^{\eta} \frac{1}{2}(\eta-s)^2 s^{\alpha-2} y(s)ds - \frac{1}{1-\eta}\int_0^1 (1-s)s^{\alpha-2} y(s)ds \\
&= \int_0^t [(t-s)-(1-s)]s^{\alpha-2} y(s)ds - \int_t^1 (1-s)s^{\alpha-2} y(s)ds \\
&\quad + \frac{1}{1-\eta}\int_0^{\eta} [\frac{1}{2}(\eta-s)^2 - (1-s)]s^{\alpha-2} y(s)ds - \frac{1}{1-\eta}\int_{\eta}^1 \eta(1-s)s^{\alpha-2} y(s)ds
\end{aligned}$$

$$= -\int_0^1 G_1(t,s)y(s)ds - \frac{1}{1-\eta}\int_0^1 G_2(\eta,s)y(s)ds$$
$$= -\int_0^1 G(t,s)y(s)ds.$$

∎

Lemma 3.3. *Fractional boundary value problem* (1) *is equivalent to the integral equation*

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \tag{16}$$

Proof. Let $y(t) = \varphi_q(x(t))$ and $h(t) = -f(t,u(t))$. Then from Lemmas 3.1 and 3.2, we get

$$y(t) = \varphi_q\left(\int_0^1 H(t,s)f(s,u(s))ds\right).$$

Hence we obtain

$$u(t) = \int_0^1 G(t,s)y(s)ds = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds. \tag{17}$$

∎

Lemma 3.4. *The function $H(t,s)$ defined by* (5) *satisfies the following conditions:*

(i) $H(t,s) > 0$ *for all* $t,s \in (0,1)$*;*
(ii) $(1-t)H(s,s) \le H(t,s) \le H(s,s)$, *for all* $t,s \in [0,1]$.

Proof. Statement (i) hold trivially. We prove statement (ii). For $0 \le s \le t$ we get

$$\begin{aligned} H(t,s) &= (1-s)s^{\alpha-2} - (t-s)s^{\alpha-2} \\ &\ge [(1-s) - t(1-\frac{s}{t})]s^{\alpha-2} \\ &\ge [(1-s) - t(1-s)]s^{\alpha-2} \\ &= (1-t)(1-s)s^{\alpha-2} \end{aligned}$$

On the other hand

$$H(t,s) \le (1-s)s^{\alpha-2} = H(s,s).$$

Hence for all $s,t \in [0,1]$

$$(1-t)H(s,s) \le H(t,s) \le H(s,s).$$

∎

Lemma 3.5. *The function $G_1(t,s)$ and $G_2(t,s)$ defined by* (9) *satisfy the following conditions*

(i) $G_1(t,s) \ge 0$, $G_2(t,s) \ge 0$ *for all* $t,s \in [0,1]$*;*
(ii) $(1-t)G_1(s,s) \le G_1(t,s) \le G_1(s,s)$ *for all* $t,s \in [0,1]$*;*

(iii) $(1-t)G_2(s,s) \leq G_2(t,s) \leq G_2(s,s)$ *for all* $t,s \in [0,1]$.

Proof. It is clear that statement (i) hold and statement (ii) is concluded from Lemma 3.4. We prove statement (iii).

If $0 \leq s \leq t \leq 1$, we obtain

$$\begin{aligned}
G_2(t,s) &= t(1-s)s^{\alpha-2} - \frac{1}{2}(t-s)^2 s^{\alpha-2} \\
&\geq s^{\alpha-2}[t(1-s) - \frac{1}{2}(t-s)^2] \\
&= s^{\alpha-2}[t(1-s) - \frac{1}{2}t^2(1-\frac{s}{t})^2] \\
&\geq s^{\alpha-2}s[(1-s) - \frac{1}{2}t(1-s)] \\
&\geq (1-t)s^{\alpha-1}(1-s)
\end{aligned}$$

On the other hand, $G_2(t,s) \leq s^{\alpha-1}(1-s)$. So for all $s,t \in [0,1]$, we have

$$(1-t)G_2(s,s) \leq G_2(t,s) \leq \Phi_\eta(s).$$

■

Lemma 3.6. *Let $\xi \in (0,1)$ be a fixed. Then for $G(t,s)$ we have the following statements:*

(i) $G(t,s) \geq 0$, *for all* $t,s \in [0,1]$,
(ii) $(1-\eta)(1-t)\Phi_\eta(s) \leq G(t,s) \leq \Phi_\eta(s)$ *for all* $0 \leq t,s \leq 1$,
(iii) $(1-\eta)(1-\xi)\Phi_\eta(s) \leq G(t,s) \leq \Phi_\eta(s)$, *for all* $(t,s) \in [0,\xi] \times [0,1]$,
where $\Phi_\eta(s) = G_1(s,s) + \frac{1}{1-\eta}G_2(s,s) = \frac{1+s-\eta}{(1-\eta)}s^{\alpha-2}(1-s)$.

Proof. It is clear that (1) holds and (3) is the direct result of (2). So we prove only statement (2). From Lemma 3.5 and relation (9), it is concluded that

$$\begin{aligned}
G(t,s) &= G_1(t,s) + \frac{1}{1-\eta}G_2(\eta,s) \\
&\leq G_1(s,s) + \frac{1}{1-\eta}G_1(s,s) = \Phi_\eta(s)
\end{aligned}$$

On the other hand from Lemma 3.5, we obtain

$$\begin{aligned}
G(t,s) &= G_1(t,s) + \frac{\eta}{1-\eta}G_2(t,s) \\
&\geq (1-t)G_1(s,s) + (1-\eta)\,\frac{1}{1-\eta}G_1(s,s) \\
&\geq (1-t)\,(1-\eta)\,G_1(s,s) + (1-t)\,(1-\eta)\,\frac{1}{1-\eta}G_2(s,s) \\
&\geq (1-t)\,(1-\eta)\,[G_1(s,s) + \frac{1}{1-\eta}G_2(s,s)] \\
&= (1-t)\,(1-\eta)\,\Phi_\eta(s).
\end{aligned}$$

■

4. Main Results

Let $E = C([0,1])$ be Banach space of all continuous function on $[0,1]$, which equipped with the maximum norm $\|u\| = \max_{0\le t\le 1} |u(t)|$. We define

$$P = \{u \in E : u(t) \ge 0, \min_{[0,\xi]} u(t) \ge \rho\|u\|\},$$

where $\rho = (1-\xi)(1-\eta)$. Now we define the nonnegative continuous concave functional ψ by $\psi(u) = \min_{[0,\xi]} |u(t)|$.

Let $f \in C([0,1] \times [0,\infty))$, we define $T, T_n : P \to E$ as

$$\begin{aligned} Tu(t) &:= \int_0^1 G(t,s)\varphi_p\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds,\\ T_n u(t) &:= \int_{\frac{1}{n}}^1 G(t,s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds, \quad n = 3,4,\dots \end{aligned}$$

Remark 4.1. By choosing $\rho = (1-\xi)(1-\eta)$ from Lemma 3.4 conclude that for $(t,s) \in [0,\xi] \times [0,1]$ we have

$$\rho H(s,s) \le H(t,s) \le H(s,s).$$

Lemma 4.2. *Operator $T : P \to P$ is completely continuous.*

Proof. The proof of this Lemma is similar to the proof of Lemma 3.1 in [12] and we present it in two steps.

Step 1: $T_n : P \to P$ are completely continuous for $n = 3,4,\dots$.

Let $u \in P$ and $u \in P$. By Lemma 3.2 and the nonnegativity of $f(t,u)$, one has

$$\begin{aligned} T_n u(t) &= \int_{\frac{1}{n}}^1 G(t,s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds\\ &\le \int_{\frac{1}{n}}^1 \Phi_\eta(s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds \end{aligned}$$

So

$$\|T_n u\| \le \int_{\frac{1}{n}}^1 \Phi_\eta(s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds$$

Now if $u \in P$ we have

$$\begin{aligned} \min_{0\le t\le\xi} T_n u(t) &= \min_{0\le t\le\xi} \int_{\frac{1}{n}}^1 G(t,s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds\\ &\ge \rho\int_{\frac{1}{n}}^1 \Phi_\eta(s)\varphi_p\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right) ds\\ &\ge \rho\|T_n u\|. \end{aligned}$$

Consequently $T_n : P \to P$. Since $f(t,u)$ and $G(t,s)$(in $[0,1]\times[\frac{1}{n},1]$) are continuous functions so T_n are continuous operator for $n = 3,4,\ldots$. Let $\Omega \subset P$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|u\| \le M$ for all $u \in \Omega$. Let

$$L = \max_{0\le t\le 1, 0\le u\le M} |f(t,u)| + 1, \quad K = \int_0^1 \Phi_\eta(s)ds + 1, H = \int_0^1 H(t,s)ds + 1.$$

Then for $u \in \Omega$, we have

$$|T_n u(t)| = \int_{\frac{1}{n}}^1 G(t,s)\varphi_q\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \le KL^{q-1}H^{q-1} < \infty$$

Hence, $T_n(\Omega)$ is bounded for $n = 3,4,\ldots$. Next we show that $T_n(\Omega)$ has equicontinuity property. For this let $\epsilon >$ is an arbitrary number. let

$$\delta = \frac{3(LH)^{q-1}}{s^{\alpha-1}}.$$

Then for $u \in \Omega, t_1,t_2 \in [0,1], t_1 < t_2$ and $t_2 - t_1 < \delta$, we show $|T_n u(t_2) - T_n u(t_1)| < \epsilon$. We consider three cases.

Case 1: $0 < t_1 < t_2 < \frac{1}{n}$.

$$\begin{aligned}
&|T_n u(t_2) - T_n u(t_1)| \\
&= \left| \int_{\frac{1}{n}}^1 G(t_2,s)\varphi_q\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \right. \\
&\quad \left. - \int_{\frac{1}{n}}^1 G(t_1,s)\varphi_q\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \right| \\
&\le L^{q-1}H^{q-1}\int_{\frac{1}{n}}^1 |G(t_2,s) - G(t_1,s)|\, ds \\
&= L^{q-1}H^{q-1}\int_{\frac{1}{n}}^1 (0)ds \\
&= 0 \le \epsilon.
\end{aligned}$$

Case 2: $0 < t_1 < \frac{1}{n} < t_2$.

$$\begin{aligned}
&|T_n u(t_2) - T_n u(t_1)| \\
&\le L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_2} |G(t_2,s) - G(t_1,s)|ds + \int_{t_2}^1 |G(t_2,s) - G(t_1,s)|ds\right) \\
&= L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_2} s^{\alpha-2}[(1-t_2) - (1-s)]ds + 0\right) \\
&\le L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_2} s^{\alpha-2}[t_2 - t_1 + t_1 - s]ds + 0\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq L^{q-1}H^{q-1}\left(2(t_2-t_1)\int_{\frac{1}{n}}^{t_2} s^{\alpha-2}ds+0\right)\\
&\leq L^{q-1}H^{q-1}2(t_2-t_1)\frac{1}{\alpha-1}\\
&< \epsilon.
\end{aligned}
$$

Case 3: $0 < t_1 < t_2 < \frac{1}{n}$.

$$
\begin{aligned}
&|T_nu(t_2)-T_nu(t_1)|\\
&\leq L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_1}|G(t_2,s)-G(t_1,s)|ds+\int_{t_1}^{t_2}|G(t_2,s)-G(t_1,s)|ds\right.\\
&\quad\left.+\int_{t_2}^{1}|G(t_2,s)-G(t_1,s)|ds\right)\\
&= L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_1}0ds+2(t_2-t_1)\int_{t_1}^{t_2}s^{\alpha-2}ds+(t_2-t_1)\int_{t_2}^{1}s^{\alpha-2}ds\right)\\
&\leq L^{q-1}H^{q-1}\frac{(t_2-t_1)}{s^{\alpha-1}}<\epsilon.
\end{aligned}
$$

By applying the Arzela-Ascoli Theorem, we conclude that $T_n : P \to P$ are completely continuous operators for all $n = 3, 4, \ldots$.

Step 2: $T_n : P \to P$ uniformly converges to T and $T : P \to P$ is completely continuous.

It is clear that $T : P \to P$. On the other hand by use of mean value theorem we can conclude

$$\varphi_q(u+v) < \varphi_q(u) + (q-1)(u+v)^{q-1}v. \tag{18}$$

Now assume $\epsilon > 0$ be an arbitrary number and let

$$N = \left(\frac{\frac{2-\eta}{(1-\eta)(\alpha-1)}+\frac{(q-1)L^{q-1}KH^{q-2}}{\beta-1}}{\epsilon}\right)^{\frac{1}{\min\{\beta-1,\alpha-1\}}}.$$

By using inequality (18) for all $n > N$ we obtain

$$
\begin{aligned}
&\|T_nu-Tu\|\\
&= \max|T_nu(t)-Tu(t)|\\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad-\int_{\frac{1}{n}}^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\leq \int_0^1\left[\varphi_q\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +(q-1)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)^{q-2}\int_0^{\frac{1}{n}} H(s,\tau)f(\tau,u(\tau))d\tau\Bigg]ds \\
&\quad -\int_{\frac{1}{n}}^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\le \int_0^{\frac{1}{n}} G(t,s)\varphi_q\left(\int_{\frac{1}{n}}^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\quad +(q-1)L^{q-1}\int_0^1 G(t,s)ds\left(\int_0^1 H(\tau,\tau)d\tau\right)^{q-2}\int_0^{\frac{1}{n}} H(\tau,\tau)d\tau \\
&\le L^{q-1}H^{q-1}\int_0^{\frac{1}{n}}\Phi_\eta(s)ds+(q-1)L^{q-1}KH^{q-2}\int_0^{\frac{1}{n}} H(\tau,\tau)d\tau \\
&\le L^{q-1}H^{q-1}\left(\frac{1}{n}\right)^{\alpha-1}\left[\frac{2-\eta}{(1-\eta)(\alpha-1)}\right] \\
&\quad +\left[(q-1)L^{q-1}KH^{q-2}\right]\frac{1}{\beta-1}\left(\frac{1}{n}\right)^{\beta-1} \\
&\le \left[\frac{2-\eta}{(1-\eta)(\alpha-1)}+\frac{(q-1)L^{q-1}KH^{q-2}}{\beta-1}\right]\left(\frac{1}{n}\right)^{\min\{\beta-1,\alpha-1\}} \\
&< \epsilon.
\end{aligned}
$$

Now in view of Steps 1 and 2 and Lemma 2.7 we conclude $T : P \to P$ is completely continuous. ■

For the convenience we introduce notations

$$
\begin{aligned}
M &= \left(\int_0^1 \Phi_\eta(s)ds\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)\right)^{-1}, \\
N &= \left(\rho^q\int_0^\xi \Phi_\eta(s)ds\varphi_q\left(\int_0^\xi H(\tau,\tau)d\tau\right)\right)^{-1}.
\end{aligned}
$$

Our first result is based on Theorem 2.4.

Theorem 4.3. *Let $f(t,u)$ be continuous on $[0,1]\times[0,\infty)$. Assume that there exist two different positive constants r_2, r_1, and $r_1 < r_2$ such that*

(A1) $f(t,u) \le \varphi(Mr_2)$, *for* $(t,u) \in [0,1]\times[0,r_2]$;

(A2) $f(t,u) \ge \varphi(Nr_1)$ *for* $(t,u) \in [0,\xi]\times[\rho r_1, r_1]$.

Then fractional boundary value problem (1) *has at least two positive solution such that $r_1 \le \|u\| \le r_2$.*

Proof. In view of Lemma 4.2, $T : P \to P$ is completely continuous. Let

$$\Omega_1 = \{u \in P : \|u\| < r_1\}, \quad \Omega_1 = \{u \in P : \|u\| < r_2\}.$$

If $u \in \partial\Omega_2$, then for all $t \in [0,1]$ we have $0 \le u(t) \le r_2$. From (A1) it is concluded that

$$\begin{aligned}\|Tu\| &= \max\left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\ &\le Mr_2\int_0^1 \Phi_\eta(s)ds\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right) \\ &= r_2 = \|u\|.\end{aligned}$$

So $\|Tu\| \le \|u\|$, for $u \in \partial\Omega_2$.
Let $u \in \partial\Omega_1$. Then by definition of P, we have

$$u(t) \ge \rho\|u\| = \rho r_1, \quad t \in [0,\xi].$$

By assumption (A2), for $t \in [0,\xi]$, we have

$$\begin{aligned}Tu(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds \\ &\ge \int_0^\xi \rho\Phi_\eta(s)\varphi_q\left(\int_0^\xi \rho H(\tau,\tau)f(\tau,u(\tau)d\tau\right)ds \\ &\ge Nr_1\left(\rho^q\int_0^\xi \Phi_\eta(s)ds\varphi_q\left(\int_0^\xi H(\tau,\tau)d\tau\right)\right) \\ &= Nr_1N^{-1} = r_1 = \|u\|\end{aligned}$$

Hence for $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$. Therefore by applying Theorem 2.4 we conclude that fractional boundary value problem (1) has at least one positive solution like $u(t)$ such that $r_1 < \|u\| < r_2$. ■

By the analogous way, one can obtain the following result.

Theorem 4.4. *Let $f(t,u)$ be continuous on $[0,1]\times[0,\infty)$. Assume that there exist two different positive constants r_2, r_1, and $r_1 < r_2$ such that*

(i) *$f(t,u) \ge \varphi_p(Nr_2)$ for all $u \in [\rho r_2, r_2]$, and*
(ii) *$f(t,u) \le \varphi_p(Mr_1)$ for all $u \in [0,r_1]$ and $t \in [0,1]$,*

then the problem (1) *has at least one positive solution $u \in P$ satisfiying $r_1 < \|u\| < r_2$.*

Theorem 4.5. *Let $f(t,u)$ be continuous on $[0,1]\times[0,\infty)$ and there exists a constant $\mu > 0$ such that*

$$\mu > \gamma^{q-1}\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)\int_0^1 \Phi_\eta(s)ds \tag{19}$$

where $\gamma = \max\{f(t,u) : (t,u) \in [0,1] \times [0,\mu]\}$. *Then, the fractional boundary value problem* (1) *has at least one positive solution.*

Proof. Let

$$\mathcal{U} = \{u \in P : \|u\| < \mu\}.$$

In view of Lemma 4.1, the operator $T : \overline{\mathcal{U}} \to P$ is completely continuous. Assume that there exists $u \in \overline{\mathcal{U}}$ and $\lambda \in (0,1)$ such that $u = \lambda Tu$. We have

$$\begin{aligned}
|u(t)| = |\lambda Tu(t)| &= \left|\lambda \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&\leq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(\tau,\tau)\gamma d\tau\right)ds \\
&\leq \gamma^{q-1}\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)\int_0^1 \Phi_\eta(s)ds
\end{aligned}$$

So

$$\|u\| \leq \gamma^{q-1}\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)\int_0^1 \Phi_\eta(s)ds.$$

Now (19), implies that $\|u\| < \mu$, that is, $u \notin \partial\mathcal{U}$. Hence it is concluded that there is no $u \in \partial\mathcal{U}$ such that $u = \lambda Tu$ for $\lambda \in (0,1)$. Therefor by Theorem 2.5, it is concluded that the fractional boundary value problem (1) has at least one positive solution. ∎

Our next results is based on Theorem 2.6.

Theorem 4.6. *Suppose* $f(t,u)$ *is continuous on* $[0,1] \times [0,\infty)$ *and there exist constants* $0 < a < \rho b < b$ *such that the following assumptions hold:*

(B1) $f(t,u) \leq \varphi_p(Ma)$, *for* $(t,u) \in [0,1] \times [0,a]$;
(B2) $f(t,u) \geq \varphi_p(\rho Nb)$, *for* $(t,u) \in [0,\xi] \times [\rho b, b]$;
(B3) $f(t,u) \leq \varphi_p(Mb)$ *for* $(t,u) \in [0,b]$.

Then the fractional boundary value problem (1) *has at least three positive solutions* u_1, u_2, u_3 *with*

$$\begin{aligned}
&\max_{0\leq t\leq 1}|u_1(t)| < a, \quad \rho b < \min_{0\leq t\leq\xi}|u_2(t)| < \max_{0\leq t\leq 1}|u_2(t)| \leq b, \\
&a < \max_{0\leq t\leq 1}|u_3(t) \leq b, \quad \min_{0\leq t\leq\xi}|u_3(t)| < \rho b.
\end{aligned}$$

Proof. We show that all the conditions of Theorem 2.6 hold. If $u \in \overline{P}_b$, then $\|u\| \leq b$. By assumption (B3) we conclude that $f(t,u(t)) \leq Mb$ for $0 \leq t \leq 1$, consequently

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&\le \int_0^1 \Phi_\eta(s)\varphi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\le Mb\int_0^1 \Phi_\eta(s)\varphi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)ds \\
&\le b.
\end{aligned}$$

Hence, $T:\overline{P}_b\to\overline{P}_b$. By the analogous way, one can prove, if $u\in\overline{P}_a$, then $\|Tu\|\le a$. Therefore, condition (ii) of Theorem 2.6 is satisfied.

Since the constant function $\frac{\rho b+b}{2}\in\{u\in P(\psi,\rho b,b):\psi(u)>\rho b\}$, we conclude that $\{u\in P(\psi,\rho b,b):\psi(u)>\rho b\}\ne\emptyset$. On the other hand, for $u\in P(\psi,\rho b,b)$, we have

$$\rho b\le\psi(u)=\min_{0\le t\le\xi}\le u(t)\le\|u\|\le b,\quad t\in[0,\xi].$$

That is $\psi(Tu)>\rho b$ for all $u\in P(\psi,\rho b,b)$. On the other hand from assumption (B2), we have $f(t,u(t))\ge N\rho b$ for $0\le t\le\xi$. So

$$\begin{aligned}
\psi(Tu) &= \min_{0\le t\le\xi}|Tu(t)|\ge\int_0^1 G(t,s)\varphi_q\left(\int_0^\xi \rho H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\ge N\rho b\left(\int_0^\xi \rho^q\Phi_\eta(s)ds\varphi_q\left(\int_0^\xi H(\tau,\tau)d\tau\right)\right) \\
&\ge \rho b
\end{aligned}$$

Thus, for all $u\in P(\psi,\rho b,b)$ we have $\psi(Tu)>\rho b$. This shows that condition (i) of Theorem 2.6 is satisfied. Thus the fractional boundary value problem (1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{0\le t\le1}|u_1(t)|,\quad \rho b<\min_{0\le t\le\xi}|u_2(t)|,\quad a<\max_{0\le t\le1}|u_3(t)|,\quad \min_{0\le t\le\xi}|u_3(t)|<\rho b$$

The proof is complete. ∎

5. Examples

Example 5.1. Consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{4}{3}}(\varphi_2(D^{\frac{5}{3}}u(t)))=\frac{1}{10}(5+\sqrt{u}+t^2), & t\in(0,1),\\ u'(0)=D^{\frac{5}{3}}u(0)=D^{\frac{5}{3}}u(1)=0, & u(1)=\int_0^\eta u(t)dt\end{cases}\tag{20}$$

where $\eta=\frac{1}{2}$. Let $\xi=0.8$. By a simple computation, we obtain $M\approx0.4938$ and $N\approx1.6390$. We choose $r_1=0.1,r_2=1$. Then

(i) $f(t,u) = \frac{1}{10}(5 + \sqrt{u} + t^2) \leq 0.7 < \varphi_p(Mr_2) = 0.702$ for all $(t,u) \in [0,1] \times [0,1]$.

(ii) $f(t,u) = \frac{1}{10}(5 + \sqrt{u} + t^2) \geq 0.5 > \varphi_p(Nr_1) \approx 0.4012$ for all $(t,u) \in [0,0.8] \times [0.02, 0.1]$.

Hence, all conditions of Theorem 4.3 are satisfied, consequently fractional boundary value problem (20) has at least one positive solution u such that $0.1 \leq |u\| \leq 1$.

Example 5.2. Consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{5}{3}}(\varphi_{\frac{7}{5}}(D^{\frac{4}{3}}u(t))) = f(t,u(t)), \quad t \in (0,1), \\ u'(0) = D^{\frac{4}{3}}u(0) = D^{\frac{4}{3}}u(1) = 0, \quad u(1) = \int_0^\eta u(t)dt \end{cases} \tag{21}$$

where $\eta = \frac{1}{2}$, and

$$f(t,u) = \begin{cases} 5u^2 + \frac{\sin^2 \pi t}{10}, & (t,u) \in [0,1] \times [0,1] \\ 2u^{\frac{1}{7}} + 3 + \frac{\sin^2 \pi t}{10}, & (t,u) \in [0,1] \times (1,\infty) \end{cases}$$

Let $\xi = 0.75$. One can see $M \approx 0.3311$ and $N \approx 5.862$. Then

(i) $f(t,u) = 5u^2 + \frac{\sin^2 \pi t}{10} \leq 0.3 < \varphi_p(Ma) = 0.3375$ for all $[0,1] \times [0,0.2]$.

(ii) $f(t,u) = 2u^{\frac{1}{7}} + 3 + \frac{\sin^2 \pi t}{10} \geq 3 > \varphi_p(\rho N b) = 1.27$, for all $f(t,u) \in [0,0.75] \times [1,512]$.

(iii) $(t,u) = 2u^{\frac{1}{7}} + 3 + \frac{\sin^2 \pi t}{10} \leq 7.1 < \varphi_p(\rho N b) \approx 8.1140$ for $(t,u) \in [0,1] \times [1,512]$.

Thus all conditions of the Theorem 4.5 are satisfied. Therefore, the fractional boundary value problem (21) has at least three positive solution u_1, u_2 and u_3 such that

$$\max_{0 \leq t \leq 1} |u(t)| < \frac{1}{5}, \quad 1 < \min_{0 \leq t \leq 0.75} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 512,$$
$$\frac{1}{5} < \max_{0 \leq t \leq 1} |u_3(t)| \leq 512, \quad \min_{0 \leq t \leq 0.75} |u_3(t)| < 32.$$

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