Southeast Asian Bulletin of Mathematics (2023) 47: 327–365

Southeast Asian
Bulletin of
Mathematics

Some Results on Relative (k,n) Valiron Defects with Respect to Integrated Moduli of Logarithmic Derivative of Entire and Meromorphic Functions*

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Received 15 February 2021
Accepted 30 July 2022

Communicated by Grigor Barsegian

AMS Mathematics Subject Classification(2020): 30D30, 30D35.

Abstract. The prime concern of this paper is to compare some relative (k, n) Nevanlinna

*The first author sincerely acknowledges the financial support rendered by DST-FIST 2022-2023 running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

defect with relative (k, n) Valiron defect from the view point of integrated moduli of logarithmic derivative of entire and meromorphic functions where k and n are two non-negative integers.

Keywords: Entire function; Meromorphic function; Integrated moduli of logarithmic derivative of entire and meromorphic functions; Relative (k, n) Nevanlinna defect; Relative (k, n) Valiron defect.

1. Introduction, Definitions and Notations

Let f be a non constant meromorphic function defined in the open complex plane $\mathbb{C}$. For $\alpha \in \mathbb{C} \cup \{\infty\}$, let $n(t,\ \alpha; f\ )$ denote the number of roots of $f = \alpha$ in $|z| \leq t$, the multiple roots being counted according to their multiplicities and $N(t, \alpha; f)$ is defined in the usual way in terms of $n(t, \alpha; f\ )$. Similarly, $\bar{n}(t, \alpha; f)$ denotes the number of distinct roots of $f = \alpha$ in $|z| \leq t$ and $\bar{N}(t, \alpha; f)$ is also defined in the usual way in terms of $\bar{n}(t, \alpha; f)$.

The Nevanlinna defect $\delta(\alpha; f)$ and the Valiron defect $\Delta(\alpha; f)$ of α are respectively defined in the following manner:

$$\delta(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}$$

and

$$\Delta(\alpha, f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, \alpha; f)}{T(r, f)}.$$

Milloux [5] introduced the concept of absolute defect of 'α' with respect to the derivative f'. Later Xiong [10] extended this definition. He introduced the term

$$\delta_R^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f)} \quad \text{for} \quad k = 1, 2, 3, \ldots.$$

and called it the relative Nevanlinna defect of 'α' with respect to $f^{(k)}$. Xiong [10] has shown various relations between the usual defects and the relative defects. Singh [7, 8] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects. In the paper we call the following two terms

$${}_R\delta_{(n)}^{(k)}(\alpha; f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \alpha; f^{(k)}\right)}{T\left(r, f^{(n)}\right)}$$

and

$${}_R\Delta_{(n)}^{(k)}(\alpha; f) = 1 - \liminf_{r\to\infty} \frac{N(r, \alpha; f^{(k)})}{T(r, f^{(n)})}$$

respectively the relative (k, n) Nevanlinna defect and the relative (k, n) Valiron defect of 'α' with respect to $f^{(k)}$ for $k = 1, 2, 3, \ldots.$ and $n = 0, 1, 2, 3, \ldots$ and prove various relations between them. For $n = 0$, the above definitions coincide with the relative Nevanlinna defect and the relative Valiron defect respectively.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution theory and the Nevanlinna theory as those are available in [3].

The following definitions are well known.

Definition 1.1. *The order ρ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

Definition 1.2. *The lower order λ_f of a meromorphic function f are defined as*

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} .$$

If f is entire, it can easily verify that

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} .$$

We may now recall the following definition.

If f is a meromorphic function in the complex plane, then the integrated moduli of the logarithmic derivative $I(r, f)$ is defined by

$$I(r, f) = \frac{r}{\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta$$

for $0 < r < +\infty$ (see [9]).

We now define the following two terms by using the concept of $I\,(r, f)$

$${}_I\delta_n^{(k)}(a; f) = 1 - \lim\sup_{r\to\infty} \frac{N(r, a; f^k)}{I(r, f^n)},$$

and

$$_{I}\Delta_{n}^{(k)}(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f^{k})}{I(r,f^{n})}.$$

These are respectively known as relative (k,n) Nevanlinna defect and relative (k,n) Valiron defect with respect to $I(r,f)$. In this paper we obtain different kind of relative (k,n) deficiencies of entire and meromorphic functions under the flavour of their integrated moduli of logarithmic derivative. Further, the estimations are sharper as ensured by suitable examples and counter examples.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [8] *Let f be a meromorphic function of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any non negative integer k,*

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}=1. \tag{1}$$

Lemma 2.2. [1] *For any meromorphic function f of finite order such that $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$,*

$$\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f^{(n)})}=1, \tag{2}$$

where k and n are any two non negative integers.

Lemma 2.3. [1] *Let f be a meromorphic function of finite order with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any $'\alpha'$,*

$$_{R}\delta_{(n)}^{(k)}(\alpha;f)=\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(n)}\right)}. \tag{3}$$

Lemma 2.4. [1] *If f be a meromorphic function of finite order with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any '$\alpha$',*

$$_{R}\Delta_{(n)}^{(k)}(\alpha;f)=\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}. \tag{4}$$

Lemma 2.5. [9] *Let f be a meromorphic function with finite lower order λ in the* *complex plane and* $f(0) = 1$. *If*

$$\gamma = \limsup_{r\to\infty} \frac{N(r,f) + N\left(r, \frac{1}{f}\right)}{T(r,f)},$$

then we have

$$\begin{aligned} &\lim_{r\to\infty} \inf \frac{I(r,f)}{T(r,f)} \\ &\leq (1+\lambda)\left(\frac{2+\lambda}{\lambda}\right)^{\lambda} \left\{ \frac{8(1+2\lambda)}{(3+4\lambda)} + \frac{\gamma(2+\log(8+8\lambda))}{\lambda \log\left(1+\frac{1}{(1+\lambda)}\right)} \right\}, \end{aligned} \tag{5}$$

when $\lambda > 0$, *and*

$$\lim_{r\to\infty} \inf \frac{I(r,f)}{T(r,f)} \leq \frac{12}{5} + \frac{3\gamma(2+\log 12)}{\log\left(\frac{5}{3}\right)} \tag{6}$$

when $\lambda = 0$.

Lemma 2.6. [9] *If for any entire function* f *of finite order* $'\rho'$ *has no zeros in* $\mathbb{C}$, *then*

$$\lim_{r\to\infty} \frac{I(r,f)}{T(r,f)} = \pi\rho \tag{7}$$

and

$$\lim_{r\to\infty} \frac{I(r,f^n)}{T(r,f^n)} = \pi\rho. \tag{8}$$

Lemma 2.7. *If* $\sum_{a\neq\infty}\delta(a;f) = 1$ *and* $\delta(\infty;f) = 1$, *where* f *is an entire function of finite order with* f *has no zeros in* $\mathbb{C}$. *Then for any non negative integer* k,

$$\lim_{r\to\infty} \frac{I(r,f^{(k)})}{I(r,f)} = 1. \tag{9}$$

Proof. In view of Lemma 2.2 and by using Equations (1), (7) and (8) we get that

$$\begin{aligned} \lim_{r\to\infty} \frac{I(r,f^{(k)})}{I(r,f)} &= \lim_{r\to\infty} \left[\frac{I(r,f^{(k)})}{T(r,f^k)} \cdot \frac{T(r,f^{(k)})}{I(r,f)} \right] \\ &= \lim_{r\to\infty} \frac{I(r,f^{(k)})}{T(r,f^k)} \cdot \lim_{r\to\infty} \frac{T(r,f^{(k)})}{I(r,f)} \\ &= \pi\rho \cdot \lim_{r\to\infty} \left[\frac{T(r,f^{(k)})}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)} \right] \\ &= \pi\rho \cdot \lim_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} \\ &= \pi\rho \cdot 1 \cdot \frac{1}{\pi\rho} = 1. \end{aligned}$$

This completes the proof. ■

Lemma 2.8. *If f is an entire function of finite order such that f has no zeros in $\mathbb{C}$ and $\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(\infty;f)=1$. Then for any two non negative integer k and n,*

$$\lim_{r\to\infty}\frac{I(r,f^{(k)})}{I(r,f^{(n)})}=1. \tag{10}$$

We omit the proof of Lemma 2.8 because it can be carried out in the line of Lemma 2.7.

Lemma 2.9. *For any entire function f of finite order such that it has no zeros in $\mathbb{C}$ with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any $'\alpha'$,*

$$ {}_I\delta_{(n)}^{(k)}(\alpha;f)=\left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}.$$

Proof. Using Equations (1), (2), (7), (8), (9), (10) and in view of the Lemma 2.3 we get that

$$\begin{aligned}
&{}_I\delta_{(n)}^{(k)}(\alpha;f)=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{I(r,f^{(n)})}\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\left(\frac{T(r,f^{(k)})}{T(r,f)}\cdot\frac{T(r,f)}{I(r,f^{(n)})}\right)\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I(r,f^{(n)})}\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\left(\frac{T(r,f)}{I\left(r,f\right)}\cdot\frac{I(r,f)}{I(r,f^{(n)})}\right)\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I\left(r,f\right)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{I(r,f^{(n)})}\\
&=1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\cdot\frac{1}{\pi\rho}\cdot 1\\
&=\left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left(1-\limsup_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T\left(r,f^{(k)}\right)}\right)\\
&=\left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{T(r,f^{(k)})}\\
&=\left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\left(\frac{I(r,f^{(n)})}{I(r,f)}\cdot\frac{I(r,f)}{T(r,f^{(k)})}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f^{(n)})}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot 1\cdot\lim_{r\to\infty}\left(\frac{I(r,f)}{T\left(r,f\right)}\cdot\frac{T\left(r,f\right)}{T(r,f^{(k)})}\right)\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T\left(r,f\right)}\cdot\lim_{r\to\infty}\frac{T\left(r,f\right)}{T(r,f^{(k)})}\\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}\cdot\pi\rho\cdot 1\\
&= \left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I\left(r,f^{(n)}\right)}.
\end{aligned}
$$

Thus the lemma is established. ■

Lemma 2.10. *Let f be an entire function of finite order such that f has no zeros in $\mathbb{C}$ with $\sum_{a\neq\infty}\delta(a;f)=\delta(\infty;f)=1$. Then for any ‘$\alpha$’,*

$$
{}_I\Delta_{(n)}^{(k)}(\alpha;f)=\left(1-\frac{1}{\pi\rho}\right)+\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}.
$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f be an entire function of non-zero finite order $'\rho'$ (i.e., $0<\rho<\infty$) such that f has no zeros in $\mathbb{C}$. Then for any two positive integers k and n,*

$$
{}_I\delta_{(n)}^{(0)}(0;f)+{}_I\delta_{(n)}^{(0)}(a;f)+{}_I\Delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi\rho}\le {}_I\Delta_{(n)}^{(0)}(\infty;f)+{}_I\Delta_{(n)}^{(k)}(0;f)+1,
$$

where ‘a’ is any non zero finite complex number.

Proof. Let us consider the following identity

$$
\frac{a}{f}=1-\frac{f-a}{f^{(k)}}.\frac{f^{(k)}}{f}.
$$

Since $m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{a}{f}\right)+O(1)$, we get from the above identity

$$
m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{f-a}{f^{(k)}}\right)+S(r,f). \tag{11}
$$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem in [3, pp. 55] it follows from Eq. (11) that

$$m\left(r,\frac{1}{f}\right)\le T\left(r,\frac{f-a}{f^{(k)}}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+S(r,f)$$

Then we have

$$m\left(r,\frac{1}{f}\right)\le T\left(r,\frac{f^{(k)}}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+S(r,f).$$

Thus,

$$m\left(r,\frac{1}{f}\right)\le N\left(r,\frac{f^{(k)}}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+S(r,f). \tag{12}$$

In view of [3, p. 34] it follows from Eq. (12) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \le\ & N(r,f^{(k)})+N\left(r,\frac{1}{f-a}\right)-N(r,f-a) \\ & -N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f). \end{aligned} \tag{13}$$

Then we have

$$\begin{aligned} & \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f^{(n)}\right)} \\ \le\ & \liminf_{r\to\infty}\left\{\frac{N\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)}-\frac{N(r,f)}{I\left(r,f^{(n)}\right)}-\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}\right\}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)} \\ \le\ & \liminf_{r\to\infty}\frac{N(r,f^{(k)})}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N(r,f)}{I\left(r,f^{(n)}\right)} \\ & -\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)}. \end{aligned}$$

Thus,

$$\begin{aligned} {}_I\delta_{(n)}^{(0)}(0;f)-\left(1-\frac{1}{\pi\rho}\right) \le\ & \{1-\ {}_I\Delta_{(n)}^{(k)}(\infty;f)\}-\{1-\ {}_I\Delta_{(n)}^{(0)}(\infty;f)\} \\ & -\{1-\ {}_I\Delta_{(n)}^{(k)}(0;f)\}+\{1-\ {}_I\delta_{(n)}^{(0)}(a;f)\}. \end{aligned}$$

Then we have

$${}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\delta_{(n)}^{(0)}(a;f)+\ {}_I\Delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi\rho}\le\ {}_I\Delta_{(n)}^{(0)}(\infty;f)+\ {}_I\Delta_{(n)}^{(k)}(0;f)+1.$$

This proves the theorem. ■

Remark 3.2. The sign $' \leq'$ in Theorem 3.1 can not be replaced by $' <'$ only as we see in the following example.

Example 3.3. Let $f(z) = \exp z$. Then $N(r, f) = 0$ and

$$T(r,f) = N(r,f) + m(r,f) = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|f(re^{i\theta})\right| d\theta = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|e^{re^{i\theta}}\right|$$

$$= \frac{1}{2\Pi}\int_0^{2\Pi} \log^+(e^{r\cos\theta})d\theta = \frac{1}{2\Pi}\int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} r\cos\theta d\theta = \frac{r}{\Pi}.$$

Now,

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{re^{i\theta}} \cdot re^{i\theta} \cdot i}{e^{re^{i\theta}}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}\left|re^{i\theta} \cdot i\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}(r)\,d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} d\theta \\ &= \frac{r^2}{2\pi} \cdot 2\pi = r^2 \neq 0. \end{aligned}$$

and

$$\rho = \limsup_{r\to\infty}\frac{\log T(r;f)}{\log r} = \limsup_{r\to\infty}\frac{\log \frac{r}{\pi}}{\log r} = 1$$

Thus,

$${}_I\Delta_{(n)}^{(0)}(\infty; f) = 1 - \lim\inf_{r\to\infty}\frac{N(r,f)}{I\left(r, f^{(n)}\right)} = 1,\ {}_I\Delta_{(n)}^{(k)}(0; f) = 1$$

and

$${}_I\delta_{(n)}^{(0)}(0; f) = \ {}_I\Delta_{(n)}^{(k)}(\infty; f) = 1.$$

Also

$$\begin{aligned} {}_I\delta_{(n)}^{(0)}(a; f) &= \left(1 - \frac{1}{\pi\rho}\right) + \lim\inf_{r\to\infty}\frac{m(r,a;f)}{I\left(r, f^{(n)}\right)} \\ &= \left(1 - \frac{1}{\pi\rho}\right) + \lim\inf_{r\to\infty}\left[\frac{m(r,a;f)}{I(r,f)} \cdot \frac{I(r,f)}{I\left(r, f^{(n)}\right)}\right] \\ &= \left(1 - \frac{1}{\pi\rho}\right) + \lim\inf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)} \\ &= \left(1 - \frac{1}{\pi\rho}\right) + \lim\inf_{r\to\infty}\frac{\frac{r}{\pi}}{r^2} = \left(1 - \frac{1}{\pi}\right). \end{aligned}$$

So,

$${}_I\Delta_{(n)}^{(0)}(\infty; f) + \ {}_I\Delta_{(n)}^{(k)}(0; f) + 1 = 1 + 1 + 1 = 3$$

and

$$ {}_I\delta_{(n)}^{(0)}(0;f) + {}_I\delta_{(n)}^{(0)}(a;f) + {}_I\Delta_{(n)}^{(k)}(\infty;f) = 1 + \left(1 - \frac{1}{\pi}\right) + 1 + \frac{1}{\pi} = 3. $$

Then

$$ {}_I\Delta_{(n)}^{(0)}(\infty;f) + {}_I\Delta_{(n)}^{(k)}(0;f) = 3 = {}_I\delta_{(n)}^{(0)}(0;f) + {}_I\delta_{(n)}^{(0)}(a;f) + {}_I\Delta_{(n)}^{(k)}(\infty;f). $$

Remark 3.4. The condition $\rho > 0$ in Theorem 3.1 is necessary as we see from the following example.

Example 3.5. Let $f(z) = z$. Then $N(r,f) = 0$ and

$$ \begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|re^{i\theta}\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \log(r\cos\theta)\, d\theta - \frac{1}{2\pi}\int_0^{-\frac{\pi}{2}} \log(r\cos\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \log(r\cos\theta)\, d\theta + \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \log(r\cos\theta)\, d\theta \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \log(r\cos\theta)\, d\theta = \frac{1}{\pi}\cdot 2\pi \log\left(\frac{r^2}{2}\right) = 2\log\left(\frac{r^2}{2}\right) \neq 0. \end{aligned} $$

Now,

$$ \rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]} r}{\log r} = \limsup_{r\to\infty} \frac{1}{\log r} = 0. $$

Thus,

$$ I(r,f) = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{re^{i\theta}\cdot i}{re^{i\theta}}\right| d\theta = \frac{r}{2\pi}\cdot 2\pi = r \neq 0. $$

Hence

$$ \infty \leq {}_I\Delta_{(n)}^{(0)}(\infty;f) + {}_I\Delta_{(n)}^{(k)}(0;f) + 1, $$

which is contrary to the assumptions of Theorem 3.1.

In the following theorem we may obtain somewhat a different estimation for meromorphic f under suitable conditions.

Theorem 3.6. *If f be any meromorphic function with finite lower order λ and $f(0) = 1$, then for any non zero finite complex number 'a' and for any two positive integers k and n,*

$$ {}_I\Delta_{(n)}^{(k)}(0;f) + {}_I\delta_{(n)}^{(k)}(\infty;f) + {}_I\delta_{(n)}^{(0)}(a;f) + \frac{1}{A} \leq {}_I\Delta_{(n)}^{(0)}(\infty;f) + {}_I\Delta_{(n)}^{(k)}(0;f) + 1 $$

holds where

$$A=(1+\lambda)\left(\frac{2+\lambda}{\lambda}\right)^{\lambda}\left\{\frac{8\,(1+2\lambda)}{(3+4\lambda)}+\frac{\gamma\,(2+\log\,(8+8\lambda))}{\lambda\log\left(1+\frac{1}{(1+\lambda)}\right)}\right\}$$

and

$$\gamma=\limsup_{r\to\infty}\frac{N\,(r,f)+N\left(r,\frac{1}{f}\right)}{T\,(r,f)}.$$

Proof. In view of Lemma 2.10 we obtain that

$$\begin{aligned}{}_{I}\Delta_{(n)}^{(k)}(\alpha;f) &= 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}\\ &= 1-\liminf_{r\to\infty}\left(\frac{N\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\cdot\frac{T(r,f^{(n)})}{I(r,f^{(n)})}\right)\\ &\leq 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\cdot\liminf_{r\to\infty}\frac{T(r,f^{(n)})}{I(r,f^{(n)})}.\end{aligned}$$

Now in view of Lemma 2.5 and by using Eq. (5) we get that

$${}_{I}\Delta_{(n)}^{(k)}(\alpha;f)\leq 1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\cdot\frac{1}{A},$$

where

$$A=(1+\lambda)\left(\frac{2+\lambda}{\lambda}\right)^{\lambda}\left\{\frac{8\,(1+2\lambda)}{(3+4\lambda)}+\frac{\gamma\,(2+\log\,(8+8\lambda))}{\lambda\log\left(1+\frac{1}{(1+\lambda)}\right)}\right\}$$

and

$$\gamma=\limsup_{r\to\infty}\frac{N\,(r,f)+N\left(r,\frac{1}{f}\right)}{T\,(r,f)}.$$

Thus,

$$\begin{aligned}{}_{I}\Delta_{(n)}^{(k)}(\alpha;f) &\leq \left(1-\frac{1}{A}\right)+\frac{1}{A}\left(1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\right)\\ &= \left(1-\frac{1}{A}\right)+\frac{1}{A}\left(1-\liminf_{r\to\infty}\frac{N\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\right)\\ &= \left(1-\frac{1}{A}\right)+\frac{1}{A}\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{T(r,f^{(n)})}\\ &= \left(1-\frac{1}{A}\right)+\frac{1}{A}\limsup_{r\to\infty}\left(\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}\cdot\frac{I(r,f^{(n)})}{T(r,f^{(n)})}\right)\end{aligned}$$

$$\leq \left(1-\frac{1}{A}\right)+\frac{1}{A}\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}\cdot A$$
$$= \left(1-\frac{1}{A}\right)+\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}.$$

Therefore,

$$_I\Delta_{(n)}^{(k)}(\alpha;f) \leq \left(1-\frac{1}{A}\right)+\limsup_{r\to\infty}\frac{m\left(r,\alpha;f^{(k)}\right)}{I(r,f^{(n)})}. \tag{14}$$

On dividing Eq. (13) by $I(r,f^{(n)})$ we get that

$$\limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f^{(n)}\right)}$$
$$\leq \limsup_{r\to\infty}\left\{\frac{N\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)}-\frac{N(r,f)}{I\left(r,f^{(n)}\right)}-\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}\right\}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)}.$$

Then we have

$$\limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{I\left(r,f^{(n)}\right)}$$
$$\leq \limsup_{r\to\infty}\frac{N\left(r,f^{(k)}\right)}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N(r,f)}{I\left(r,f^{(n)}\right)}$$
$$-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I\left(r,f^{(n)}\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)}.$$

From Eq. (14) we obtain that

$$_I\Delta_{(n)}^{(k)}(0;f)-\left(1-\frac{1}{A}\right) \leq \{1-\ _I\delta_{(n)}^{(k)}(\infty;f)\}-\{1-\ _I\Delta_{(n)}^{(0)}(\infty;f)\}$$
$$-\{1-\ _I\Delta_{(n)}^{(k)}(0;f)\}+\{1-\ _I\delta_{(n)}^{(0)}(a;f)\}.$$

Thus,

$$_I\Delta_{(n)}^{(k)}(0;f)+\ _I\delta_{(n)}^{(k)}(\infty;f)+\ _I\delta_{(n)}^{(0)}(a;f)+\frac{1}{A}\leq\ _I\Delta_{(n)}^{(0)}(\infty;f)+\ _I\Delta_{(n)}^{(k)}(0;f)+1.$$

Thus the theorem is established. ∎

Remark 3.7. In Theorem 3.6, the inequality $'\leq'$ can not be removed by $''<'$ only as is evident from the following example.

Example 3.8. Let $f=\frac{1}{z+1}$ and $k=n=0$. Then as $\frac{1}{|re^{i\theta}+1|}\leq\frac{1}{r+1}\to 0$ for $r\to\infty$, we get for all large values of r that

$$\log^+\frac{1}{|re^{i\theta}+1|}=0$$

and therefore

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|re^{i\theta}+1|}d\theta=0,$$

for all sufficiently large values of r.

Also,

$$N(r,f)=\int_0^r\frac{n(t,f)}{t}dt=\log r$$

and

$$N\left(r,\frac{1}{f}\right)=\int_0^r\frac{n\left(t,\frac{1}{f}\right)}{t}dt=0.$$

Therefore,

$$T(r,f)=m(r,f)+N(r,f)=\log r+O(1).$$

Now,

$$\begin{aligned}
I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{\frac{-ire^{i\theta}}{(re^{i\theta}+1)^2}}{\frac{1}{(re^{i\theta}+1)}}\right|d\theta=\frac{r}{2\pi}\cdot\frac{r}{r^2+1}\int_0^{2\pi}d\theta=\frac{r^2}{r^2+1}\neq 0,\\
\lambda &= \lim_{r\to\infty}\inf\frac{\log^{[2]}r+O(1)}{\log r}=0,\\
\gamma &= \lim_{r\to\infty}\sup\frac{N(r,f)+N\left(r,\frac{1}{f}\right)}{T(r,f)}=1.
\end{aligned}$$

Thus,

$$\begin{aligned}
{}_I\Delta_{(n)}^{(0)}(0;f) &= 1-\lim_{r\to\infty}\inf\frac{N\left(r,\frac{1}{f}\right)}{I(r,f^n)}=1,\\
{}_I\delta_{(n)}^{(0)}(\infty;f) &= 1-\lim_{r\to\infty}\sup\frac{N(r,f)}{I(r,f^n)}=1-\lim_{r\to\infty}\sup\frac{\log r+O(1)}{\log r}=1,\\
{}_I\delta_{(n)}^{(0)}(a;f) &= {}_I\Delta_{(n)}^{(0)}(\infty;f)={}_I\Delta_{(n)}^{(k)}(0;f)=1.
\end{aligned}$$

Hence from Eq. (5) we get that

$${}_I\Delta_{(n)}^{(k)}(0;f)+{}_I\delta_{(n)}^{(k)}(\infty;f)+{}_I\delta_{(n)}^{(0)}(a;f)+\frac{1}{A}=1+1+1+0=3$$

and

$${}_I\Delta_{(n)}^{(0)}(\infty;f)+{}_I\Delta_{(n)}^{(k)}(0;f)+1=1+1+1=3.$$

Therefore,

$${}_I\Delta_{(n)}^{(k)}(0;f)+{}_I\delta_{(n)}^{(k)}(\infty;f)+{}_I\delta_{(n)}^{(0)}(a;f)+\frac{1}{A}=3={}_I\Delta_{(n)}^{(0)}(\infty;f)+{}_I\Delta_{(n)}^{(k)}(0;f)+1.$$

Theorem 3.9. *If for any entire function f of finite order $'\rho'$ with no zeros in $\mathbb{C}$, then for any two positive integers k and n,*

$$ {}_I\delta^{(0)}_{(n)}(0;f) + {}_I\delta^{(0)}_{(n)}(\infty;f) \leq {}_I\Delta^{(k)}_{(n)}(0;f) + {}_I\Delta^{(0)}_{(n)}(\infty;f). $$

Proof. Since $f = f^{(k)}.\frac{f}{f^{(k)}}$, we get

$$ m(r,f) \leq m(r,f^{(k)}) + m\left(r, \frac{f}{f^{(k)}}\right). \tag{15} $$

Now by Nevanlinna's first fundamental theorem and Milloux's theorem on [3, pp. 55] we obtain from Eq. (15) that

$$ m(r,f) \leq m(r,f^{(k)}) + T\left(r, \frac{f}{f^{(k)}}\right) - N\left(r, \frac{f}{f^{(k)}}\right). $$

Then we have

$$ m(r,f) \leq m(r,f^{(k)}) + T\left(r, \frac{f^{(k)}}{f}\right) - N\left(r, \frac{f}{f^{(k)}}\right) + O(1). $$

Thus,

$$ m(r,f) \leq m(r,f^{(k)}) + N\left(r, \frac{f^{(k)}}{f}\right) - N\left(r, \frac{f}{f^{(k)}}\right) + S(r,f). \tag{16} $$

Now in view of [3, p. 34] it follows from Eq. (16) that

$$ \begin{aligned} m(r,f) \leq\ & m(r,f^{(k)}) + N(r,f^{(k)}) + N\left(r, \frac{1}{f}\right) \\ & -N(r,f) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r,f). \end{aligned} $$

Then we have

$$ \begin{aligned} \liminf_{r\to\infty} \frac{m(r,f)}{I(r,f^{(n)})} \leq\ & \liminf_{r\to\infty} \left\{ \frac{N(r,f^{(k)})}{I(r,f^{(n)})} - \frac{N(r,f)}{I(r,f^{(n)})} - \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{I(r,f^{(n)})} \right\} \\ & + \limsup_{r\to\infty} \left\{ \frac{N\left(r, \frac{1}{f}\right)}{I(r,f^{(n)})} + \frac{m(r,f^{(k)})}{I(r,f^{(n)})} \right\} \\ \leq\ & \liminf_{r\to\infty} \frac{N(r,f^{(k)})}{I(r,f^{(n)})} - \liminf_{r\to\infty} \frac{N(r,f)}{I(r,f^{(n)})} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{I(r,f^{(n)})} \\ & + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f}\right)}{I(r,f^{(n)})} + \limsup_{r\to\infty} \frac{m(r,f^{(k)})}{I(r,f^{(n)})} \end{aligned} \tag{17} $$

and

$$
{}_I\delta_{(n)}^{(0)}(\infty;f)-\left(1-\frac{1}{\pi\rho}\right)\leq\{1-\ {}_I\Delta_{(n)}^{(k)}(\infty;f)\}-\{1-\ {}_I\Delta_{(n)}^{(0)}(\infty;f)\}
$$
$$
-\{1-\ {}_I\Delta_{(n)}^{(k)}(0;f)\}+\{1-\ {}_I\delta_{(n)}^{(0)}(0;f)\}
$$
$$
+{}_I\Delta_{(n)}^{(k)}(\infty;f)-\left(1-\frac{1}{\pi\rho}\right).
$$

Thus,

$$
{}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\delta_{(n)}^{(0)}(\infty;f)\leq\ {}_I\Delta_{(n)}^{(k)}(0;f)+\ {}_I\Delta_{(n)}^{(0)}(\infty;f).
$$

Thus the theorem is established. ■

Remark 3.10. The inequality $'\leq'$ in Theorem 3.9 can not be removed by $'<'$ only as is evident from the following example.

Example 3.11. Let us suppose that $f=z^2$. Then $N(r,f)=0$.

So,

$$
T(r,f)=m(r,f)=\frac{1}{2\Pi}\int_0^{2\Pi}\log^+\left|(r^2e^{2i\theta})\right|d\theta=\frac{1}{2\Pi}\int_0^{2\Pi}\log^+\left|r^2\right|d\theta
$$
$$
=\frac{1}{2\Pi}\log^+\left|r^2\right|\int_0^{2\Pi}d\theta=\frac{1}{2\Pi}\log^+\left|r^2\right|\cdot 2\pi=\log^+\left|r^2\right|\neq 0,
$$

$$
I(r,f)=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{r^2\cdot 2e^{2i\theta}}{r^2\cdot e^{2i\theta}}\right|d\theta
$$
$$
=\frac{r}{2\pi}\int_0^{2\pi}|2|\,d\theta=\frac{r}{2\pi}\cdot 2\int_0^{2\pi}d\theta=2r\neq 0.
$$

Then

$$
{}_I\delta_{(n)}^{(0)}(0;f)=\ {}_I\delta_{(n)}^{(0)}(\infty;f)=1
$$

and

$$
{}_I\Delta_{(n)}^{(k)}(0;f)=\ {}_I\Delta_{(n)}^{(0)}(\infty;f)=1.
$$

Thus,

$$
{}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\delta_{(n)}^{(0)}(\infty;f)=2=\ {}_I\Delta_{(n)}^{(k)}(0;f)+\ {}_I\Delta_{(n)}^{(0)}(\infty;f).
$$

Theorem 3.12. *Let a,b be any two distinct finite complex numbers. Then for any two positive integers k and n,*

$$
{}_I\Delta_{(n)}^{(k)}(0;f)+\ {}_I\Delta_{(n)}^{(0)}(\infty;f)+1\geq\ {}_I\Delta_{(n)}^{(k)}(\infty;f)+\ {}_I\delta_{(n)}^{(0)}(a;f)+\frac{1}{2}\cdot\ {}_I\delta_{(n)}^{(0)}(b;f)+\frac{1}{2\pi\rho},
$$

where f is an entire function of non-zero finite order $'\rho'$ (i.e., $0<\rho<\infty$) such that f has no zeros in $\mathbb{C}$.

Proof. Considering the identity

$$\frac{b-a}{f-a}=\frac{f^{(k)}}{f-a}\left\{\frac{f-a}{f^{(k)}}-\frac{f-b}{f^{(k)}}\right\},$$

we obtain in view of Milloux's theorem [3, p. 55],

$$m\left(r,\frac{b-a}{f-a}\right)\le m\left(r,\frac{f-a}{f^{(k)}}\right)+m\left(r,\frac{f-b}{f^{(k)}}\right)+S(r,f).$$

Then we have

$$\begin{aligned}m\left(r,\frac{b-a}{f-a}\right)\le\ & T\left(r,\frac{f-a}{f^{(k)}}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)\\ &+T\left(r,\frac{f-b}{f^{(k)}}\right)-N\left(r,\frac{f-b}{f^{(k)}}\right)+S(r,f).\end{aligned}\tag{18}$$

Since $m\left(r,\frac{1}{f-a}\right)\le m\left(r,\frac{b-a}{f-a}\right)+O(1)$ and $T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)$, it follows from Eq. (18) that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right)\le\ & N\left(r,\frac{f^{(k)}}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+N\left(r,\frac{f^{(k)}}{f-b}\right)\\ &-N\left(r,\frac{f-b}{f^{(k)}}\right)+S(r,f).\end{aligned}\tag{19}$$

In view of [3, p. 34] we get from Eq. (19) that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right)\le\ & N(r,f^{(k)})+N\left(r,\frac{1}{f-a}\right)-N(r,f-a)\\ &-N\left(r,\frac{1}{f^{(k)}}\right)+N(r,f^{(k)})+N\left(r,\frac{1}{f-b}\right)-N(r,f-b)\\ &-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).\end{aligned}\tag{20}$$

Then we have

$$\begin{aligned}&\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I(r,f^{(n)})}\\ &\le 2\left\{\liminf_{r\to\infty}\frac{N(r,f^{(k)})}{I(r,f^{(n)})}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}}\right)}{I(r,f^{(n)})}-\liminf_{r\to\infty}\frac{N(r,f)}{I(r,f^{(n)})}\right\}\\ &\quad+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I(r,f^{(n)})}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{I(r,f^{(n)})}.\end{aligned}$$

Thus,

$$\begin{aligned}
& {}_{I}\delta_{(n)}^{(0)}(a;f) - \left(1 - \frac{1}{\pi\rho}\right) \\
\leq\ & 2\{1 - {}_{I}\Delta_{(n)}^{(k)}(\infty;f)\} - 2\{1 - {}_{I}\Delta_{(n)}^{(k)}(0;f)\} - 2\{1 - {}_{I}\Delta_{(n)}^{(0)}(\infty;f)\} \\
& + \{1 - {}_{I}\delta_{(n)}^{(0)}(a;f)\} + \{1 - {}_{I}\delta_{(n)}^{(0)}(b;f)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& 2\ {}_{I}\delta_{(n)}^{(0)}(a;f) + \frac{1}{\pi\rho} \\
\leq\ & 2\ {}_{I}\Delta_{(n)}^{(k)}(0;f) + 2\ {}_{I}\Delta_{(n)}^{(0)}(\infty;f) - 2\ {}_{I}\Delta_{(n)}^{(k)}(\infty;f) - {}_{I}\delta_{(n)}^{(0)}(b;f) + 1.
\end{aligned}$$

Then we have

$$\begin{aligned}
& {}_{I}\Delta_{(n)}^{(k)}(0;f) + {}_{I}\Delta_{(n)}^{(0)}(\infty;f) + 1 \\
\geq\ & {}_{I}\Delta_{(n)}^{(k)}(\infty;f) + {}_{I}\delta_{(n)}^{(0)}(a;f) + \frac{1}{2} \cdot {}_{I}\delta_{(n)}^{(0)}(b;f) + \frac{1}{2\pi\rho}.
\end{aligned}$$

This proves the theorem. ■

Remark 3.13. The condition that $'a'$ and $'b'$ be any two distinct finite complex numbers in Theorem 3.12 is essential as we see from the following examples.

Example 3.14. Let $f = \exp z^2$ and $a = b = 0$. Then $N(r,f) = 0$,

$$\begin{aligned}
T(r,f) &= m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r^2 e^{2i\theta}}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r^2(\cos 2\theta + i\sin 2\theta)}\right| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(e^{r^2\cos 2\theta}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^2 \cos 2\theta d\theta = \frac{r^2}{\pi}
\end{aligned}$$

and

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]} e^{r^2}}{\log r} = \limsup_{r\to\infty} \frac{2\log r}{\log r} = 2.$$

Thus,

$$\begin{aligned}
I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{\left|e^{r^2 e^{2i\theta}}\right| \cdot \left|2ir^2 e^{2i\theta}\right|}{\left|e^{r^2 e^{2i\theta}}\right|} d\theta \\
&= \frac{r}{2\pi} \cdot 2r^2 \int_0^{2\pi} \frac{e^{r^2\cos 2\theta} \cdot e^{c\cos 2\theta}}{e^{r^2\cos 2\theta}} d\theta = \frac{r^3}{\pi}\int_0^{2\pi} e^{\cos 2\theta} d\theta \\
&= \frac{r^3}{\pi} \cdot \frac{1}{2}\int_0^{4\pi} e^{\cos\eta} d\eta = \frac{r^3}{2\pi} \cdot 4\pi I_0(1) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0.
\end{aligned}$$

where $I_n(z)$ is the Modified Bessel Function of the first kind such that

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cdot\cos n\theta d\theta.$$

So,

$${}_I\Delta^{(k)}_{(n)}(0;f) = {}_I\Delta^{(0)}_{(n)}(\infty;f) = 1$$

and

$${}_I\Delta^{(k)}_{(n)}(\infty;f) = {}_I\delta^{(0)}_{(n)}(0;f) = 1.$$

Therefore,

$${}_I\Delta^{(k)}_{(n)}(0;f) + {}_I\Delta^{(0)}_{(n)}(\infty;f) + 1 = 1+1+1 = 3$$

and

$${}_I\Delta^{(k)}_{(n)}(\infty;f) + {}_I\delta^{(0)}_{(n)}(0;f) + \frac{1}{2}\cdot {}_I\delta^{(0)}_{(n)}(0;f) + \frac{1}{4\pi} = 1+1+\frac{1}{2}+\frac{1}{4\pi} = \frac{5}{2}+\frac{1}{4\pi},$$

which is contradictory to Theorem 3.12.

Example 3.15. Let $f = \exp z, a = \infty$ and $b = 0$. Then $N(r,f) = 0$, $I(r,f) = r^2 \neq 0$ and $\rho = 1$.

So,

$${}_I\Delta^{(k)}_{(n)}(0;f) = {}_I\Delta^{(0)}_{(n)}(\infty;f) = 1$$

and

$${}_I\Delta^{(k)}_{(n)}(\infty;f) = {}_I\delta^{(0)}_{(n)}(\infty;f) = {}_I\delta^{(0)}_{(n)}(0;f) = 1.$$

Thus,

$${}_I\Delta^{(k)}_{(n)}(0;f) + {}_I\Delta^{(0)}_{(n)}(\infty;f) + 1 = 1+1+1 = 3$$

and

$${}_I\Delta^{(k)}_{(n)}(\infty;f) + {}_I\delta^{(0)}_{(n)}(\infty;f) + \frac{1}{2}\cdot {}_I\delta^{(0)}_{(n)}(0;f) + \frac{1}{4\pi} = 1+1+\frac{1}{2}+\frac{1}{2\pi} = \frac{5}{2}+\frac{1}{4\pi},$$

which is a contradiction.

Example 3.16. Let $f = \exp z$, $a = 0$ and $b = \infty$. Then $N(r,f) = 0$, $I(r,f) = r^2 \neq 0$ and $\rho = 1$.

So,

$${}_I\Delta^{(k)}_{(n)}(0;f) = {}_I\Delta^{(0)}_{(n)}(\infty;f) = 1$$

and

$${}_I\Delta^{(k)}_{(n)}(\infty;f) = {}_I\delta^{(0)}_{(n)}(0;f) = {}_I\delta^{(0)}_{(n)}(\infty;f) = 1.$$

Thus,

$${}_I\Delta^{(k)}_{(n)}(0;f) + {}_I\Delta^{(0)}_{(n)}(\infty;f) + 1 = 1+1+1 = 3$$

and

$$ {}_{I}\Delta_{(n)}^{(k)}(\infty; f) + {}_{I}\delta_{(n)}^{(0)}(0; f) + \frac{1}{2} \cdot {}_{I}\delta_{(n)}^{(0)}(\infty; f) + \frac{1}{4\pi} = 1 + 1 + \frac{1}{2} + \frac{1}{2\pi} = \frac{5}{2} + \frac{1}{4\pi}. $$

So, we arrive at a contradiction.

Remark 3.17. The following example ensures the necessity of the condition $\rho > 0$ in Theorem 3.12.

Example 3.18. Let $f(z) = z^2$ and $a = b = 0$. Then $N(r, f) = 0$,

$$ \begin{aligned} I(r, f) &= \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} \right| d\theta = \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{r^2 e^{2i\theta} \cdot 2i}{r^2 e^{2i\theta}} \right| d\theta \\ &= \frac{r}{2\pi} \dot{2} \int_0^{2\pi} d\theta = 2r \neq 0 \end{aligned} $$

and

$$ \rho = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^{[2]} r^2}{\log r} = \limsup_{r \to \infty} \frac{2}{\log(r^2)} = 0. $$

So,

$$ {}_{I}\Delta_{(n)}^{(k)}(0; f) = {}_{I}\Delta_{(n)}^{(0)}(\infty; f) = 1 $$

and

$$ {}_{I}\Delta_{(n)}^{(k)}(\infty; f) = {}_{I}\delta_{(n)}^{(0)}(0; f) = {}_{I}\delta_{(n)}^{(0)}(\infty; f) = 1. $$

Thus,

$$ {}_{I}\Delta_{(n)}^{(k)}(0; f) + {}_{I}\Delta_{(n)}^{(0)}(\infty; f) + 1 \geq \infty, $$

which is contrary to Theorem 3.12.

If we consider f to be a meromorphic function in Theorem 3.12 we have the next theorem.

Theorem 3.19. *Let f be a meromorphic function of finite lower order $'\lambda'$ with $f(0) = 1$ and $'a','b'$ be any two distinct finite complex numbers. Then for any two positive integers k and n,*

$$ \begin{aligned} &{}_{I}\Delta_{(n)}^{(0)}(a; f) + 2 \cdot {}_{I}\delta_{(n)}^{(k)}(\infty; f) + {}_{I}\delta_{(n)}^{(0)}(a; f) + {}_{I}\delta_{(n)}^{(0)}(b; f) + \frac{1}{A} \\ &\leq 2 \cdot {}_{I}\Delta_{(n)}^{(k)}(0; f) + 2 \cdot {}_{I}\Delta_{(n)}^{(0)}(\infty; f) + 1, \end{aligned} $$

where

$$ A = (1 + \lambda) \left(\frac{2 + \lambda}{\lambda} \right)^{\lambda} \left\{ \frac{8(1 + 2\lambda)}{(3 + 4\lambda)} + \frac{\gamma(2 + \log(8 + 8\lambda))}{\lambda \log\left(1 + \frac{1}{(1+\lambda)}\right)} \right\} $$

and

$$\gamma = \limsup_{r\to\infty} \frac{N(r,f) + N\left(r, \frac{1}{f}\right)}{T(r,f)}.$$

Proof. On dividing Eq. (20) by $I\left(r, f^{(n)}\right)$ we obtain that

$$\limsup_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{I(r,f^{(n)})} \leq 2\limsup_{r\to\infty} \left\{ \frac{N(r,f^{(k)})}{I(r,f^{(n)})} - \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{I(r,f^{(n)})} - \frac{N(r,f)}{I(r,f^{(n)})} \right\}$$
$$+ \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I(r,f^{(n)})} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-b}\right)}{I(r,f^{(n)})}.$$

That is,

$$\limsup_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{I(r,f^{(n)})}$$
$$\leq 2\left\{ \limsup_{r\to\infty} \frac{N(r,f^{(k)})}{I(r,f^{(n)})} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{f^{(k)}}\right)}{I(r,f^{(n)})} - \liminf_{r\to\infty} \frac{N(r,f)}{I(r,f^{(n)})} \right\}$$
$$+\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I(r,f^{(n)})} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-b}\right)}{I(r,f^{(n)})}.$$

By using Eq. (14) we get that

$${}_I\Delta_{(n)}^{(0)}(a;f) - \left(1 - \frac{1}{A}\right)$$
$$\leq 2\{1 - {}_I\delta_{(n)}^{(k)}(\infty;f)\} - 2\{1 - {}_I\Delta_{(n)}^{(k)}(0;f)\} - 2\{1 - {}_I\Delta_{(n)}^{(0)}(\infty;f)\}$$
$$+\{1 - {}_I\delta_{(n)}^{(0)}(a;f)\} + \{1 - {}_I\delta_{(n)}^{(0)}(b;f)\}.$$

That is,

$${}_I\Delta_{(n)}^{(0)}(a;f) + 2\cdot {}_I\delta_{(n)}^{(k)}(\infty;f) + {}_I\delta_{(n)}^{(0)}(a;f) + {}_I\delta_{(n)}^{(0)}(b;f) + \frac{1}{A}$$
$$\leq 2\cdot {}_I\Delta_{(n)}^{(k)}(0;f) + 2\cdot {}_I\Delta_{(n)}^{(0)}(\infty;f) + 1.$$

This completes the proof. ■

Remark 3.20. The sign $'\leq'$ in Theorem 3.19 can not be replaced by $'<'$ only as we see in the following example.

Example 3.21. Let $f(z) = \frac{1}{2z+1}$. Then $m(r,f) = 0$, $N(r,f) = \log r + O(1)$ and $N\left(r, \frac{1}{f}\right) = 0$. So, $T(r,f) = \log r + O(1)$.

Also, $I(r,f) = \frac{2r^2}{2r+1} \neq 0, \lambda = 0$ and $\gamma = 1$.
Now,

$$ {}_I\Delta^{(0)}_{(n)}(a;f) = {}_I\delta^{(k)}_{(n)}(\infty;f) = {}_I\delta^{(0)}_{(n)}(a;f) = {}_I\delta^{(0)}_{(n)}(b;f) = 1 $$

and

$$ {}_I\Delta^{(k)}_{(n)}(0;f) = {}_I\Delta^{(0)}_{(n)}(\infty;f) = 1. $$

Hence, we have

$$ \begin{aligned} &{}_I\Delta^{(0)}_{(n)}(a;f) + 2\cdot {}_I\delta^{(k)}_{(n)}(\infty;f) + {}_I\delta^{(0)}_{(n)}(a;f) + {}_I\delta^{(0)}_{(n)}(b;f) + \frac{1}{A} = 5 \\ &= 2\cdot {}_I\Delta^{(k)}_{(n)}(0;f) + 2\cdot {}_I\Delta^{(0)}_{(n)}(\infty;f) + 1. \end{aligned} $$

Theorem 3.22. *Let f be an entire function of non-zero finite order $'\rho'$ (i.e., $0 < \rho < \infty$) such that f has no zeros in $\mathbb{C}$. Then for any three positive integers n, k and p with $n > k$*

$$ {}_I\Delta^{(k)}_{(p)}(\infty;f) + {}_I\Delta^{(n)}_{(p)}(0;f) + 1 \geq {}_I\Delta^{(n)}_{(p)}(\infty;f) + {}_I\delta^{(k)}_{(p)}(a;f) + {}_I\delta^{(0)}_{(p)}(a;f) + \frac{1}{\pi\rho}, $$

where $'a'$ is any finite non zero complex number.

Proof. From the identity

$$ \frac{1}{f-a} = \frac{1}{a}\left\{\frac{f^{(k)}}{f-a} - \frac{f^{(k)}-a}{f^{(n)}}.\frac{f^{(n)}}{f-a}\right\}. \tag{21} $$

and by Milloux's theorem [3, p. 55] we get that

$$ m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{f^{(k)}-a}{f^{(n)}}\right) + S(r,f). \tag{22} $$

Now by Nevanlinna's first fundamental theorem it follows from Eq. (22),

$$ \begin{aligned} & m\left(r,\frac{1}{f-a}\right) \leq T\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) - N\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) + S(r,f) \\ \Rightarrow\ & m\left(r,\frac{1}{f-a}\right) \leq T\left(r,\frac{f^{(n)}}{f^{(k)}-a}\right) - N\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) + S(r,f) \\ \Rightarrow\ & m\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{f^{(n)}}{f^{(k)}-a}\right) - N\left(r,\frac{f^{(k)}-a}{f^{(n)}}\right) + S(r,f). \end{aligned} \tag{23} $$

Now in view of [3, p. 34] we obtain from Eq. (23) that

$$ \begin{aligned} m\left(r,\frac{1}{f-a}\right) \leq & N(r,f^{(n)}) + N\left(r,\frac{1}{f^{(k)}-a}\right) - N(r,f^{(k)}-a) \\ & - N\left(r,\frac{1}{f^{(n)}}\right) + S(r,f). \end{aligned} \tag{24} $$

Then

$$\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I(r,f^{(p)})} \leq \liminf_{r\to\infty}\left\{\frac{N(r,f^{(n)})}{I(r,f^{(p)})}-\frac{N(r,f^{(k)})}{I(r,f^{(p)})}-\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{I(r,f^{(p)})}\right\}$$
$$+\limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(p)})}\right\}.$$

Then

$$\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I(r,f^{(p)})} \leq \liminf_{r\to\infty}\frac{N(r,f^{(n)})}{I(r,f^{(p)})}-\liminf_{r\to\infty}\frac{N(r,f^{(k)})}{I(r,f^{(p)})}$$
$$-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{I(r,f^{(p)})}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(p)})}.$$

And so

$${}_I\delta_{(p)}^{(0)}(a;f)-\left(1-\frac{1}{\pi\rho}\right) \leq \{1-{}_I\Delta_{(p)}^{(n)}(\infty;f)\}-\{1-{}_I\Delta_{(p)}^{(k)}(\infty;f)\}$$
$$-\{1-{}_I\Delta_{(p)}^{(n)}(0;f)\}+\{1-{}_I\delta_{(p)}^{(k)}(a;f)\}.$$

Thus,

$${}_I\Delta_{(p)}^{(k)}(\infty;f)+{}_I\Delta_{(p)}^{(n)}(0;f)+1$$
$$\geq {}_I\Delta_{(p)}^{(n)}(\infty;f)+{}_I\delta_{(p)}^{(k)}(a;f)+{}_I\delta_{(p)}^{(0)}(a;f)+\frac{1}{\pi\rho}.$$

Thus the theorem is established. ∎

Remark 3.23. The condition that $''a'$ any finite non zero complex number in Theorem 3.22 is necessary as we see in the following examples.

Example 3.24. Let $f=\exp z$ and $a=0$. Also let $n=3, k=2$ and $p=1$. Then $N(r,f)=0$, $I(r,f)=r^2\neq 0$ and $\rho=1$.

Now,

$${}_I\delta_{(1)}^{(2)}(0;f)={}_I\delta_{(1)}^{(0)}(0;f)=1.$$

Thus

$${}_I\Delta_{(p)}^{(k)}(\infty;f)+{}_I\Delta_{(p)}^{(n)}(0;f)=1+1=2$$

and

$${}_I\Delta_{(p)}^{(n)}(\infty;f)+{}_I\delta_{(p)}^{(k)}(0;f)+{}_I\delta_{(p)}^{(0)}(0;f)=1+1+1=3,$$

which is contrary to the conclusion of Theorem 3.22.

Example 3.25. Let $f = \exp z$ and $a = \infty$. Also let $n = 3, k = 2$ and $p = 1$. Then $N(r, f) = 0, I(r, f) = r^2 \neq 0$ and $\rho = 1$.

Now,

$$_{I}\Delta_{(1)}^{(2)}(\infty; f) = \ _{I}\Delta_{(1)}^{(3)}(0; f) = \ _{I}\Delta_{(1)}^{(3)}(\infty; f) = 1$$

and

$$_{I}\delta_{(1)}^{(2)}(\infty; f) = \ _{I}\delta_{(1)}^{(0)}(\infty; f) = 1.$$

Thus

$$_{I}\Delta_{(p)}^{(k)}(\infty; f) + \ _{I}\Delta_{(p)}^{(n)}(0; f) = 1 + 1 = 2.$$

and

$$_{I}\Delta_{(p)}^{(n)}(\infty; f) + \ _{I}\delta_{(p)}^{(k)}(\infty; f) + \ _{I}\delta_{(p)}^{(0)}(\infty; f) = 1 + 1 + 1 = 3,$$

which is contrary to Theorem 3.22

Remark 3.26. The condition $\rho > 0$ in Theorem 3.22 is necessary as we see below.

Considering $f(z) = z$, $a = 0$, $n = 3, k = 2$ and $p = 1$,

we see that $N(r, f) = 0$, $I(r, f) = 2 \log\left(\frac{r^2}{2}\right) \neq 0$ and $\rho = 0$.

So,

$$_{I}\Delta_{(1)}^{(2)}(\infty; f) = \ _{I}\Delta_{(1)}^{(3)}(0; f) = \ _{I}\Delta_{(1)}^{(3)}(\infty; f) = 1$$

and

$$_{I}\delta_{(1)}^{(2)}(\infty; f) =_{I} \delta_{(1)}^{(0)}(\infty; f) = 1.$$

Thus,

$$_{I}\Delta_{(p)}^{(k)}(\infty; f) + \ _{I}\Delta_{(p)}^{(n)}(0; f) + 1 \geq \infty.$$

So we arrive at a contradiction.

Theorem 3.22 may take an alternative shape under meromorphic f as we see below.

Theorem 3.27. *If f be any meromorphic function with finite lower order λ and $f(0) = 1$, then for any finite non zero complex number $'a'$ and for any three positive integers n, k and p with $n > k$,*

$$_{I}\Delta_{(p)}^{(0)}(a; f) + \ _{I}\delta_{(p)}^{(n)}(\infty; f) + \ _{I}\delta_{(p)}^{(k)}(a; f) + \frac{1}{A} \leq \ _{I}\Delta_{(p)}^{(k)}(\infty; f) + \ _{I}\Delta_{(p)}^{(n)}(0; f) + 1,$$

where

$$A = (1 + \lambda)\left(\frac{2 + \lambda}{\lambda}\right)^{\lambda}\left\{\frac{8(1 + 2\lambda)}{(3 + 4\lambda)} + \frac{\gamma(2 + \log(8 + 8\lambda))}{\lambda \log\left(1 + \frac{1}{(1+\lambda)}\right)}\right\}$$

and

$$\gamma = \limsup_{r \to \infty} \frac{N(r, f) + N\left(r, \frac{1}{f}\right)}{T(r, f)}.$$

Proof. From Eq. (24) we get that

$$m\left(r,\frac{1}{f-a}\right)\leq N(r,f^{(n)})+N\left(r,\frac{1}{f^{(k)}-a}\right)-N(r,f^{(k)}-a)$$
$$-N\left(r,\frac{1}{f^{(n)}}\right)+S(r,f).$$

That is,

$$\limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)}$$
$$\leq\limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I\left(r,f^{(n)}\right)}-\frac{N(r,f^{(k)}-a)}{I\left(r,f^{(n)}\right)}-\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{I\left(r,f^{(n)}\right)}\right\}$$
$$+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I\left(r,f^{(n)}\right)}.$$

Thus,

$$\limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I\left(r,f^{(n)}\right)}\leq\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I\left(r,f^{(n)}\right)}-\liminf_{r\to\infty}\frac{N(r,f^{(k)}-a)}{I\left(r,f^{(n)}\right)}$$
$$-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(n)}}\right)}{I\left(r,f^{(n)}\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I\left(r,f^{(n)}\right)}.$$

Then

$${}_I\Delta^{(0)}_{(p)}(a;f)-\left(1-\frac{1}{A}\right)\leq\{1-\ {}_I\delta^{(n)}_{(p)}(\infty;f)\}-\{1-\ {}_I\Delta^{(k)}_{(p)}(\infty;f)\}$$
$$-\{1-\ {}_I\Delta^{(n)}_{(p)}(0;f)\}+\{1-\ {}_I\delta^{(k)}_{(p)}(a;f)\}.$$

Then

$${}_I\Delta^{(0)}_{(p)}(a;f)+\ {}_I\delta^{(n)}_{(p)}(\infty;f)+\ {}_I\delta^{(k)}_{(p)}(a;f)+\frac{1}{A}$$
$$\leq\ {}_I\Delta^{(k)}_{(p)}(\infty;f)+\ {}_I\Delta^{(n)}_{(p)}(0;f)+1.$$

This completes the proof. ■

Remark 3.28. In Theorem 3.27, the inequality $'\leq'$ can not be removed by $''<'$ only as is evident from the following example.

Example 3.29. Let $f(z)=\frac{1}{5z+1}$, $a=0$, $n=3,k=2$ and $p=1$. Then $m(r,f)=0$, $N(r,f)=\log r+O(1)$ and $N\left(r,\frac{1}{f}\right)=0$. So, $T(r,f)=\log r+O(1)$.

Also, $I\left(r,f\right)=\frac{5r^2}{5r+1}\neq 0, \lambda=0$ and $\gamma=1$.
Now,

$${}_I\Delta^{(0)}_{(p)}(a;f) = {}_I\delta^{(n)}_{(p)}(\infty;f) = {}_I\delta^{(k)}_{(p)}(a;f) = 1$$

and

$${}_I\Delta^{(k)}_{(p)}(\infty;f) = {}_I\Delta^{(n)}_{(p)}(0;f) = 1.$$

Hence,

$$\begin{aligned}&{}_I\Delta^{(0)}_{(p)}(a;f) + {}_I\delta^{(n)}_{(p)}(\infty;f) + {}_I\delta^{(k)}_{(p)}(a;f) + \frac{1}{A}\\ &= 3 = {}_I\Delta^{(k)}_{(p)}(\infty;f) + {}_I\Delta^{(n)}_{(p)}(0;f) + 1.\end{aligned}$$

Theorem 3.30. *If f be an entire function of non-zero finite order $'\rho'$ (i.e., $0<\rho<\infty$) such that f has no zeros in $\mathbb{C}$ with $'a'$ be a finite complex number and $'b','c'$ be two distinct non zero complex numbers, then for any two positive integers k and n,*

$${}_I\Delta^{(0)}_{(n)}(a;f) + {}_I\delta^{(k)}_{(n)}(b;f) + {}_I\delta^{(k)}_{(n)}(c;f) + \frac{1}{\pi\rho} \leq 3.$$

Proof. Since $\frac{1}{f-a}=\frac{f^{(k)}}{f-a}.\frac{1}{f^{(k)}}$ by Milloux's theorem [3, p. 55] we obtain

$$m\left(r,\frac{1}{f-a}\right) \leq m\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \tag{25}$$

Applying Nevanlinna's first fundamental theorem we get from Eq. (25) that

$$m\left(r,\frac{1}{f-a}\right) \leq T(r,f^{(k)}) - N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \tag{26}$$

Now by Nevanlinna's second fundamental theorem and Lemma 2.2 it follows from Eq. (26) that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right) \leq{}& \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right)\\ &+\bar{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f).\end{aligned} \tag{27}$$

Since $\bar{N}\left(r,\frac{1}{f^{(k)}}\right) - N\left(r,\frac{1}{f^{(k)}}\right) \leq 0$, we obtain from Eq. (27) that

$$m\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{1}{f^{(k)}-b}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) + S(r,f).$$

Then we have

$$m\left(r,\frac{1}{f-a}\right)\leq T\left(r,\frac{1}{f^{(k)}-b}\right)-m\left(r,\frac{1}{f^{(k)}-b}\right)+T\left(r,\frac{1}{f^{(k)}-c}\right)$$
$$-m\left(r,\frac{1}{f^{(k)}-c}\right)+S(r,f).$$

Thus,

$$m\left(r,\frac{1}{f-a}\right)\leq 2T(r,f^{(k)})-m\left(r,\frac{1}{f^{(k)}-b}\right)-m\left(r,\frac{1}{f^{(k)}-c}\right)+S(r,f).$$

Then we have

$$\limsup_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I(r,f^{(n)})}\leq 2\limsup_{r\to\infty}\frac{T(r,f^{(k)})}{I(r,f^{(n)})}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-b}\right)}{I(r,f^{(n)})}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-c}\right)}{I(r,f^{(n)})}.$$

Now by using Lemmas 2.2 and 2.3 we get from Eq. (22) that

$$_{I}\Delta_{(n)}^{(0)}(a;f)-\left(1-\frac{1}{\pi\rho}\right)$$
$$\leq\frac{2}{\pi\rho}-\ {}_{I}\delta_{(n)}^{(k)}(b;f)+\left(1-\frac{1}{\pi\rho}\right)-\ {}_{I}\delta_{(n)}^{(k)}(c;f)+\left(1-\frac{1}{\pi\rho}\right).$$

Then we have

$$_{I}\Delta_{(n)}^{(0)}(a;f)+\ {}_{I}\delta_{(n)}^{(k)}(b;f)+\ {}_{I}\delta_{(n)}^{(k)}(c;f)+\frac{1}{\pi\rho}\leq 3.$$

This proves the theorem. ■

Remark 3.31. The condition that $'b'$ and $'c'$ are two distinct non zero complex numbers in Theorem 3.30 is essential as is evident from the following examples.

Example 3.32. Let $f=\exp z$, $a=0$ and $b=c=\infty$. Also let $n=1$ and $k=2$. Then $N\left(r,f\right)=0$, $I\left(r,f\right)=r^{2}\neq 0$ and $\rho=1$.

So,

$$_{I}\Delta_{(n)}^{(0)}(0;f)=\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)=\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)=1.$$

Thus,

$$_{I}\Delta_{(n)}^{(0)}(0;f)+\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)+\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi}=1+1+1+\frac{1}{\pi}=3+\frac{1}{\pi}.$$

Hence

$$_{I}\Delta_{(n)}^{(0)}(0;f)+\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)+\ {}_{I}\delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi}=3+\frac{1}{\pi}<3,$$

which is contrary to Theorem 3.30.

Example 3.33. Let $f = \exp z$, $a = 0, b = 0$ and $c = \infty$. Also let $n = 1$ and $k = 2$. Then $N(r, f) = 0, I(r, f) = r^2 \neq 0$ and $\rho = 1$

So,

$$_I\Delta_{(n)}^{(0)}(0; f) = {}_I\delta_{(n)}^{(k)}(0; f) = {}_I\delta_{(n)}^{(k)}(\infty; f) = 1.$$

Thus,

$$_I\Delta_{(n)}^{(0)}(0; f) + {}_I\delta_{(n)}^{(k)}(0; f) + {}_I\delta_{(n)}^{(k)}(\infty; f) + \frac{1}{\pi} = 1 + 1 + 1 + \frac{1}{\pi} = 3 + \frac{1}{\pi}.$$

Hence

$$_I\Delta_{(n)}^{(0)}(0; f) + {}_I\delta_{(n)}^{(k)}(0; f) + {}_I\delta_{(n)}^{(k)}(\infty; f) + \frac{1}{\pi} = 3 + \frac{1}{\pi} < 3,$$

So, we arrive at a contradiction.

Example 3.34. Let $f = \exp z$, $a = 0, b = \infty$ and $c = 0$. Also let $n = 1$ and $k = 2$. Then $N(r, f) = 0, I(r, f) = r^2 \neq 0$ and $\rho = 1$.

So,

$$_I\Delta_{(n)}^{(0)}(0; f) = {}_I\delta_{(n)}^{(k)}(\infty; f) = {}_I\delta_{(n)}^{(k)}(0; f) = 1.$$

Thus,

$$_I\Delta_{(n)}^{(0)}(0; f) + {}_I\delta_{(n)}^{(k)}(\infty; f) + {}_I\delta_{(n)}^{(k)}(0; f) + \frac{1}{\pi} = 1 + 1 + 1 + \frac{1}{\pi} = 3 + \frac{1}{\pi}.$$

Hence

$$_I\Delta_{(n)}^{(0)}(0; f) + {}_I\delta_{(n)}^{(k)}(\infty; f) + {}_I\delta_{(n)}^{(k)}(0; f) + \frac{1}{\pi} = 3 + \frac{1}{\pi} < 3,$$

which is contradictory to Theorem 3.30.

Remark 3.35. The condition $\rho > 0$ is necessary in Theorem 3.30. This is evident by considering $f(z) = z$, $a = 0, b = c = \infty, n = 1$ and $k = 2$.

Then we see that $N(r, f) = 0$, $I(r, f) = 2\log\left(\frac{r^2}{2}\right) \neq 0$ and $\rho = 0$.

Now,

$$_I\Delta_{(n)}^{(0)}(0; f) = {}_I\delta_{(n)}^{(k)}(\infty; f) = {}_I\delta_{(n)}^{(k)}(0; f) = 1.$$

Hence,

$$_I\Delta_{(n)}^{(0)}(a; f) + {}_I\delta_{(n)}^{(k)}(b; f) + {}_I\delta_{(n)}^{(k)}(c; f) + \frac{1}{\pi\rho} \geq \infty,$$

which is contrary to Theorem 3.30.

Theorem 3.36. *Let f be an entire function of non-zero finite order $'\rho'$ (i.e., $0 < \rho < \infty$) such that f has no zeros in $\mathbb{C}$ satisfying $\sum_{a\neq\infty}\delta(a; f) = 1$ and $\delta(\infty; f) = 1$. Then for any two positive integers k and n*

$$_I\delta_{(n)}^{(0)}(0; f) + {}_I\Delta_{(n)}^{(k)}(\alpha; f) + \frac{1}{\pi\rho} \leq 2,$$

where $'\alpha'$ is a non zero finite complex number.

Proof. Considering the identity

$$\frac{\alpha}{f} = \frac{f^{(k)}}{f} - \frac{f^{(k)} - \alpha}{f^{(k+1)}} . \frac{f^{(k+1)}}{f},$$

we get in view of Milloux's theorem [3, p. 55] and Nevanlinna's first fundamental theorem,

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}\right) + S(r, f).$$

Then we have

$$m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}\right) + S(r, f).$$

Thus,

$$m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}\right) - N\left(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}\right) + S(r, f).$$

Then we have

$$m\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - \alpha}\right) - N\left(r, \frac{f^{(k)} - \alpha}{f^{(k+1)}}\right) + S(r, f). \tag{28}$$

Now in view of [3, p. 34] it follows from Eq. (28) that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) \leq\ & N(r, f^{(k+1)}) + N\left(r, \frac{1}{f^{(k)} - \alpha}\right) - N(r, f^{(k)} - \alpha) \\ & -N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \end{aligned}$$

Then we have

$$\begin{aligned} m\left(r, \frac{1}{f}\right) \leq\ & \left\{N\left(r, \frac{1}{f^{(k)} - \alpha}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right)\right\} \\ & +\{N(r, f^{(k+1)}) - N(r, f^{(k)})\} + S(r, f). \end{aligned}$$

Thus,

$$m\left(r, \frac{1}{f}\right) \leq N(r, \alpha; f^{(k)}) + \bar{N}(r, f) + S(r, f)$$

Since $\delta(\infty; f) = 1$, it follows that

$$\lim_{r \to \infty} \frac{\bar{N}(r, f)}{T(r, f^{(n)})} = 0$$

and

$$\begin{aligned}\lim_{r\to\infty}\frac{\bar{N}(r,f)}{I(r,f^{(n)})} &= \lim_{r\to\infty}\left[\frac{\bar{N}(r,f)}{T(r,f^{(n)})}\cdot\frac{T(r,f^{(n)})}{I(r,f^{(n)})}\right]\\ &= \lim_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f^{(n)})}\cdot\lim_{r\to\infty}\frac{T(r,f^{(n)})}{I(r,f^{(n)})}\\ &= 0\cdot\frac{1}{\pi\rho}=0.\end{aligned}$$

So from the above we get that

$$\liminf_{r\to\infty}\frac{m(r,0;f)}{I(r,f^{(n)})}\le\liminf_{r\to\infty}\frac{N(r,\alpha;f^{(k)})}{I(r,f^{(n)})}.$$

Then we have

$${}_I\delta_{(n)}^{(0)}(0;f)-\left(1-\frac{1}{\pi\rho}\right)+\ {}_I\Delta_{(n)}^{(k)}(\alpha;f)\le 1.$$

Thus,

$${}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\Delta_{(n)}^{(k)}(\alpha;f)+\frac{1}{\pi\rho}\le 2.$$

Thus the theorem is established. ■

Remark 3.37. The condition that $'\alpha'$ is a non zero finite complex number in Theorem 3.36 is essential as is evident from the following examples.

Example 3.38. Let $f=\exp z$, $k=2, n=1$ and $\alpha=0$. Then $N(r,f)=0, I(r,f)=r^2\neq 0$ and $\rho=1$.

So,

$${}_I\delta_{(1)}^{(0)}(0;f)=\ {}_I\Delta_{(1)}^{(2)}(0;f)=1.$$

Thus,

$${}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\Delta_{(n)}^{(k)}(\alpha;f)+\frac{1}{\pi\rho}=1+1+\frac{1}{\pi}=2+\frac{1}{\pi}>2,$$

which is contrary to Theorem 3.36.

Example 3.39. Let $f=\exp z$, $k=2, n=1$ and $\alpha=\infty$. Then $N(r,f)=0, I(r,f)=r^2\neq 0$ and $\rho=1$.

So,

$${}_I\delta_{(1)}^{(0)}(0;f)=\ {}_I\Delta_{(1)}^{(2)}(\infty;f)=1.$$

Thus,

$${}_I\delta_{(n)}^{(0)}(0;f)+\ {}_I\Delta_{(n)}^{(k)}(\infty;f)+\frac{1}{\pi\rho}=1+1+\frac{1}{\pi}=2+\frac{1}{\pi}>2,$$

which is contrary to the assumtion of Theorem 3.36.

Remark 3.40. In Theorem 3.36, the inequality $'\leq'$ can not be removed by $''<'$ only which can be seen from the following example.

Example 3.41. Let $f = \exp z$. Then $N(r, f) = 0$, $I(r, f) = r^2 \neq 0$ and $\rho = 1$.

Now,

$$_I\delta^{(0)}_{(n)}(0; f) = 1.$$

Now by Nevanlinna's second fundamental theorem and in view of above we get that

$$\begin{aligned} T\left(r, f^{(k)}\right) &\leq N\left(r, a; f^{(k)}\right) + S(r, f) \\ &\leq T\left(r, f^{(k)}\right) + S(r, f). \end{aligned}$$

Then we have

$$\frac{T(r, f^{(k)})}{I(r, f^{(n)})} \leq \frac{N(r, a; f^{(k)})}{I(r, f^{(n)})} + \frac{S(r, f^{(k)})}{I(r, f^{(k)})}.\frac{I(r, f^{(k)})}{I(r, f^{(n)})}$$

Thus,

$$\frac{T(r, f^{(k)})}{I(r, f^{(n)})} \leq \frac{T(r, f^{(k)})}{I(r, f^{(n)})} + \frac{S(r, f^{(k)})}{I(r, f^{(k)})}.\frac{I(r, f^{(k)})}{I(r, f^{(n)})}.$$

By Lemma 2.8 it follows from above that

$$\lim_{r\to\infty} \frac{N(r, a; f^{(k)})}{I(r, f^{(n)})} = 1.$$

Therefore,

$$_I\Delta^{(k)}_{(n)}(\alpha; f) = \left(1 - \frac{1}{\pi}\right).$$

Thus,

$$_I\delta^{(0)}_{(n)}(0; f) + {}_I\Delta^{(k)}_{(n)}(\alpha; f) + \frac{1}{\pi\rho} = 1 + \left(1 - \frac{1}{\pi}\right) + \frac{1}{\pi} = 2.$$

Remark 3.42. The condition $\rho > 0$ is essential in Theorem 3.36 as we see by taking $f(z) = z$, $k = 2, n = 1$ and $\alpha = \infty$.

Then $N(r, f) = 0$, $I(r, f) = 2\log\left(\frac{r^2}{2}\right) \neq 0$ and $\rho = 0$.

Now,

$$_I\delta^{(0)}_{(1)}(0; f) = {}_I\Delta^{(2)}_{(1)}(\infty; f) = 1.$$

Hence,

$$_{I}\delta^{(0)}_{(n)}(0;f) + {}_{I}\Delta^{(k)}_{(n)}(\alpha;f) + \frac{1}{\pi\rho} \leq 2.$$

Then we have

$$\infty \leq 2,$$

which is contradictory to Theorem 3.36.

Theorem 3.43. *Let k and n be any two positive integers and $'a'$ be a finite complex number. Then for any entire function f of finite order $'\rho'$ (i.e., $0 < \rho < \infty$) such that f has no zeros in $\mathbb{C}$ and $\sum_{a\neq\infty}\delta(a;f) = \delta(\infty;f) = 1$,*

$$_{I}\Delta^{(k)}_{(n)}(a;f) + 1 \geq {}_{I}\delta^{(k)}_{(n)}(a;f) + {}_{I}\delta^{(0)}_{(n)}(0;f) + \frac{1}{\pi\rho}.$$

Proof. Let $b \neq a$ be a finite complex number. Since

$$\frac{a-b}{f^{(k)}-a} = \frac{f}{f^{(k)}-a}\left\{\frac{f^{(k)}-b}{f} - \frac{f^{(k)}-a}{f}\right\},$$

we obtain in view of Milloux's theorem [3, p. 55] and Nevanlinna's first fundamental theorem,

$$m\left(r, \frac{a-b}{f^{(k)}-a}\right) \leq m\left(r, \frac{f}{f^{(k)}-a}\right) + S(r,f).$$

Then we have

$$m\left(r, \frac{1}{f^{(k)}-a}\right) \leq T\left(r, \frac{f}{f^{(k)}-a}\right) - N\left(r, \frac{f}{f^{(k)}-a}\right) + S(r,f).$$

Thus,

$$m\left(r, \frac{1}{f^{(k)}-a}\right) \leq T\left(r, \frac{f^{(k)}-a}{f}\right) - N\left(r, \frac{f}{f^{(k)}-a}\right) + S(r,f).$$

Then we have

$$m\left(r, \frac{1}{f^{(k)}-a}\right) \leq N\left(r, \frac{f^{(k)}-a}{f}\right) - N\left(r, \frac{f}{f^{(k)}-a}\right) + S(r,f). \tag{29}$$

In view of [3, p. 34] it follows from Eq. (29) that

$$\begin{aligned} m\left(r, \frac{1}{f^{(k)}-a}\right) \leq\ & N(r, f^{(k)}-a) + N\left(r, \frac{1}{f}\right) - N(r,f) \\ & -N\left(r, \frac{1}{f^{(k)}-a}\right) + S(r,f). \end{aligned}$$

Then we have

$$\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(n)})} \leq \liminf_{r\to\infty}\left\{\frac{N(r,f^{(k)})}{I(r,f^{(n)})}-\frac{N(r,f)}{I(r,f^{(n)})}-\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(n)})}\right\}$$
$$+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{I(r,f^{(n)})}.$$

Thus,

$$\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(n)})} \leq \liminf_{r\to\infty}\frac{N(r,f^{(k)})}{I(r,f^{(n)})}-\liminf_{r\to\infty}\frac{N(r,f)}{I(r,f^{(n)})}$$
$$-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a}\right)}{I(r,f^{(n)})}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{I(r,f^{(n)})}. \quad (30)$$

Since $\delta(\infty;f)=1$,

$$\lim_{r\to\infty}\frac{N(r,f)}{T(r,f^{(n)})}=\lim_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f^{(n)})}=0$$

and

$$\lim_{r\to\infty}\frac{N(r,f)}{I(r,f^{(n)})}=\lim_{r\to\infty}\frac{\bar{N}(r,f)}{I(r,f^{(n)})}=0$$

and so

$$\lim_{r\to\infty}\frac{N(r,f^{(k)})}{I(r,f^{(n)})}=\lim_{r\to\infty}\frac{N(r,f)}{I(r,f^{(n)})}+k\lim_{r\to\infty}\frac{\bar{N}(r,f)}{I(r,f^{(n)})}=0.$$

Thus by Lemma 2.3 it follows from Eq. (30) that

$${}_I\delta_{(n)}^{(k)}(a;f)-\left(1-\frac{1}{\pi\rho}\right)\leq {}_I\Delta_{(n)}^{(k)}(a;f)-1+\{1-{}_I\delta_{(n)}^{(0)}(0;f)\}.$$

Then we have

$${}_I\Delta_{(n)}^{(k)}(a;f)+1\geq {}_I\delta_{(n)}^{(k)}(a;f)+{}_I\delta_{(n)}^{(0)}(0;f)+\frac{1}{\pi\rho}.$$

This proves the theorem. ■

Remark 3.44. The condition that $'a'$ is a finite complex number in Theorem 3.43 is necessary as we see in the next example.

Example 3.45. Let $f=\exp z$, $k=2, n=1$ and $a=\infty$. Then $N(r,f)=0$, $I(r,f)=r^2\neq 0$ and $\rho=1$.

So,

$$_I\Delta_{(n)}^{(k)}(\infty; f) = 1$$

and

$$_I\delta_{(n)}^{(k)}(\infty; f) = \ _I\delta_{(n)}^{(0)}(0; f) = 1.$$

Thus,

$$_I\delta_{(n)}^{(k)}(\infty; f) + \ _I\delta_{(n)}^{(0)}(0; f) = 1 + 1 = 2 < 2 + \frac{1}{\pi}.$$

which is contradictory to Theorem 3.43.

Remark 3.46. The condition $\rho > 0$ is necessary in Theorem 3.43 as we see by taking $f = z$, $k = 2, n = 1$ and $a = \infty$. Then we get that $N(r, f) = 0$, $I(r, f) = 2\log\left(\frac{r^2}{2}\right) \neq 0$ and $\rho = 0$.

So,

$$_I\Delta_{(n)}^{(k)}(\infty; f) = 1$$

and

$$_I\delta_{(n)}^{(k)}(\infty; f) = \ _I\delta_{(n)}^{(0)}(0; f) = 1.$$

Hence,

$$2 \geq \infty.$$

So, we arrive at a contradiction.

Theorem 3.47. *Let f be an entire function of non-zero finite order $'\rho'$ (i.e., $0 < \rho < \infty$) such that f has no zeros in $\mathbb{C}$ $\sum_{a\neq\infty}\delta(\alpha; f) = \delta(\infty; f) = 1$ and $a_1, a_2, \ldots, a_q$ are all distinct finite complex numbers. Then for any three positive integers n, k and p with $k > n$,*

$$\sum_{i=1}^{q} \ _I\delta_{(p)}^{(k)}(a_i; f) + q \cdot \ _I\delta_{(p)}^{(n)}(0; f)$$
$$\leq \sum_{i=1}^{q} \ _I\Delta_{(p)}^{(k)}(a_i; f) + \ _I\Delta_{(p)}^{(n)}(0; f) + (q-1)\left(1 - \frac{1}{\pi\rho}\right).$$

Proof. Let $F = \sum_{i=1}^{q} \frac{1}{f^{(k)} - a_i}$ for $i = 1, 2, \ldots, q$. Then we get that

$$\sum_{i=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_i}\right) \leq m(r, F) + O(1).$$

Then we have

$$\sum_{i=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_i}\right) \leq m\left(r, \sum_{i=1}^{q} \frac{f^{(n)}}{f^{(k)} - a_i}\right) + m\left(r, \frac{1}{f^{(n)}}\right) + O(1).$$

Thus,

$$\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)\leq m\left(r,\frac{1}{f^{(n)}}\right)+\sum_{i=1}^{q} m(r,\frac{f^{(n)}}{f^{(k)}-a_i})+O(1).$$

Then we have

$$\begin{aligned}&\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)\\ &\leq m\left(r,\frac{1}{f^{(n)}}\right)+\sum_{i=1}^{q}\left\{T\left(r,\frac{f^{(n)}}{f^{(k)}-a_i}\right)-N\left(r,\frac{f^{(n)}}{f^{(k)}-a_i}\right)\right\}+O(1).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)\\ &\leq m\left(r,\frac{1}{f^{(n)}}\right)+\sum_{i=1}^{q}\left\{T\left(r,\frac{f^{(k)}-a_i}{f^{(n)}}\right)-N\left(r,\frac{f^{(n)}}{f^{(k)}-a_i}\right)\right\}+O(1).\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)\\ &\leq m\left(r,\frac{1}{f^{(n)}}\right)+\sum_{i=1}^{q}\left\{N\left(r,\frac{f^{(k)}-a_i}{f^{(n)}}\right)-N\left(r,\frac{f^{(n)}}{f^{(k)}-a_i}\right)\right\}+S(r,f).\end{aligned}\tag{31}$$

In view of [3, p. 34] we obtain from Eq. (31) that

$$\begin{aligned}&\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)\\ &\leq m\left(r,\frac{1}{f^{(n)}}\right)+\sum_{i=1}^{q}\left[N(r,f^{(k)}-a_i)+N\left(r,\frac{1}{f^{(n)}}\right)\right.\\ &\quad\left.-N(r,f^{(n)})-N\left(r,\frac{1}{f^{(k)}-a_i}\right)\right]+S(r,f)\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)&\leq m\left(r,\frac{1}{f^{(n)}}\right)+qN(r,f^{(k)})+qN\left(r,\frac{1}{f^{(n)}}\right)\\ &\quad -qN(r,f^{(n)})-\sum_{i=1}^{q} N\left(r,\frac{1}{f^{(k)}-a_i}\right)+S(r,f).\end{aligned}$$

Then

$$\liminf_{r\to\infty}\frac{\sum\limits_{i=1}^{q} m\left(r,\frac{1}{f^{(k)}-a_i}\right)}{T(r,f^{(p)})}$$

$$\leq \liminf_{r\to\infty}\left\{\frac{qN(r,f^{(k)})}{T(r,f^{(p)})}-\frac{qN(r,f^{(n)})}{T(r,f^{(p)})}-\frac{\sum\limits_{i=1}^{q} N\left(r,\frac{1}{f^{(k)}-a_i}\right)}{T(r,f^{(p)})}\right\}$$

$$+\limsup_{r\to\infty}\left\{\frac{m\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f^{(p)})}+\frac{qN\left(r,\frac{1}{f^{(n)}}\right)}{T(r,f^{(p)})}\right\}.$$

Then we have

$$\sum_{i=1}^{q}\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f^{(k)}-a_i}\right)}{T(r,f^{(p)})}$$

$$\leq q\liminf_{r\to\infty}\frac{N(r,f^{(k)})}{T(r,f^{(p)})}-q\liminf_{r\to\infty}\frac{N(r,f^{(n)})}{T(r,f^{(p)})}-\sum_{i=1}^{q}\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f^{(k)}-a_i}\right)}{T\left(r,f^{(p)}\right)} \qquad (32)$$

$$+\limsup_{r\to\infty}\frac{m(r,\frac{1}{f^{(n)}})}{T(r,f^{(p)})}+q\limsup_{r\to\infty}\frac{N(r,\frac{1}{f^{(n)}})}{T(r,f^{(p)})}.$$

Since $\delta(\infty;f)=1$,

$$\lim_{r\to\infty}\frac{N(r,f)}{T(r,f)}=\lim_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)}=0.$$

and

$$\lim_{r\to\infty}\frac{N(r,f)}{I(r,f^{(n)})}=\lim_{r\to\infty}\frac{\bar{N}(r,f)}{I(r,f^{(n)})}=0.$$

So

$$\lim_{r\to\infty}\frac{N(r,f^{(k)})}{I(r,f^{(p)})}=\lim_{r\to\infty}\frac{N(r,f)}{I(r,f^{(p)})}+k\lim_{r\to\infty}\frac{\bar{N}(r,f)}{I(r,f^{(p)})}=0.$$

Similarly

$$\lim_{r\to\infty}\frac{N(r,f^{(n)})}{I(r,f^{(p)})}=0.$$

Now by Lemma 2.9 and Lemma 2.10 it follows from Eq. (27) that

$$\sum_{i=1}^{q}\ {}_I\delta_{(p)}^{(k)}(a_i;f)-q\left(1-\frac{1}{\pi\rho}\right)$$

$$\leq\sum_{i=1}^{q}\ {}_I\Delta_{(p)}^{(k)}(a_i;f)-q+\ {}_I\Delta_{(p)}^{(n)}(0;f)-\left(1-\frac{1}{\pi\rho}\right)+q\{1-\ {}_I\delta_{(p)}^{(n)}(0;f)\}.$$

Then we have

$$\sum_{i=1}^{q} {}_{I}\delta_{(p)}^{(k)}(a_i; f) + q \cdot {}_{I}\delta_{(p)}^{(n)}(0; f)$$
$$\leq \sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) + {}_{I}\Delta_{(p)}^{(n)}(0; f) + (q-1)\left(1 - \frac{1}{\pi\rho}\right).$$

Thus the theorem is established. ■

Remark 3.48. The condition that $a_1, a_2, \ldots, a_q$ are all distinct finite complex numbers in Theorem 3.47 is necessary as is evident from the following three examples.

Example 3.49. Let $f = \exp z$, $q = 2, a_1 = a_2 = 0$, $k = 3, n = 2$, and $p = 1$. Then $N(r, f) = 0$, $I(r, f) = r^2 \neq 0$ and $\rho = 1$.

Now

$$\sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) = \sum_{i=1}^{2} {}_{I}\Delta_{(1)}^{(3)}(a_i; f) = {}_{I}\Delta_{(1)}^{(3)}(a_1; f) + {}_{I}\Delta_{(1)}^{(3)}(a_2; f)$$
$$= {}_{I}\Delta_{(1)}^{(3)}(0; f) + {}_{I}\Delta_{(1)}^{(3)}(0; f) = \left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1}{\pi}\right) = 2 - \frac{2}{\pi},$$
$${}_{I}\Delta_{(p)}^{(n)}(0; f) = {}_{I}\Delta_{(1)}^{(2)}(0; f) = 1,$$
$$\sum_{i=1}^{q} {}_{I}\delta_{(p)}^{(k)}(a_i; f) = \sum_{i=1}^{2} {}_{I}\delta_{(1)}^{(3)}(a_i; f) = {}_{I}\delta_{(1)}^{(3)}(a_2; f) + {}_{I}\delta_{(1)}^{(3)}(a_2; f)$$
$$= {}_{I}\delta_{(1)}^{(3)}(0; f) + {}_{I}\delta_{(1)}^{(3)}(0; f) = \left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1}{\pi}\right) = 2 - \frac{2}{\pi},$$

and

$$q \cdot {}_{I}\delta_{(p)}^{(n)}(0; f) = 2 \cdot {}_{I}\delta_{(1)}^{(2)}(0; f) = 2.1 = 2.$$

Therefore

$$\sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) + {}_{I}\Delta_{(p)}^{(n)}(0; f) + (q-1)\left(1 - \frac{1}{\pi\rho}\right)$$
$$= 2 - \frac{2}{\pi} + 1 + 1 - \frac{1}{\pi} = 4 - \frac{3}{\pi}$$

and

$$\sum_{i=1}^{q} {}_{I}\delta_{(p)}^{(k)}(a_i; f) + q \cdot {}_{I}\delta_{(p)}^{(n)}(0; f) = \left(2 - \frac{2}{\pi}\right) + 2 \cdot 1 = 4 - \frac{2}{\pi},$$

which contradicts Theorem 3.47.

Example 3.50. Let $f = \exp z$, $q = 2, a_1 = 0$ and $a_2 = \infty$. Also let $k = 3, n = 2$ and $p = 1$. Then $N(r, f) = 0, I(r, f) = r^2 \neq 0$ and $\rho = 1$.

Now

$$\begin{aligned}\sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) &= \sum_{i=1}^{2} {}_{I}\Delta_{(1)}^{(3)}(a_i; f) = {}_{I}\Delta_{(1)}^{(3)}(a_1; f) + {}_{I}\Delta_{(1)}^{(3)}(a_2; f) \\ &= {}_{I}\Delta_{(1)}^{(3)}(0; f) + {}_{I}\Delta_{(1)}^{(3)}(\infty; f) \\ &= \left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1}{\pi}\right) = 2 - \frac{2}{\pi}, \\ {}_{I}\Delta_{(p)}^{(n)}(0; f) &= {}_{I}\Delta_{(1)}^{(2)}(0; f) = 1, \\ \sum_{i=1}^{q} {}_{I}\delta_{(p)}^{(k)}(a_i; f) &= \sum_{i=1}^{2} {}_{I}\delta_{(1)}^{(3)}(a_i; f) = {}_{I}\delta_{(1)}^{(3)}(a_1; f) + {}_{I}\delta_{(1)}^{(3)}(a_2; f) \\ &= {}_{I}\delta_{(1)}^{(3)}(0; f) + {}_{I}\delta_{(1)}^{(3)}(\infty; f) \\ &= \left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1}{\pi}\right) = 2 - \frac{2}{\pi},\end{aligned}$$

and

$$q \cdot {}_{I}\delta_{(p)}^{(n)}(0; f) = 2 \cdot {}_{I}\delta_{(1)}^{(2)}(0; f) = 2 \cdot 1 = 2.$$

Therefore

$$\begin{aligned}&\sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) + {}_{I}\Delta_{(p)}^{(n)}(0; f) + (q-1)\left(1 - \frac{1}{\pi\rho}\right) \\ &= 2 - \frac{2}{\pi} + 1 + 1 - \frac{1}{\pi} = 4 - \frac{3}{\pi}\end{aligned}$$

and

$$\sum_{i=1I}^{q} \delta_{(p)}^{(k)}(a_i; f) + q \cdot {}_{I}\delta_{(p)}^{(n)}(0; f) = 2 - \frac{2}{\pi} + 2 = 4 - \frac{2}{\pi}$$

which is contrary to Theorem 3.47.

Example 3.51. Let $f = \exp z$, $q = 2, a_1 = \infty$ and $a_2 = 0$. Also let $k = 3, n = 2$ and $p = 1$. Then $N(r, f) = 0, I(r, f) = r^2 \neq 0$ and $\rho = 1$.

Now

$$\begin{aligned}\sum_{i=1}^{q} {}_{I}\Delta_{(p)}^{(k)}(a_i; f) &= \sum_{i=1}^{2} {}_{I}\Delta_{(1)}^{(3)}(a_i; f) = {}_{I}\Delta_{(1)}^{(3)}(a_1; f) + {}_{I}\Delta_{(1)}^{(3)}(a_2; f) \\ &= {}_{I}\Delta_{(1)}^{(3)}(\infty; f) + {}_{I}\Delta_{(1)}^{(3)}(0; f) \\ &= \left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1}{\pi}\right) = 2 - \frac{2}{\pi},\end{aligned}$$

$$
\begin{aligned}
{}_I\Delta_{(p)}^{(n)}(0;f) &= {}_I\Delta_{(1)}^{(2)}(0;f) = 1, \\
\sum_{i=1}^{q} {}_I\delta_{(p)}^{(k)}(a_i;f) &= \sum_{i=1}^{2} {}_I\delta_{(1)}^{(3)}(a_i;f) = {}_I\delta_{(1)}^{(3)}(a_1;f) + {}_I\delta_{(1)}^{(3)}(a_2;f) \\
&= {}_I\delta_{(1)}^{(3)}(\infty;f) + {}_I\delta_{(1)}^{(3)}(0;f) \\
&= \left(1-\frac{1}{\pi}\right) + \left(1-\frac{1}{\pi}\right) = 2-\frac{2}{\pi},
\end{aligned}
$$

and

$$q \cdot {}_R\delta_{(p)}^{(n)}(0;f) = 2 \cdot {}_I\delta_{(1)}^{(2)}(0;f) = 2.1 = 2.$$

Therefore

$$\sum_{i=1}^{q} {}_I\Delta_{(p)}^{(k)}(a_i;f) + {}_I\Delta_{(p)}^{(n)}(0;f) = 2-\frac{2}{\pi}+1+(2-1)\left(1-\frac{1}{\pi}\right) = 4-\frac{3}{\pi}$$

and

$$\sum_{i=1}^{q} {}_I\delta_{(p)}^{(k)}(a_i;f) + q \cdot {}_I\delta_{(p)}^{(n)}(0;f) = 2-\frac{2}{\pi}+2 = 4-\frac{2}{\pi},$$

which contradicts Theorem 3.47.

Remark 3.52. The condition that $\rho > 0$ in Theorem 3.47 is necessary as is evident by considering $f = z$, $q = 2, a_1 = \infty$, $a_2 = 0, k = 3, n = 2$ and $p = 1$.

Here we see that $N(r,f) = 0, I(r,f) = 2\log\left(\frac{r^2}{2}\right) \neq 0$ and $\rho = 0$.

Thus

$$\sum_{i=1}^{q} {}_I\delta_{(p)}^{(k)}(a_i;f) + q \cdot {}_I\delta_{(p)}^{(n)}(0;f) + \infty \leq 0,$$

which is a contradiction.

4. Future Prospect

In the line of the works as carried out in the paper, one may think of relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them. As a consequence, the derivation of relevant results in this field may be an active area of research to the future workers of this branch.

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