Southeast Asian Bulletin of Mathematics (2023) 47: 367–380

Southeast Asian
Bulletin of
Mathematics

On Fractional Integrodifferential Equation Involving Caputo-Hadamard Derivative with Hadamard Fractional Integral Boundary Conditions

Subramanian Muthaiah
Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore, India
Email: subramanianmcbe@gmail.com

Manigandan Murugesan
Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, India
Email: yogimani22@gmail.com

Sivasamy Ramasamy
Department of Science and Humanities, M. Kumarasamy College of Engineering, Karur, India
Email: sivasamymaths@gmail.com

Nandha Gopal Thangaraj
Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, India
Email: nandhu792002@yahoo.co.in

Received 13 July 2019
Accepted 25 August 2021

Communicated by Peichu Hu

AMS Mathematics Subject Classification(2020): 26A33, 34A08, 34B15

Abstract. The primary objective of this paper is to validate the existence and uniqueness of solutions for boundary value problem (BVP) of Caputo-Hadamard fractional integro-differential equation (CHFIDE) supplemented with nonlocal fractional integral boundary conditions. The convergence of the problem has been validated with suitable examples.

Keywords: Caputo-Hadamard fractional derivative (CHFD); Integrodifferential equation; Hadamard fractional integral (HFI); Integral boundary conditions; Existence; Fixed point.

1. Introduction

Ever since the evolution of fractional differential equations (FDEs) there have been intense efforts to raise the theoretical and application aspects in various fields like physics, chemistry, biology, engineering sciences etc. Our researchers have indeed given numerous contributions in these areas, for details we refer the reader to the papers [11, 12, 13, 19, 20] and references therein. The BVP has been established in recent years with a strong connection to the development of classical calculus. Moreover, some analytical results and applications of fractional calculus have been outlined in their historical context, for instance, see [3, 18, 21, 24, 16, 7, 23, 6, 5] and the references cited therein. It has been seen that the greater part of the work on the point is concerned about Riemann-Liouville (RL) or Caputo type FDEs. Other than these fractional derivatives, another sort of fractional derivatives established in the literature is the fractional derivative because Hadamard made it known in 1892 [9], which contrasts from the previously mentioned derivatives as in the kernel of the integral in the delineation of Hadamard fractional derivative (HFD) comprises logarithmic function of arbitrary exponent. A point by point depiction of HFD and integral have been discovered in [2, 26, 25]. Recently, much interest has been created in establishing the existence of solutions for fractional BVP with multipoint, HFI, RLFI, Erdelyi-Kober fractional integral conditions. Ma et.al in [14] discussed a Lyapunovtype inequality with the HFD. Similarly, Wang et al. in [27] studied nonlocal Hadamard fractional BVP with Hadamard integral and discrete boundary conditions. Recently, in [1], the author discussed FDEs involving HFD with three-point boundary conditions. In 2012, Jarad et al. modified the fractional derivative of Hadamard type into a more suitable one with physically interpretable initial conditions comparable to the singles in the Caputo setting and named it fractional derivative Caputo-Hadamard type. Refer to [10] for defining the properties of the modified derivative. We refer the reader to the articles of [22, 15, 17, 28, 8, 4] for certain concepts in the theory of Caputo-Hadamard FDEs, and the references cited therein. In this paper, we introduce nonlocal integral boundary conditions on the CHFIDE of the form:

$$\begin{aligned} &{}^{C}\mathscr{D}^{\varrho} z(\iota) = \xi g(\iota, z(\iota)) + \zeta\, {}^{H}\mathscr{I}^{\vartheta} h(\iota, z(\iota)), \qquad \iota \in [1, T], \\ &z(1) = 0, \quad z^{'}(1) = 0, \quad z(T) = \omega\, {}^{H}\mathscr{I}^{\varsigma} z(\varphi), \ 1 < \varphi < T, \end{aligned} \tag{1}$$

where ${}^{C}\mathscr{D}^{\varrho}$ denotes the CHFD of order $2 < \varrho \leq 3$, ${}^{H}\mathscr{I}^{\vartheta}$, ${}^{H}\mathscr{I}^{\varsigma}$ denotes the HFIs of order $1 < \vartheta, \varsigma < 2$, and g, h are given continuous functions and ω is positive real constant. The rest of the paper is organised as follows: The Section 2 is devoted to some fundamental concepts of fractional calculus with basic lemmas related to the given problem. The existence and uniqueness results, based on Leray-Schauder nonlinear alternative (LSNA), Krasnoselskii's fixed point theorem (KFPT), Banach fixed point theorem (BFPT) are obtained in Section 3. The validation of the results is done by providing examples in Section 4.

2. Preliminaries

We begin with some basic definitions, properties and lemmas with results [8, 10].

Definition 2.1. *The left and right HFIs of order $\varrho > 0$ are respectively defined by*

$$({}^H\mathscr{I}^{\varrho}_{b+}g)(\iota) = \frac{1}{\Gamma(\varrho)}\int_b^{\iota}\Big(\log\frac{\iota}{\sigma}\Big)^{\varrho-1}g(\sigma)\frac{d\sigma}{\sigma}, \quad b<\iota<c$$

and

$$({}^H\mathscr{I}^{\varrho}_{c-}g)(\iota) = \frac{1}{\Gamma(\varrho)}\int_{\iota}^{c}\Big(\log\frac{\sigma}{\iota}\Big)^{\varrho-1}g(\sigma)\frac{d\sigma}{\sigma}, \quad b<\iota<c.$$

Definition 2.2. *The left and right-sided HFDs of order ϱ with $\mathbb{R}(\varrho)\geq 0$ on (b,c) and $b<\iota<c$ are defined by*

$$({}^H\mathscr{D}^{\varrho}_{b+}g)(\iota) = \Big(\iota\frac{d}{d\iota}\Big)^n\frac{1}{\Gamma(n-\varrho)}\int_b^{\iota}\Big(\log\frac{\iota}{\sigma}\Big)^{n-\varrho-1}\frac{g(\sigma)}{\sigma}d\sigma,$$

and

$$({}^H\mathscr{D}^{\varrho}_{c-}g)(\iota) = \Big(-\iota\frac{d}{d\iota}\Big)^n\frac{1}{\Gamma(n-\varrho)}\int_b^{\iota}\Big(\log\frac{\sigma}{\iota}\Big)^{n-\varrho-1}\frac{g(\sigma)}{\sigma}d\sigma,$$

where $n=[\mathbb{R}(\varrho)+1]$.

Lemma 2.3. *If $\mathbb{R}(\varrho)>0$, $\mathbb{R}(\varsigma)>0$ and $0<b<c<\infty$, then we have*

$$\Big({}^H\mathscr{I}^{\varrho}_{b+}\Big(\log\frac{\sigma}{b}\Big)^{\varsigma-1}\Big)(\iota) = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)}\Big(\log\frac{\iota}{b}\Big)^{\varsigma+\varrho-1}$$

and

$$\Big({}^H\mathscr{I}^{\varrho}_{c-}\Big(\log\frac{c}{\sigma}\Big)^{\varsigma-1}\Big)(\iota) = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)}\Big(\log\frac{c}{\iota}\Big)^{\varsigma+\varrho-1}.$$

Lemma 2.4. *Let $\varrho,\varsigma\ni\mathbb{R}(\varrho)>\mathbb{R}(\varsigma)>0$. If $0<b<c<\infty$ and $1\leq p<\infty$, then for $g\in\mathscr{L}^p(b,c)$,*

$${}^H\mathscr{I}^{\varrho}_{b+}\,{}^H\mathscr{I}^{\varsigma}_{b+}g = {}^H\mathscr{I}^{\varrho+\varsigma}_{b+}g$$

and

$$ {}^{H}\mathscr{I}^{\varrho}_{c-}\,{}^{H}\mathscr{I}^{\varsigma}_{c-}g = {}^{H}\mathscr{I}^{\varrho+\varsigma}_{c-}g. $$

Definition 2.5. *Let $0 < b < c < \infty$, $\mathbb{R}(\varrho) \geq 0$, $n = [\mathbb{R}(\varrho) + 1]$. The left and right CHFDs of order ϱ are respectively defined by*

$$ ({}^{C}\mathscr{D}^{\varrho}_{b+}g)(\iota) = \mathscr{D}^{\varrho}_{b+}\Big[g(\sigma) - \sum_{k=0}^{n-1}\frac{\delta^k g(b)}{k!}\Big(\log\frac{\sigma}{b}\Big)^k\Big](\iota), $$

and

$$ ({}^{C}\mathscr{D}^{\varrho}_{c-}g)(\iota) = \mathscr{D}^{\varrho}_{c-}\Big[g(\sigma) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k g(c)}{k!}\Big(\log\frac{c}{\sigma}\Big)^k\Big](\iota). $$

Lemma 2.6. *Let $\mathbb{R}(\varrho) > 0$, $n = [\mathbb{R}(\varrho) + 1]$ and $g \in \mathscr{C}[b, c]$. If $\mathbb{R}(\varrho) \neq 0$ or $\varrho \in \mathbb{N}$, then*

$$ ({}^{C}\mathscr{D}^{\varrho}_{b+}\mathscr{I}^{\varrho}_{b+}g)(\iota) = g(\iota), \quad ({}^{C}\mathscr{D}^{\varrho}_{c-}\mathscr{I}^{\varrho}_{c-}g)(\iota) = g(\iota). $$

Lemma 2.7. *Let $g \in \mathscr{AC}^n_\delta[b, c]$ or $\mathscr{C}^n_\delta[b, c]$ and $\varrho > 0$. Then*

$$ \mathscr{I}^{\varrho}_{b+}({}^{C}\mathscr{D}^{\varrho}_{b+}g)(\iota) = \left[g(\iota) - \sum_{k=0}^{n-1}\frac{\delta^k g(b)}{k!}\Big(\log\frac{\iota}{b}\Big)^k\right], $$

and

$$ \mathscr{I}^{\varrho}_{c-}({}^{C}\mathscr{D}^{\varrho}_{c-}g)(\iota) = \left[g(\iota) - \sum_{k=0}^{n-1}\frac{\delta^k g(c)}{k!}\Big(\log\frac{c}{\iota}\Big)^k\right]. $$

Lemma 2.8. *For any $\hat{f} \in \mathscr{C}([1, T], \mathbb{R})$, $z \in \mathscr{C}([1, T], \mathbb{R})$, the function z is the solution of the problem*

$$ \begin{aligned} {}^{C}\mathscr{D}^{\varrho}z(\iota) &= \hat{f}(\iota), \quad \iota \in [1, T], \\ z(1) = 0, \quad z'(1) &= 0, \quad z(T) = \omega\,{}^{H}\mathscr{I}^{\varsigma}z(\varphi), \quad 1 < \varphi < T, \end{aligned} \tag{2} $$

if and only if

$$ z(\iota) = {}^{H}\mathscr{I}^{\varrho}\hat{f}(\iota) + \frac{(\log\iota)^2}{\Lambda}\left[\omega\,{}^{H}\mathscr{I}^{\varrho+\varsigma}\hat{f}(\varphi) - {}^{H}\mathscr{I}^{\varrho}\hat{f}(T)\right], \tag{3} $$

where

$$\Lambda = (\log T)^2 - \frac{2\omega(\log\varphi)^{\varsigma+2}}{\Gamma(\varsigma+3)}, \tag{4}$$

Proof. Applying the operator ${}^{H}\mathscr{I}^{\varrho}$ on the linear FDE in (2), we obtain

$$z(\iota) = {}^{H}\mathscr{I}^{\varrho}\hat{f}(\iota) + a_0 + a_1\log\iota + a_2(\log\iota)^2, \tag{5}$$

where a_0,a_1 and $a_2 \in \mathbb{R}$, are arbitrary unknown constants. Using the boundary conditions (2) in (5), we get $a_0, a_1 = 0$,

$$a_2 = \frac{1}{\Lambda}\Bigg[\omega^{H}\mathscr{I}^{\varrho+\varsigma}\hat{f}(\varphi) - {}^{H}\mathscr{I}^{\varrho}\hat{f}(T)\Bigg].$$

Substituting the value of a_2 in (5), we get the solution (3). ■

Theorem 2.9. *(LSNA) Let $\mathscr{G}:\mathscr{Q}\to\mathscr{Q}$ be completely continuous operator (i.e., a map restricted to any bounded set in $\mathscr{Q}$ is compact). Let*

$$\mathscr{Q}(\mathscr{G}) = \{z\in\mathscr{Q} : z = \nu\mathscr{G}(z) \text{ for some } 0<\nu<1\}.$$

Then either the set $\mathscr{Q}(\mathscr{G})$ is unbounded, or $\mathscr{G}$ has atleast one fixed point.

Theorem 2.10. *(KPFT) Let $\mathscr{W}$ be a closed, bounded, convex and nonempty subset of a Banach space $\mathscr{Y}$. Let $\mathscr{U}$, $\mathscr{V}$ be the operators $\ni$. Then the following statements hold:*

(i) *$\mathscr{U}x + \mathscr{V}y \in \mathscr{W}$ whenever $x, y \in \mathscr{W}$;*
(ii) *$\mathscr{U}$ is a compact and continuous;*
(iii) *$\mathscr{V}$ is a contraction mapping. Then there exists $\iota\in\mathscr{W} \ni \iota = \mathscr{U}\iota + \mathscr{V}\iota$.*

Theorem 2.11. *(BFPT) Let $\mathscr{Q}$ be a Banach space, $\mathscr{F}\subset\mathscr{Q}$ be closed and $\mathscr{G}:\mathscr{F}\to\mathscr{F}$ a strict contraction, i.e., $|\mathscr{G}x - \mathscr{G}y| \le \kappa|x-y|$ for some $\kappa\in(0,1)$ and all $x,y\in\mathscr{F}$ Then $\mathscr{G}$ has a unique fixed point.*

3. Main Results

We define space $\mathscr{P} = \mathscr{C}([1,T],\mathbb{R})$ endowed with the norm $\|z\| = \sup\{|z(\iota)|, \iota\in[1,T]\}$. Obviously $(\mathscr{P}, \|\cdot\|)$ is a Banach space. In view of Lemma 2.8, we interpret an operator $\mathscr{T}:\mathscr{P}\to\mathscr{P}$ as

$$\begin{aligned}\mathscr{T}(z)(\iota) = {}&{}^{H}\mathscr{I}^{\varrho}\phi_z(\sigma,z(\sigma))(\iota) + \frac{(\log\iota)^2}{\Lambda}\Bigg[\omega^{H}\mathscr{I}^{\varrho+\varsigma}\phi_z(\sigma,z(\sigma))(\varphi)\\ &-{}^{H}\mathscr{I}^{\varrho}\phi_z(\sigma,z(\sigma))(T)\Bigg],\end{aligned}$$

where

$$\phi_z(\sigma, z(\sigma)) = \xi g(\sigma, z(\sigma)) + \zeta^H \mathscr{I}^{\vartheta} h(\sigma, z(\sigma)),$$

$$\begin{aligned}\mathscr{T}(z)(\iota) = {} & \xi^H \mathscr{I}^{\varrho} g(\sigma, z(\sigma))(\iota) + \zeta^H \mathscr{I}^{\varrho+\vartheta} h(\sigma, z(\sigma))(\iota) \\ & + \frac{(\log \iota)^2}{\Lambda}\Bigg[\xi\omega^H \mathscr{I}^{\varrho+\varsigma} g(\sigma, z(\sigma))(\varphi) + \zeta\omega^H \mathscr{I}^{\varrho+\varsigma+\vartheta} h(\sigma, z(\sigma))(\varphi) \\ & - \xi^H \mathscr{I}^{\varrho} g(\sigma, z(\sigma))(T) - \zeta^H \mathscr{I}^{\varrho+\vartheta} h(\sigma, z(\sigma))(T)\Bigg]. \end{aligned} \tag{6}$$

In the sequel, we use the following expressions:

$$\begin{aligned}
{}^H\mathscr{I}^{\varrho} g(\sigma, z(\sigma))(\iota) &= \frac{1}{\Gamma(\varrho)} \int_1^{\iota} \left(\log \frac{\iota}{\sigma}\right)^{\varrho-1} g(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}, \\
{}^H\mathscr{I}^{\varrho+\vartheta} h(\sigma, z(\sigma))(\iota) &= \frac{1}{\Gamma(\varrho+\vartheta)} \int_1^{\iota} \left(\log \frac{\iota}{\sigma}\right)^{\varrho+\vartheta-1} h(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}, \\
{}^H\mathscr{I}^{\varrho+\varsigma} g(\sigma, z(\sigma))(\varphi) &= \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\varphi} \left(\log \frac{\varphi}{\sigma}\right)^{\varrho+\varsigma-1} g(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}, \\
{}^H\mathscr{I}^{\varrho+\varsigma+\vartheta} h(\sigma, z(\sigma))(\varphi) &= \frac{1}{\Gamma(\varrho+\varsigma+\vartheta)} \int_1^{\varphi} \left(\log \frac{\varphi}{\sigma}\right)^{\varrho+\varsigma+\vartheta-1} h(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}, \\
{}^H\mathscr{I}^{\varrho} g(\sigma, z(\sigma))(T) &= \frac{1}{\Gamma(\varrho)} \int_1^{T} \left(\log \frac{T}{\sigma}\right)^{\varrho-1} g(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}, \\
{}^H\mathscr{I}^{\varrho+\vartheta} h(\sigma, z(\sigma))(T) &= \frac{1}{\Gamma(\varrho+\vartheta)} \int_1^{T} \left(\log \frac{T}{\sigma}\right)^{\varrho+\vartheta-1} h(\sigma, z(\sigma)) \frac{d\sigma}{\sigma}.
\end{aligned}$$

Suitable for computation, we represent:

$$\begin{aligned}\Delta = {} & \left(\frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}\right)\left[\frac{(\log T)^2}{\Lambda} + 1\right] \\ & + \left[\frac{\omega(\log T)^2}{\Lambda}\left(\frac{\xi(\log \varphi)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)} + \frac{\zeta(\log \varphi)^{\varrho+\varsigma+\vartheta}}{\Gamma(\varrho+\varsigma+\vartheta+1)}\right)\right],\end{aligned} \tag{7}$$

$$\eta_1 = \left(\frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)}\right)\left[\frac{(\log T)^2}{\Lambda} + 1\right] + \frac{\omega(\log T)^2}{\Lambda}\left[\frac{\xi(\log \varphi)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)}\right], \tag{8}$$

$$\eta_2 = \left(\frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}\right)\left[\frac{(\log T)^2}{\Lambda} + 1\right] + \frac{\omega(\log T)^2}{\Lambda}\left[\frac{\zeta(\log \varphi)^{\varrho+\varsigma+\vartheta}}{\Gamma(\varrho+\varsigma+\vartheta+1)}\right]. \tag{9}$$

First, we prove the existence result is based on LSNA.

Theorem 3.1. *Let us speculate that $g, h : [1, T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and the following conditions hold:*

($\mathscr{E}_1$) *There exists a function* $q_1, q_2 \in \mathscr{C}([1,T], \mathbb{R}^+)$, *and* $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *nondecreasing* $\ni$ $|g(\iota, z)| \leq q_1(\iota)\psi_1(\|z\|)$, $|h(\iota, z)| \leq q_2(\iota)\psi_2(\|z\|)$ *for each* $(\iota, z) \in [1,T] \times \mathbb{R}$*;*

($\mathscr{E}_2$) *There exists a number* $\mathscr{L} > 0$ *such that*

$$\frac{\mathscr{L}}{\|q_1\|\psi_1(\mathscr{L}) + \|q_2\|\psi_2(\mathscr{L})} > \xi^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(T) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(T) + \mathscr{G},$$

$$\begin{aligned}\mathscr{G} = \frac{(\log T)^2}{\Lambda}\Bigg[&\xi\omega^H\mathscr{I}^{\varrho+\varsigma}|g(\sigma, z(\sigma))|(\varphi) + \zeta\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma, z(\sigma))|(\varphi) \\ &+\xi^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(T) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(T)\Bigg].\end{aligned}$$

Then there exists at least one solution for BVP (1) on $[1,T]$.

Proof. To begin with, the operator $\mathscr{T} : \mathscr{P} \to \mathscr{P}$ is described by (6). Next, we demonstrate that $\mathscr{T}$ maps bounded sets into bounded sets in $\mathscr{C}([1,T], \mathbb{R})$. For a positive number θ, let $\mathscr{B}_\theta = \{z \in \mathscr{C}([1,T], \mathbb{R}) : \|z\| \leq \theta\}$ be a bounded set in $\mathscr{C}([1,T], \mathbb{R})$. Then, for each $z \in \mathscr{B}_\theta$, we have

$$\begin{aligned}|(\mathscr{T}z)(\iota)| \leq\ & \xi^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(\iota) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(\iota) \\ &+\frac{(\log \iota)^2}{\Lambda}\Bigg[\xi\omega^H\mathscr{I}^{\varrho+\varsigma}|g(\sigma, z(\sigma))|(\varphi) + \zeta\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma, z(\sigma))|(\varphi) \\ &+\xi^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(T) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(T)\Bigg] \\ \leq\ & \psi_1(\|z\|)\xi\Bigg\{{}^H\mathscr{I}^{\varrho}q_1(\sigma)(T) + \frac{(\log T)^2}{\Lambda}\Bigg[\omega^H\mathscr{I}^{\varrho+\varsigma}q_1(\sigma)(\varphi) \\ &+{}^H\mathscr{I}^{\varrho}q_1(\sigma)(T)\Bigg]\Bigg\}\psi_2(\|z\|)\zeta\Bigg\{{}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T) \\ &+\frac{(\log T)^2}{\Lambda}\Bigg[\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}q_2(\sigma)(\varphi) + {}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T)\Bigg]\Bigg\} \\ \leq\ & |\xi|\|q_1\|\psi_1(\|z\|)\eta_1 + |\zeta|\|q_2\|\psi_2(\|z\|)\eta_2.\end{aligned}$$

We shall proceed to prove that the operator $\mathscr{T}$ maps bounded sets into equicontinuous sets of $\mathscr{C}([1,T], \mathbb{R})$. For $\iota_1, \iota_2 \in [1,T]$ with $\iota_1 < \iota_2$, and $z \in \mathscr{B}_\theta$

is a bounded set of $\mathscr{C}([1,T],\mathbb{R})$. Then we have

$$
\begin{aligned}
&|(\mathscr{T}z)(\iota_2)-(\mathscr{T}z)(\iota_1)|\\
\leq\ & |{}^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))|(\iota_2)-{}^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))|(\iota_1)|\\
&+|{}^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))|(\iota_2)-{}^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))|(\iota_1)|\\
&+\frac{|(\log\iota_2)^2-(\log\iota_1)^2|}{\Lambda}\Bigg[\xi\omega{}^H\mathscr{I}^{\varrho+\varsigma}|g(\sigma,z(\sigma))|(\varphi)\\
&+\zeta\omega{}^H\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma,z(\sigma))|(\varphi)+\xi{}^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))|(T)\\
&+\zeta{}^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))|(T)\Bigg]\\
\leq\ & \frac{\psi_1(\|\theta\|)}{\Gamma(\varrho)}\Bigg|\int_0^{\iota_1}\Big[\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho-1}-\Big(\log\frac{\iota_1}{\sigma}\Big)^{\varrho-1}\Big]q_1(\sigma)\frac{d\sigma}{\sigma}\\
&+\int_{\iota_1}^{\iota_2}\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho-1}q_1(\sigma)\frac{d\sigma}{\sigma}\Bigg|\\
&\frac{\psi_2(\|\theta\|)}{\Gamma(\varrho+\vartheta)}\Bigg|\int_0^{\iota_1}\Big[\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho+\vartheta-1}-\Big(\log\frac{\iota_1}{\sigma}\Big)^{\varrho+\vartheta-1}\Big]q_2(\sigma)\frac{d\sigma}{\sigma}\\
&+\int_{\iota_1}^{\iota_2}\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho+\vartheta-1}q_2(\sigma)\frac{d\sigma}{\sigma}\Bigg|\\
&+\frac{|(\log\iota_2)^2-(\log\iota_1)^2|}{\Lambda}\Bigg[\xi\omega{}^H\mathscr{I}^{\varrho+\varsigma}q_1(\sigma)(\varphi)+\zeta\omega{}^H\mathscr{I}^{\varrho+\varsigma+\vartheta}q_2(\sigma)(\varphi)\\
&+\xi{}^H\mathscr{I}^{\varrho}q_1(\sigma)(T)+\zeta{}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T)\Bigg].
\end{aligned}
$$

Hence we have that RHS of the above inequality tends to zero independent of $z\in\mathscr{B}_\theta$ as $\iota_2-\iota_1\to 0$. Therefore, the operator $\mathscr{T}(z)$ is equicontinuous and consequently, by Arzela-Ascoli theorem, it is completely continuous. Next, we demonstrate that the boundedness of the set of all solutions to equations $z=\nu\mathscr{T}(z),\ 0<\nu<1$. Let z be a solution. Then, for $\iota\in[1,T]$, and using the computations in proving that $\mathscr{T}$ is bounded, we have

$$
\begin{aligned}
|(\mathscr{T}z)(\iota)|\leq\ & \psi_1(\|z\|)\xi\Bigg\{{}^H\mathscr{I}^{\varrho}q_1(\sigma)(T)+\frac{(\log T)^2}{\Lambda}\Bigg[\omega{}^H\mathscr{I}^{\varrho+\varsigma}q_1(\sigma)(\varphi)\\
&+{}^H\mathscr{I}^{\varrho}q_1(\sigma)(T)\Bigg]\Bigg\}+\psi_2(\|z\|)\zeta\Bigg\{{}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T)\\
&+\frac{(\log T)^2}{\Lambda}\Bigg[\omega{}^H\mathscr{I}^{\varrho+\varsigma+\vartheta}q_2(\sigma)(\varphi)+{}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T)\Bigg]\Bigg\}\\
=\ & \psi_1(\|z\|){}^H\mathscr{I}^{\varrho}q_1(\sigma)(T)+\psi_2(\|z\|){}^H\mathscr{I}^{\varrho+\vartheta}q_2(\sigma)(T)+\mathscr{G}.
\end{aligned}
$$

In view of $(\mathscr{E}_2)$, there exists $\mathscr{L} \ni \|z\| \neq \mathscr{L}$. Let us set

$$\mathscr{M} = \{z \in \mathscr{C}([1,T],\mathbb{R}) : \|z\| < \mathscr{L}\}.$$

Bearing in mind that the operator $\mathscr{T} : \overline{\mathscr{W}} \to \mathscr{C}([1,T],\mathbb{R})$ is continuous and completely continuous. From the of choice of $\mathscr{W}$, there is no $z \in \partial\mathcal{W} \ni z = \nu\mathscr{T}(z),\ 0 < \nu < 1$. Consequently, by Theorem 2.9, we deduce that $\mathscr{T}$ has a fixed point $z \in \overline{\mathscr{W}}$ which is a solution of the problem (1). ■

Next, we prove the existence result is based on KFPT.

Theorem 3.2. *Let $g, h : [1,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then, the following conditions hold:*

- $(\mathscr{E}_4)$ $|g(\iota,p_1) - g(\iota,p_2)| \leq \mathscr{S}_1|p_1 - p_2|$, $|h(\iota,p_1) - g(\iota,p_2)| \leq \mathscr{S}_2|p_1 - p_2|$, $\forall\ \iota \in [1,T],\ p_1,p_2 \in \mathbb{R}, \mathscr{S}_1,\mathscr{S}_2 > 0$ *with* $\mathscr{S} = \max\{\mathscr{S}_1,\mathscr{S}_2\}$.
- $(\mathscr{E}_5)$ $|g(\iota,z(\iota))| \leq \varpi_1(\iota)$, $|h(\iota,z(\iota))| \leq \varpi_2(\iota)$ *for* $(\iota,z) \in [1,T]\times\mathbb{R}$, *and* $\varpi_1,\varpi_2 \in \mathscr{C}([1,T],\mathbb{R}^+)$ *with* $\|\varpi\| = \max_{\iota\in[1,T]}|\varpi_i(\iota)|,\ i = 1,2$.

Then the BVP (1) has at least one solution on $[1,T]$ if $\mathscr{S}\hat{\Delta} < 1$, where

$$\hat{\Delta} = \Delta - \left(\frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}\right), \tag{10}$$

Δ *is given by (7) and* $\sup_{\iota\in[1,T]}|\varpi_1(\iota)| = \|\varpi_i\|, i = 1,2.$

Proof. Let us interpret $\mathscr{B}_\theta = \{z \in \mathscr{P} : \|z\| \leq \theta\}$, where $\theta \geq \|\varpi\|\Delta$. To prove the hypothesis of Theorem 2.10, we split the operator $\mathscr{T}$ given by (6) as $\mathscr{T} = \mathscr{T}_1 + \mathscr{T}_2$ on $\mathscr{B}_\theta$, where

$$\begin{aligned}
(\mathscr{T}_1 z)(\iota) &= \xi^H\mathscr{I}^{\varrho}g(\sigma,z(\sigma))(\iota) + \zeta^H\mathscr{I}^{\varrho+\vartheta}h(\sigma,z(\sigma))(\iota),\\
(\mathscr{T}_2 z)(\iota) &= \frac{(\log\iota)^2}{\Lambda}\Bigg[\xi\omega^H\mathscr{I}^{\varrho+\varsigma}g(\sigma,z(\sigma))(\varphi) + \zeta\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}h(\sigma,z(\sigma))(\varphi)\\
&\qquad -\xi^H\mathscr{I}^{\varrho}g(\sigma,z(\sigma))(T) - \zeta^H\mathscr{I}^{\varrho+\vartheta}h(\sigma,z(\sigma))(T)\Bigg].
\end{aligned}$$

For $\hat{z}_1, \hat{z}_2 \in \mathscr{B}_\theta$, we have

$$\begin{aligned}
&|(\mathscr{T}_1\hat{z}_1)(\iota) + (\mathscr{T}_2\hat{z}_2)(\iota)|\\
&\leq \sup_{\iota\in[1,T]}\Bigg\{\xi^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))|(\iota) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))|(\iota)\\
&\quad + \frac{(\log\iota)^2}{\Lambda}\Bigg[\xi\omega^H\mathscr{I}^{\varrho+\varsigma}|g(\sigma,z(\sigma))|(\varphi) + \zeta\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma,z(\sigma))|(\varphi)\\
&\quad + \xi^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))|(T) + \zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))|(T)\Bigg]\Bigg\}
\end{aligned}$$

$$\leq \|\varpi\| \Bigg\{ \Bigg(\frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} \Bigg) + \Bigg[\frac{(\log T)^2}{\Lambda} \Bigg(\frac{\xi\omega(\log\varphi)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)}$$
$$+ \frac{\zeta\omega(\log\varphi)^{\varrho+\varsigma+\vartheta}}{\Gamma(\varrho+\varsigma+\vartheta+1)} + \frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} \Bigg) \Bigg] \Bigg\}$$
$$\leq \|\varpi\|\Delta \leq \theta,$$

which imply that $\mathscr{T}_1\hat{z}_1 + \mathscr{T}_2\hat{z}_2 \in \mathscr{B}_\theta$.

Now, we will show that $\mathscr{T}_2$ is a contraction. Let $p_1, p_2 \in \mathbb{R}$, $\iota \in [1,T]$. Then, using the assumption $(\mathscr{E}_4)$ together with (10), we get

$$\|\mathscr{T}_2p_1 - \mathscr{T}_2p_2\| \leq \frac{\mathscr{S}(\log T)^2}{\Lambda} \Bigg[\frac{\xi\omega(\log\varphi)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)} + \frac{\zeta\omega(\log\varphi)^{\varrho+\varsigma+\vartheta}}{\Gamma(\varrho+\varsigma+\vartheta+1)}$$
$$+ \frac{\xi(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{\zeta(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} \Bigg] \|p_1 - p_2\|.$$

By the assumption $(\mathscr{E}_4)$, it follows that the operator $\mathscr{T}_2$ is contraction. Next, we will show that $\mathscr{T}_1$ is compact and continuous. Continuity of g, h implies that the operator $\mathscr{T}_1$ is continuous. Also, $\mathscr{T}_1$ is uniformly bounded on B_θ as

$$\|\mathscr{T}_1 z\| \leq \|\varpi\| \Bigg\{ \frac{|\xi|(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{|\zeta|(\log T)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} \Bigg\}.$$

Moreover, with $\sup_{(\iota,z)\in[1,T]\times\mathscr{B}_\theta} |g(\iota,z)| = \hat{g} < \infty$, $\sup_{(\iota,z)\in[1,T]\times\mathscr{B}_\theta} |h(\iota,z)| = \hat{h} < \infty$ and $\iota_1 < \iota_2$, $\iota_1, \iota_2 \in [1,T]$, we have

$$|(\mathscr{T}_1 z)(\iota_2) - (\mathscr{T}_1 z)(\iota_1)| \tag{11}$$
$$= |{}^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(\iota_2) - {}^H\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(\iota_1)|$$
$$|{}^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(\iota_2) - {}^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(\iota_1)|$$
$$\leq \frac{\hat{g}}{\Gamma(\varrho)} \Bigg| \int_0^{\iota_1} \Big[\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho-1} - \Big(\log\frac{\iota_1}{\sigma}\Big)^{\varrho-1} \Big] \frac{d\sigma}{\sigma}$$
$$+ \int_{\iota_1}^{\iota_2} \Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho-1} \frac{d\sigma}{\sigma} \Bigg|$$
$$+ \frac{\hat{h}}{\Gamma(\varrho+\vartheta)} \Bigg| \int_0^{\iota_1} \Big[\Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho+\vartheta-1} - \Big(\log\frac{\iota_1}{\sigma}\Big)^{\varrho+\vartheta-1} \Big] \frac{d\sigma}{\sigma}$$
$$+ \int_{\iota_1}^{\iota_2} \Big(\log\frac{\iota_2}{\sigma}\Big)^{\varrho+\vartheta-1} \frac{d\sigma}{\sigma} \Bigg|. \tag{12}$$

Clearly, the RHS of (11) tends to zero independent of z as $\iota_2 - \iota_1 \to 0$. Thus, $\mathscr{T}_1$ is relatively compact on $\mathscr{B}_\theta$. Hence, by the Arzela-Ascoli Theorem, $\mathscr{T}_1$ is compact on $\mathscr{B}_\theta$. Thus, all the assumptions of Theorem 2.10 are satisfied. Therefore, there exists at least one solution for BVP (1) on $[1, T]$. ∎

Next, we establish the uniqueness of solution using BFPT for the problem (1).

Theorem 3.3. *Let $g, h : [1, T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the assumptions ($\mathscr{E}_4$). In addition, it is assumed that $\mathscr{S}\Delta < 1$, where Δ is described by (7). Then there exists a unique solution for BVP (1) on $[1, T]$.*

Proof. Let us interpret $\sup_{\iota\in[1,T]} |g(\iota, 0)| = \mathscr{Q}_1 < \infty$, $\sup_{\iota\in[1,T]} |h(\iota, 0)| = \mathscr{Q}_2 < \infty$, with $\mathscr{Q} = \max\{\mathscr{Q}_1, \mathscr{Q}_2\}$. Nominating $\theta \geq \frac{\mathscr{Q}\Delta}{1-\mathscr{S}\Delta}$. We demonstrate that $\mathscr{T}\mathscr{B}_\theta \subset \mathscr{B}_\theta$, where $\mathscr{B}_\theta = \{z \in \mathscr{P} : \|z\| \leq \theta\}$. For $z \in \mathscr{B}_\theta$, we have

$$
\begin{aligned}
&|(\mathscr{T}z)(\iota)| \\
&\leq \sup_{\iota\in[1,T]} \Bigg\{ \xi^{H}\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(\iota) + \zeta^{H}\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(\iota) \\
&\quad + \frac{(\log\iota)^2}{\Lambda}\Bigg[\xi\omega^{H}\mathscr{I}^{\varrho+\varsigma}|g(\sigma, z(\sigma))|(\varphi) + \zeta\omega^{H}\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma, z(\sigma))|(\varphi) \\
&\quad + \xi^{H}\mathscr{I}^{\varrho}|g(\sigma, z(\sigma))|(T) + \zeta^{H}\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma))|(T)\Bigg]\Bigg\} \\
&\leq |\xi|(\mathscr{S}_1\theta + \mathscr{Q}_1) \sup_{\iota\in[1,T]} \Bigg\{ {}^{H}\mathscr{I}^{\varrho}(1)(\iota) + \frac{(\log\iota)^2}{\Lambda}\Bigg[\omega^{H}\mathscr{I}^{\varrho+\varsigma}(1)(\varphi) \\
&\quad + {}^{H}\mathscr{I}^{\varrho}(1)(T)\Bigg]\Bigg\} |\zeta|(\mathscr{S}_2\theta + \mathscr{Q}_2) \sup_{\iota\in[1,T]} \Bigg\{ \zeta^{H}\mathscr{I}^{\varrho+\vartheta}(1)(\iota) \\
&\quad + \frac{(\log\iota)^2}{\Lambda}\Bigg[\zeta\omega^{H}\mathscr{I}^{\varrho+\varsigma+\vartheta}(1)(\varphi) + \zeta^{H}\mathscr{I}^{\varrho+\vartheta}(1)(T)\Bigg]\Bigg\} \\
&\leq (\mathscr{S}\theta + \mathscr{Q})\Delta.
\end{aligned} \tag{13}
$$

Thus, it follows from (13) that $\|(\mathscr{T}z)\| \leq \theta$.

Now, for $z, \hat{z} \in \mathscr{P}$, we obtain

$$
\begin{aligned}
&|\mathscr{T}z(\iota) - \mathscr{T}\hat{z}(\iota)| \\
&\leq \sup_{\iota\in[1,T]} \Bigg\{ \xi^{H}\mathscr{I}^{\varrho}|g(\sigma, z(\sigma)) - g(\sigma, \hat{z}(\sigma))|(\iota) \\
&\quad + \zeta^{H}\mathscr{I}^{\varrho+\vartheta}|h(\sigma, z(\sigma)) - h(\sigma, \hat{z}(\sigma))|(\iota) \\
&\quad + \frac{(\log\iota)^2}{\Lambda}\Bigg[\xi\omega^{H}\mathscr{I}^{\varrho+\varsigma}|g(\sigma, z(\sigma)) - g(\sigma, \hat{z}(\sigma))|(\varphi) \\
&\quad + \zeta\omega^{H}\mathscr{I}^{\varrho+\varsigma+\vartheta}|h(\sigma, z(\sigma)) - h(\sigma, \hat{z}(\sigma))|(\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&+\xi^H\mathscr{I}^{\varrho}|g(\sigma,z(\sigma))-g(\sigma,\hat{z}(\sigma))|(T)\\
&+\zeta^H\mathscr{I}^{\varrho+\vartheta}|h(\sigma,z(\sigma))-h(\sigma,\hat{z}(\sigma))|(T)\Bigg]\Bigg\}\\
\leq\ &\mathscr{S}_1\Delta\|z-\hat{z}\|\Bigg\{\xi^H\mathscr{I}^{\varrho}(1)(T)+\frac{(\log\iota)^2}{\Lambda}\Bigg[\xi\omega^H\mathscr{I}^{\varrho+\varsigma}(1)(\varphi)+\xi^H\mathscr{I}^{\varrho}(1)(T)\Bigg]\\
&+\mathscr{S}_2\Delta\|z-\hat{z}\|\Bigg\{\zeta^H\mathscr{I}^{\varrho+\vartheta}(1)(T)+\frac{(\log\iota)^2}{\Lambda}\Bigg[\zeta\omega^H\mathscr{I}^{\varrho+\varsigma+\vartheta}(1)(\varphi)\\
&+\zeta^H\mathscr{I}^{\varrho+\vartheta}(1)(T)\Bigg]\Bigg\}\\
=\ &\mathscr{S}\Delta\|z-\hat{z}\|.
\end{aligned}
$$

Thus,

$$\|\mathscr{T}z-\mathscr{T}\hat{z}\|\leq\mathscr{S}\Delta\|z-\hat{z}\|.$$

Since $\mathscr{S}\Delta<1$ by the given assumption, therefore $\mathscr{T}$ is a contraction. Hence it follows from Theorem 2.11 that Eq. (1) has a unique solution on $[1,T]$. ■

4. Numerical Examples

Example 4.1. Consider a BVP of CHFIDE given by

$$
\begin{aligned}
{}^C\mathscr{D}^{\varrho}z(\iota) &= \xi g(\iota,z(\iota))+\zeta^H\mathscr{I}^{\vartheta}h(\iota,z(\iota)),\quad \iota\in[1,T],\\
z(1) &= 0,\quad z'(1)=0,\quad z(T)=\omega^H\mathscr{I}^{\varsigma}z(\varphi),
\end{aligned}\tag{14}
$$

where

$$
\begin{aligned}
g(\iota,z) &= \frac{1}{(\iota+5)^2}\left(\frac{|z|}{1+|z|}+\frac{\cos z}{(e^{\log\iota})}+\frac{\iota}{1+\iota}\right),\\
h(\iota,z) &= \frac{1}{4}\sin z+\iota.
\end{aligned}
$$

Here, $\varrho=\frac{12}{5}$, $\vartheta=\frac{6}{5}$, $\varsigma=\frac{7}{5}$, $\xi=\frac{1}{4}$, $\zeta=\frac{1}{6}$, $\omega=\frac{3}{10}$, $\varphi=\frac{7}{4}$.

In addition, we find that

$$
\begin{aligned}
|g(\iota,p_1(\iota))-g(\iota,p_2(\iota))| &\leq \frac{1}{36}\|p_1-p_2\|,\\
|h(\iota,p_1(\iota))-h(\iota,p_2(\iota))| &\leq \frac{1}{4}\|p_1-p_2\|.
\end{aligned}
$$

With the above specifics, we find that $\mathscr{S}_1=\frac{1}{36}$, $\mathscr{S}_2=\frac{1}{4}$, so $\mathscr{S}=\max\{\mathscr{S}_1,\mathscr{S}_2\}=\frac{1}{4}$. Next, we find that $\Lambda=0.47222857758628856$, $\Delta=0.10248515987315399$.

Thus, $\mathscr{S}\Delta < 1$, the presumptions of Theorem 3.3 are satisfied. Hence, by Theorem 3.3, the BVP (14) has a unique solution on $[1, T]$.

Example 4.2. Consider the BVP of CHFIDE (14) with $|g(\iota, z(\iota))| \leq \varpi_1(\iota)$, $|h(\iota, z(\iota))| \leq \varpi_2(\iota)$ for $(\iota, z) \in [1, T] \times \mathbb{R}$, and $\varpi_1, \varpi_2 \in \mathscr{C}([1, T], \mathbb{R}^+)$. Clearly,

$$\begin{aligned} |g(\iota, z)| &\leq \frac{1}{1+\log\iota} + \frac{|z|}{1+|z|}\frac{e^{\log\iota}}{(4+\iota)^2}, \\ |h(\iota, z)| &\leq \frac{1}{1+\iota^2} + \frac{\cos z}{(2+\iota)^2}. \end{aligned}$$

Here, $\varrho = \frac{12}{5}$, $\vartheta = \frac{6}{5}$, $\varsigma = \frac{7}{5}$, $\xi = \frac{1}{4}$, $\zeta = \frac{1}{6}$, $\omega = \frac{3}{10}$, $\varphi = \frac{7}{4}$. With the above specifics, we find that $\Lambda = 0.47222857758628856$, $\Delta = 0.05276145334211216$. Thus, $\hat{\Delta}\mathscr{S} < 1$, the presumptions of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the BVP (14) has at least one solution on $[1, T]$.

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