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Some Properties and Characterizations of Binary Soft Functions

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Abstract. The purpose of this paper is to introduce the concepts of binary soft functions namely binary soft continuous functions, binary soft open maps, binary soft closed maps and binary soft homeomorphisms in binary soft topological spaces. Further, their properties and characterizations are studied, and interrelationship between these functions are investigated.

Keywords: Binary soft functions; Binary soft continuous functions; Binary soft open maps; Binary soft closed maps.

1. Introduction

Mathematical modeling constitutes an important role in analyzing and solving the real world problems. To deal problems with incomplete knowledge, Molodtsov [4] introduced the theory of soft sets. Acikgoz et al. [1] introduced the concept of binary soft sets on two universal sets and a parameter set. The concept of topological spaces for binary soft sets was initiated by Benchalli et al. [2]. Also, they have defined binary soft separation axioms, binary soft closure and interior operators. Patil et al. [5] introduced and studied new binary soft separation axioms in binary soft topological spaces.

In this paper, the concepts of binary soft functions namely binary soft continuous functions, binary soft open maps, binary soft closed maps and binary soft homeomorphisms in binary soft topological spaces are introduced. Further, their properties and characterizations are studied, and interrelationship between these binary soft functions are investigated.

2. Preliminaries

In this section, we recollect some concepts that are helpful for the new consequences.

Definition 2.1. [1] *A structure (Ψ, E) is called a binary soft set over the two universal sets U_1 and U_2 with power sets $P(U_1)$ and $P(U_2)$ respectively if and only if $\Psi : E \to P(U_1) \times P(U_2)$ defined as $\Psi(e) = (M, N)$ for each $e \in E$ such that $M \subseteq U_1$, $N \subseteq U_2$, where E is a parameter set.*

Definition 2.2. [1] *Let U_1, U_2 be two universal sets and E be a set of parameters. Also, let A, B, $C \subseteq E$ and (Ψ_1, A), (Ψ_2, B) be two binary soft sets over U_1, U_2 respectively such that $\Psi_1(e) = (M_1, N_1)$ for each $e \in A$, $\Psi_2(e) = (M_2, N_2)$ for each $e \in B$. Then,*

(1) *$(\Psi_1, A) \subseteq (\Psi_2, B)$, read as (Ψ_1, A) is binary soft subset of (Ψ_2, B), if $A \subseteq B$ and $M_1 \subseteq M_2$, $N_1 \subseteq N_2$ for each $e \in A$.*
(2) *$(\Psi_1, A) = (\Psi_2, B)$, read as (Ψ_1, A) is binary soft equal of (Ψ_2, B), if $(\Psi_1, A) \subseteq (\Psi_2, B)$ and $(\Psi_2, B) \subseteq (\Psi_1, A)$.*
(3) *union of (Ψ_1, A) and (Ψ_2, B), denoted by $(\Psi_1, A) \cup (\Psi_2, B)$ is a binary soft set (Ψ, C), where $C = A \cup B$ and for each $e \in C$,*

$$\Psi(e) = \begin{cases} (M_1, N_1) & e \in A \setminus B, \\ (M_2, N_2) & e \in B \setminus A, \\ (M_1 \cup M_2, N_1 \cup N_2) & e \in A \cap B. \end{cases}$$

(4) *intersection of (Ψ_1, A) and (Ψ_2, B), denoted by $(\Psi_1, A) \cap (\Psi_2, B)$ is a binary soft set (Ψ, C), where $C = A \cap B$ and $\Psi(e) = (M_1 \cap M_2, N_1 \cap N_2)$ for each $e \in C$.*
(5) *(Ψ_1, A) is a binary null soft set, denoted by $\widetilde{\widetilde{\emptyset}}$, if $\Psi_1(e) = (\emptyset, \emptyset)$ for each $e \in A$.*
(6) *(Ψ_1, A) is a binary absolute soft set, denoted by $\widetilde{\widetilde{A}}$, if $\Psi_1(e) = (U_1, U_2)$ for each $e \in A$.*

Definition 2.3. [2] *(U_1, U_2, τ_b, E) is said to be a binary soft topological space over U_1, U_2 with the set of parameters E, if a collection τ_b of binary soft sets over U_1, U_2 satisfies the following axioms:*

(1) $\widetilde{\widetilde{\emptyset}}$, $\widetilde{\widetilde{E}} \in \tau_b$.
(2) *The union of any members of τ_b belongs to τ_b.*

(3) *The intersection of any two members of* τ_b *belongs to* τ_b.

Here, τ_b *is called a binary soft topology on* U_1, U_2. *The members of* τ_b *are binary soft open sets and their relative complements are binary soft closed sets.*

Definition 2.4. [2] *Let* (Ψ, E) *be a binary soft subset of a binary soft topological space* (U_1, U_2, τ_b, E). *Then*

(1) $\overline{\overline{(\Psi, E)}}$, *read as binary soft closure of* (Ψ, E), *is the intersection of all binary soft closed sets containing* (Ψ, E).
(2) $(\Psi, E)^{\circledcirc}$, *read as binary soft interior of* (Ψ, E), *is the union of all binary soft open sets contained in* (Ψ, E).

Proposition 2.5. [5] *Let* (U_1, U_2, τ_b, E) *be a binary soft topological space. Then for each* $e \in E$,

(1) $\tau_e = \{M \subseteq U_1 : (\Psi, E) \in \tau_b \text{ and } \Psi(e) = (M, N)\}$ *is a topology on* U_1.
(2) $\tau_e = \{N \subseteq U_2 : (\Psi, E) \in \tau_b \text{ and } \Psi(e) = (M, N)\}$ *is a topology on* U_2.

3. Binary Soft Functions

In this section, we have defined binary soft functions, binary soft images and binary soft inverse images. Further, their properties are investigated.

Definition 3.1. *Let* (G, E) *and* (H, E') *be two non-empty binary soft sets with* $G(e) = (X_1, Y_1)$ *for each* $e \in E$, $H(e') = (X_2, Y_2)$ *for each* $e' \in E'$ *and* E, E' *are sets of parameters. Let* $u_1 : X_1 \to X_2$, $u_2 : Y_1 \to Y_2$ *and* $p : E \to E'$ *be functions. Then, the binary soft function* $f : (G, E) \to (H, E')$ *is defined as:*

For any binary soft subset (F, A) *of* (G, E), $(f(F, A), B)$ *with* $B = p(A) \subseteq E'$ *is a binary soft subset of* (H, E') *given by* $(f(F, A))(b) = (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A} F(a)\right)$ *where* $b \in B \subseteq E'$, u_1 *operates only on the first ordinate of* $\cup_{a \in p^{-1}(b) \cap A} F(a)$ *and* u_2 *operates only on the second ordinate of* $\cup_{a \in p^{-1}(b) \cap A} F(a)$.

Here, $(f(F, A), B)$ *is called a binary soft image of* (F, A).

Remark 3.2. Step-wise procedure to find the binary soft image $(f(F, A), B)$ of (F, A):

Step 1: Obtain $B = p(A)$.

Step 2: Find a, where $a \in p^{-1}(b) \cap A$.

Suppose $p(a_1) = p(a_2) = \cdots = p(a_n) = b$ for a finite positive integer n and $a_1, a_2, \ldots, a_n \in A$, then $a_1, a_2, \ldots, a_n \in (p^{-1}(b) \cap A) \subseteq E$.

Step 3: Now, for $b \in B \subseteq E'$,

$$\begin{aligned}(f(F,A))(b) &= (u_1u_2)\left(\cup_{a\in p^{-1}(b)\cap A}F(a)\right)\\ &= (u_1u_2)\left(F(a_1)\cup F(a_2)\cup\ldots\cup F(a_n)\right)\\ &= (u_1u_2)\left(F(a_m)\right)\\ &= (u_1u_2)(M,N), \quad for \quad M\subseteq X_1 \quad and \quad N\subseteq Y_1\\ &= (R,S), \qquad\qquad for \quad u_1(M)=R\subseteq X_2 \quad and \quad u_2(N)=S\subseteq Y_2.\end{aligned}$$

Remark 3.3. If $B = E'$, then $(f(F,A),B)$ can be simply written as $f(F,A)$.

Definition 3.4. *Let $f : (G,E) \to (H,E')$ be a binary soft function with $G(e) = (X_1,Y_1)$ for each $e \in E$, $H(e') = (X_2,Y_2)$ for each $e' \in E'$ and E, E' are sets of parameters. Let $u_1 : X_1 \to X_2$, $u_2 : Y_1 \to Y_2$ and $p : E \to E'$ be functions. Let (F,B) ba any binary soft subset of (H,E'). Then, the binary soft inverse image of (F,B) denoted by $(f^{-1}(F,B),A)$, $A = p^{-1}(B) \subseteq E$ is a binary soft subset of (G,E) defined as*

$$(f^{-1}(F,B))(a) = (u_1^{-1}u_2^{-1})\left(F(p(a))\right)$$

where $a \in A \subseteq E$.

Remark 3.5. If $A = E$, then $(f^{-1}(F,B),A)$ can be simply written as $f^{-1}(F,B)$.

Example 3.6. Let $f : (G,E) \to (H,E')$ be a binary soft function, where $(G,E) = \{(e_1,(\{x_1^1,x_1^2, \ x_1^3\},\{y_1^1,y_1^2\})),(e_2,(\{x_1^2,x_1^3,x_1^4,x_1^5\},\{y_1^3\})),(e_3,(\{x_1^2, x_1^4\},\{y_1^1,y_1^2,y_1^3\})),(e_4,(\{x_1^1\},\{y_1^2\}))\}$ and $(H,E') = \{(e_1',(\{x_2^1,x_2^2\},\{y_2^1,y_2^2, y_2^3\})),(e_2',(\{x_2^3\},\{y_2^3,y_2^4\})),(e_3',(\{x_2^1,x_2^2,x_2^3\},\emptyset))\}$.

Define functions u_1, u_2 and p as:
$u_1(x_1^1) = x_2^2$, $u_1(x_1^2) = x_2^3$, $u_1(x_1^3) = x_2^1$, $u_1(x_1^4) = x_2^2$, $u_1(x_1^5) = x_2^3$;
$u_2(y_1^1) = y_2^3$, $u_2(y_1^2) = y_2^2$, $u_2(y_1^3) = y_2^1$;
$p(e_1) = e_1'$, $p(e_2) = e_3'$, $p(e_3) = e_2'$, $p(e_4) = e_1'$.

Let $(F,A) = \{(e_1,(\{x_1^1,x_1^3\},\{y_1^1\})),(e_3,(\{x_1^2,x_1^4\},\{y_1^3\})),(e_4,(\emptyset,\{y_1^2\}))\}$ be a binary soft subset of (G,E). Then $B = p(A) = \{e_1', \ e_2'\}$ and $(f(F,A))(e_1') = (\{x_2^1,x_2^2\},\{y_2^2,y_2^3\})$, $(f(F,A))(e_2') = (\{x_2^2,x_2^3\},\{y_2^1\})$.

Therefore, $(f(F,A),B) = \{(e_1',(\{x_2^1,x_2^2\},\{y_2^2,y_2^3\})),(e_2',(\{x_2^2,x_2^3\},\{y_2^1\}))\}$ is the binary soft image of (F,A). Again, let $(F,B) = \{(e_2',(\{x_2^3\},\{y_2^3\})),(e_3',(\{x_2^1, x_2^2,x_2^3\},\emptyset))\}$ be a binary soft subset of (H,E'). Then $A = p^{-1}(B) = \{e_2,e_3\}$ and $(f^{-1}(F,B))(e_2) = (\{x_1^1,x_1^2,x_1^3,x_1^4,x_1^5\},\emptyset)$, $(f^{-1}(F,B))(e_3) = (\{x_1^2,x_1^5\},\{y_1^1\})$.

Therefore, $(f^{-1}(F,B),A) = \{(e_2,(\{x_1^1,x_1^2,x_1^3,x_1^4,x_1^5\},\emptyset)),(e_3,(\{x_1^2,x_1^5\}, \{y_1^1\}))\}$ is the binary soft inverse image of (F,B).

Theorem 3.7. *Let $f : (G,E) \to (H,E')$ be a binary soft function, where $G(e) = (X_1,Y_1)$ for each $e \in E$, $H(e') = (X_2,Y_2)$ for each $e' \in E'$ and E, E' are sets of* *parameters. Let* $u_1 : X_1 \to X_2$, $u_2 : Y_1 \to Y_2$ *and* $p : E \to E'$ *be functions. Then for any binary soft subsets* (F_1, A_1), (F_2, A_2) *of* (G, E), *the following results are true:*

(1) $f\left(\widetilde{\widetilde{\emptyset}}\right) = \widetilde{\widetilde{\emptyset}}$.

(2) $f(G, E) \subseteq (H, E')$.

(3) $f((F_1, A_1) \cup (F_2, A_2)) = f(F_1, A_1) \cup f(F_2, A_2)$.

(4) $f((F_1, A_1) \cap (F_2, A_2)) \subseteq f(F_1, A_1) \cap f(F_2, A_2)$.

(5) *If* $(F_1, A_1) \subseteq (F_2, A_2)$, *then* $f(F_1, A_1) \subseteq f(F_2, A_2)$.

Proof. (1) and (2) are obvious.

(3) Let $(F_1, A_1) \cup (F_2, A_2) = (F, A)$ with $A = A_1 \cup A_2$ and for each $a \in A$,

$$F(a) = \begin{cases} F_1(a) & a \in A_1 \setminus A_2, \\ F_2(a) & a \in A_2 \setminus A_1, \\ F_1(a) \cup F_2(a) & a \in A_1 \cap A_2, \end{cases}$$

where $F_1(a) = (N_1, M_1)$ for each $a \in A_1$ such that $N_1 \subseteq X_1$, $M_1 \subseteq Y_1$, and $F_2(a) = (N_2, M_2)$ for each $a \in A_2$ such that $N_2 \subseteq X_2$, $M_2 \subseteq Y_2$.

Then, for each $b \in E'$,

$$\begin{aligned} f(F, A)(b) &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A} F(a)\right) \\ &= (u_1 u_2)\left(\cup \begin{cases} F_1(a) & a \in p^{-1}(b) \cap (A_1 \setminus A_2), \\ F_2(a) & a \in p^{-1}(b) \cap (A_2 \setminus A_1), \\ F_1(a) \cup F_2(a) & a \in p^{-1}(b) \cap (A_1 \cap A_2). \end{cases}\right) \end{aligned}$$

On the other hand, for each $b \in E'$,

$$\begin{aligned} & f(F_1, A_1)(b) \cup f(F_2, A_2)(b) \\ &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_1} F_1(a)\right) \cup (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_2} F_2(a)\right) \\ &= (u_1 u_2)\left[\left(\cup_{a \in p^{-1}(b) \cap A_1} F_1(a)\right) \cup \left(\cup_{a \in p^{-1}(b) \cap A_2} F_2(a)\right)\right] \end{aligned}$$

From the above equations (3) holds.

(4) Let $(F_1, A_1) \cap (F_2, A_2) = (F, A)$ with $A = A_1 \cap A_2$ and for each $a \in A$, $F(a) = F_1(a) \cap F_2(a)$, where $F_1(a) = (N_1, M_1)$ for each $a \in A_1$ such that $N_1 \subseteq X_1$, $M_1 \subseteq Y_1$, and $F_2(a) = (N_2, M_2)$ for each $a \in A_2$ such that $N_2 \subseteq X_1$, $M_2 \subseteq Y_1$, implies $F_1(a) \cap F_2(a) = (N_1 \cap N_2, M_1 \cap M_2)$ for all $a \in A$.

Now, for $b \in E'$,

$$\begin{aligned} f(F, A)(b) &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A} F(a)\right) \\ &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A} F_1(a) \cap F_2(a)\right). \end{aligned}$$

On the other hand, for each $b \in E'$,

$$\begin{aligned} & f(F_1, A_1)(b) \cap f(F_2, A_2)(b) \\ &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_1} F_1(a)\right) \cap (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_2} F_2(a)\right) \\ &\supseteq (u_1 u_2)\left[\left(\cup_{a \in p^{-1}(b) \cap A_1} F_1(a)\right) \cap \left(\cup_{a \in p^{-1}(b) \cap A_2} F_2(a)\right)\right]. \end{aligned}$$

From the above equations (4) holds.

(5) Let $(F_1, A_1) \subseteq (F_2, A_2)$ where $F_1(a) = (N_1, M_1)$ for each $a \in A_1$ such that $N_1 \subseteq X_1$, $M_1 \subseteq Y_1$, and $F_2(a) = (N_2, M_2)$ for each $a \in A_2$ such that $N_2 \subseteq X_2$, $M_2 \subseteq Y_2$. Therefore, $A_1 \subseteq A_2$, $N_1 \subseteq N_2$ and $M_1 \subseteq M_2$, which implies $F_1(a) \subseteq F_2(a)$ for each $a \in A_1$. Now, for $b \in E'$,

$$\begin{aligned} f(F_1, A_1)(b) &= (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_1} F_1(a)\right) \\ &\subseteq (u_1 u_2)\left(\cup_{a \in p^{-1}(b) \cap A_2} F_2(a)\right). \end{aligned}$$

Therefore, (5) holds. ∎

Remark 3.8. In Theorem 3.7, equalities in (2) and (4) does not hold in general.

Example 3.9. In Example 3.6, $f(G,E) = \{(e'_1, (\{x_2^1, x_2^2, x_2^3\}, \{y_2^2, y_2^3\})), (e'_2, (\{x_2^1, x_2^2, x_2^3\}, \{y_2^1\})), (e'_3, (\{x_2^2, x_2^3\}, \{y_2^1, y_2^2, y_2^3\}))\}$. Therefore, $(H, E') \nsubseteq f(G,E)$. So $f(G,E) \neq (H, E')$.

Also, let $(F_1, A_1) = \{(e_1, (\{x_1^1, x_1^3\}, \{y_1^2\})), (e_2, (\{x_1^2\}, \emptyset)), (e_3, (\{x_1^2\}, \{y_1^3\})), (e_4, (\{x_1^1\}, \{y_1^2\}))\}$ and $(F_2, A_2) = \{(e_2, (\{x_1^5\}, \{y_1^3\})), (e_3, (\{x_1^4\}, \{y_1^2, y_1^3\})), (e_4, (\emptyset, \{y_1^2\}))\}$ be two binary soft subsets of (G, E).

Now, $(F_1, A_1) \cap (F_2, A_2) = \{(e_2, (\emptyset, \emptyset)), (e_3, (\emptyset, \{y_1^3\})), (e_4, (\emptyset, \{y_1^2\}))\}$. Therefore,

$$f((F_1, A_1) \cap (F_2, A_2)) = \{(e'_1, (\emptyset, \{y_2^2\})), (e'_2, (\emptyset, \{y_2^1\})), (e'_3, (\emptyset, \emptyset))\} \qquad (1)$$

Again, $f(F_1, A_1) = \{(e'_1, (\{x_2^1, x_2^2\}, \{y_2^2\})), (e'_2, (\{x_2^3\}, \{y_2^1\})), (e'_3, (\{x_2^3\}, \emptyset))\}$ and $f(F_2, A_2) = \{(e'_1, (\emptyset, \{y_2^2\})), (e'_2, (\{x_2^2\}, \{y_2^1, y_2^2\})), (e'_3, (\{x_2^3\}, \{y_2^1\}))\}$. Therefore,

$$f(F_1, A_1) \cap f(F_2, A_2) = \{(e'_1, (\emptyset, \{y_2^2\})), (e'_2, (\emptyset, \{y_2^1\})), (e'_3, (\{x_2^3\}, \emptyset))\} \qquad (2)$$

From equations (1) and (2), $f((F_1, A_1) \cap (F_2, A_2)) \neq f(F_1, A_1) \cap f(F_2, A_2)$.

Theorem 3.10. *Let $f : (G,E) \to (H, E')$ be a binary soft function where $G(e) = (X_1, Y_1)$ for each $e \in E$, $H(e') = (X_2, Y_2)$ for each $e' \in E'$ and E, E' are sets of parameters. Let $u_1 : X_1 \to X_2$, $u_2 : Y_1 \to Y_2$ and $p : E \to E'$ be functions. Then for any binary soft subsets (F_1, B_1), (F_2, B_2) of (H, E'), the following results are true:*

(1) $f^{-1}\left(\widetilde{\widetilde{\emptyset}}\right) = \widetilde{\widetilde{\emptyset}}$.
(2) $f^{-1}((F_1, B_1) \cup (F_2, B_2)) = f^{-1}(F_1, B_1) \cup f^{-1}(F_2, B_2)$.
(3) $f^{-1}((F_1, B_1) \cap (F_2, B_2)) = f^{-1}(F_1, B_1) \cap f^{-1}(F_2, B_2)$.
(4) *If $(F_1, B_1) \subseteq (F_2, B_2)$, then $f^{-1}(F_1, B_1) \subseteq f^{-1}(F_2, B_2)$.*

Proof. (1) is obvious.

(2) Let $(F_1, B_1) \cup (F_2, B_2) = (F, B)$ with $B = B_1 \cup B_2$ and for each $b \in B$,

$$F(b) = \begin{cases} (N_1, M_1) & b \in B_1 \setminus B_2 \\ (N_2, M_2) & b \in B_2 \setminus B_1 \\ (N_1 \cup N_2 M_1 \cup M_2) & b \in B_1 \cap B_2 \end{cases}$$
$$= \begin{cases} F_1(b) & b \in B_1 \setminus B_2, \\ F_2(b) & b \in B_2 \setminus B_1, \\ F_1(b) \cup F_2(b) & b \in B_1 \cap B_2. \end{cases}$$

Now, for $a \in E$, $f^{-1}(F, B)(a) = (u_1^{-1} u_2^{-1})\,(F(p(a)))$.

Take $p(a) = b \in B$. Therefore,

$$f^{-1}(F, B)(a) = (u_1^{-1} u_2^{-1}) \begin{cases} F_1(b) & b \in B_1 \setminus B_2, \\ F_2(b) & b \in B_2 \setminus B_1, \\ F_1(b) \cup F_2(b) & b \in B_1 \cap B_2. \end{cases}$$

On the other hand, for $a \in E$,

$$\begin{aligned} f^{-1}(F_1, B_1)(a) \cup f^{-1}(F_2, B_2)(a) &= (u_1^{-1} u_2^{-1})\,(F_1(p(a))) \cup (u_1^{-1} u_2^{-1})\,(F_2(p(a))) \\ &= (u_1^{-1} u_2^{-1}) \begin{cases} F_1(b) & b \in B_1 \setminus B_2, \\ F_2(b) & b \in B_2 \setminus B_1, \\ F_1(b) \cup F_2(b) & b \in B_1 \cap B_2, \end{cases} \end{aligned}$$

where $b = p(a)$. Therefore, (2) follows.

(3) Let $(F_1, B_1) \cap (F_2, B_2) = (F, B)$ with $B = B_1 \cap B_2$ and for each $b \in B$, $F(b) = (N_1 \cap N_2, M_1 \cap M_2)$, where $F_1(b) = (N_1, M_1)$ for each $b \in B_1$ such that $N_1 \subseteq X_2$, $M_1 \subseteq Y_2$, and $F_2(b) = (N_2, M_2)$ for each $b \in B_2$ such that $N_2 \subseteq X_2$, $M_2 \subseteq Y_2$.

Now, for $a \in E$,

$$\begin{aligned} f^{-1}(F, B)(a) &= (u_1^{-1} u_2^{-1})\,(F(p(a))) \\ &= (u_1^{-1} u_2^{-1})\,(F(b)) \qquad \text{(since } b = p(a) \in B) \\ &= (u_1^{-1} u_2^{-1})\,(F_1(b) \cap F_2(b)) \\ &= (u_1^{-1} u_2^{-1})\,(F_1(b)) \cap (u_1^{-1} u_2^{-1})\,(F_2(b)) \\ &= f^{-1}(F_1, B_1)(a) \cap f^{-1}(F_2, B_2)(a). \end{aligned}$$

Therefore, (4) holds. ■

Remark 3.11. In Theorem 3.10, $f^{-1}(H, E') \neq (G, E)$.

Example 3.12. In Example 3.6, $f^{-1}(H, E') = \{(e_1, (\{x_1^1, x_1^3, x_1^4\}, \{y_1^2, y_1^3\})), (e_2, (\{x_1^1, x_1^2, x_1^3, x_1^4, x_1^5\}, \emptyset)), (e_3, (\{x_1^2, x_1^5\}, \{y_1^1\})), (e_4, (\{x_1^1, x_1^3, x_1^4\}, \{y_1^2, y_1^3\}))\} \neq (G, E)$.

Definition 3.13. *A binary soft function* $f : (G, E) \to (H, E')$ *with* $G(e) = (X_1, Y_1)$ *for each* $e \in E$, $H(e') = (X_2, Y_2)$ *for each* $e' \in E'$ *along with functions* $u_1 : X_1 \to X_2$, $u_2 : Y_1 \to Y_2$ *and* $p : E \to E'$ *is said to be*

(1) *a binary soft injective (respectively binary soft surjective, binary soft constant) if all* u_1, u_2, p *are one-one (respectively onto, constant) functions.*

(2) *a binary soft bijection if* f *is both binary soft injective and binary soft surjective.*

Remark 3.14. A binary soft function $f : (G, E) \to (H, E')$ is not binary soft injective (respectively binary soft surjective, binary soft constant) if any one of the functions u_1, u_2, p is not one-one (respectively onto, constant).

Definition 3.15. *A binary soft function* $f : (G, E) \to (G, E)$ *with* $G(e) = (X, Y)$ *for each* $e \in E$ *along with functions* $u_1 : X \to X$, $u_2 : Y \to Y$ *and* $p : E \to E$ *is said to be binary soft identity if all* u_1, u_2, p *are identity functions. Or equivalently,* $f(F, A) = (F, A)$ *for any* $(F, A) \subseteq (G, E)$.

4. Binary Soft Functions in Binary Soft Topological Spaces

Binary soft functions such as binary soft continuous functions, binary soft open maps, binary soft closed maps and binary soft homeomorphisms are defined in this section. Further, their properties and characterizations are obtained.

Definition 4.1. *Let* (Ψ_1, E) *be a binary soft subset of* (U_1, U_2, τ_b, E) *and* $(x, y) \in U_1 \times U_2$. *Then* (Ψ_1, E) *is said to be a binary soft neighbourhood (briefly binary soft nbd) of* (x, y), *if there exists* $(\Psi_2, E) \in \tau_b$ *such that* $(x, y) \in (\Psi_2, E) \subseteq (\Psi_1, E)$. *Here, the binary point* (x, y) *is called a binary soft interior point of* (Ψ_1, E).

Definition 4.2. *For any binary soft subsets* (Ψ_1, E) *and* (Ψ_2, E) *of* (U_1, U_2, τ_b, E), (Ψ_1, E) *is said to be a binary soft nbd of* (Ψ_2, E), *if there exists* $(\Psi, E) \in \tau_b$ *such that* $(\Psi_2, E) \subseteq (\Psi, E) \subseteq (\Psi_1, E)$.

Definition 4.3. *Let* (U_1, U_2, τ_{b1}, E) *and* $(V_1, V_2, \tau_{b2}, E')$ *be any two binary soft topological spaces. A binary soft function* $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ *is said to be a binary soft continuous function, if for every* $(H, E') \in \tau_{b2}$, $f^{-1}(H, E') \in \tau_{b1}$.

Example 4.4. Let $U_1 = \{a_1, a_2, a_3\}$, $U_2 = \{b_1, b_2\}$, $E = \{1, 2\}$, $\tau_{b1} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E}}, (F_1,$

$E), (F_2, E), (F_3, E), (F_4, E)\}$ with

$$(F_1, E) = \{(1, (\{a_2\}, \{b_1, b_2\})), (2, (\{a_2\}, \{b_2\}))\},$$
$$(F_2, E) = \{(1, (\{a_1, a_3\}, \emptyset)), (2, (\{a_1, a_3\}, \{b_1\}))\},$$
$$(F_3, E) = \{(1, (\{a_1, a_2\}, \{b_1, b_2\})), (2, (\{a_1, a_2\}, \{b_1, b_2\}))\},$$
$$(F_4, E) = \{(1, (\{a_1\}, \emptyset)), (2, (\{a_1\}, \{b_1\}))\}.$$

Also, let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $E' = \{i, ii\}$, $\tau_{b2} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E'}}, (G_1, E'), (G_2, E')\}$ with $(G_1, E') = \{(i, (\{x_2\}, \{y_2\})), (ii, (\{x_2\}, \{y_1, y_2\}))\}$, $(G_2, E') = \{(i, (\{x_1\}, \{y_1\})), (ii, (\{x_1\}, \emptyset))\}$.

Then, $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ with $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$ and $p : E \to E'$, is binary soft continuous, which is defined as:

$u_1(a_1) = u_1(a_3) = x_1$, $u_1(a_2) = x_2$; $u_2(b_1) = y_1$, $u_2(b_2) = y_2$; $p(1) = ii$, $p(2) = i$.

Theorem 4.5. *Let $f : (U_1, U_2, \tau_{b1}, E_1) \to (V_1, V_2, \tau_{b2}, E_2)$ be a binary soft function. Then the following statements are equivalent:*

(1) *f is binary soft continuous.*

(2) *For each $(x, y) \in U_1 \times U_2$ and for each binary soft nbd (H, E_2) of $f(x, y)$, there exists a binary soft nbd (G, E_1) of (x, y) such that $f(G, E_1) \subseteq (H, E_2)$.*

(3) *For each binary soft closed set (F, E_2) in $(V_1, V_2, \tau_{b2}, E_2)$, $f^{-1}(F, E_2)$ is binary soft closed in $(U_1, U_2, \tau_{b1}, E_1)$.*

(4) *$f\left(\overline{\overline{(A, E_1)}}\right) \subseteq \overline{\overline{f(A, E_1)}}$ for any binary soft subset (A, E_1) over U_1, U_2.*

(5) *$\overline{\overline{f^{-1}(B, E_2)}} \subseteq f^{-1}\left(\overline{\overline{(B, E_2)}}\right)$ for any binary soft subset (B, E_2) over V_1, V_2.*

(6) *$f^{-1}\left((B, E_2)^{\circledcirc}\right) \subseteq \left(f^{-1}(B, E_2)\right)^{\circledcirc}$ for any binary soft subset (B, E_2) over V_1, V_2.*

Proof. (1)⇒(2) Let $(x, y) \in U_1 \times U_2$ and (H, E_2) be a binary soft nbd of $f(x, y)$ in $(V_1, V_2, \tau_{b2}, E_2)$. Therefore, $(x, y) \in f^{-1}(H, E_2)$, which implies there exists some $(G, E_1) \in \tau_{b1}$ such that $(x, y) \in (G, E_1) \subseteq f^{-1}(H, E_2)$. Hence, $f(G, E_1) \subseteq (H, E_2)$.

(2)⇒(1) Let $(H, E_2) \in \tau_{b2}$ and $(x, y) \in f^{-1}(H, E_2)$. Therefore, $f(x, y) \in (H, E_2)$. By (2), $(x, y) \in (G, E_1)$ and $f(G, E_1) \subseteq (H, E_2)$ for some (G, E_1) over U_1, U_2. That is, $(x, y) \in (G, E_1) \subseteq (H, E_2)$. Hence, $f^{-1}(H, E_2) \in \tau_{b1}$.

(1)⇒(3) Let (F, E_2) be a binary soft closed set in $(V_1, V_2, \tau_{b2}, E_2)$. Therefore, $(F, E_2)' \in \tau_{b2}$. From (1), $f^{-1}(F, E_2)' \in \tau_{b1}$, which proves (3).

(3)⇒(4) Let (A, E_1) be any binary soft subset over U_1, U_2. We know that $(A, E_1) \subseteq f^{-1}(f(A, E_1))$ and $f(A, E_1) \subseteq \overline{\overline{f(A, E_1)}}$. Therefore, $(A, E_1) \subseteq f^{-1}\left(\overline{\overline{f(A, E_1)}}\right)$. Since $f^{-1}\left(\overline{\overline{f(A, E_1)}}\right)$ is binary soft closed in $(U_1, U_2, \tau_{b1}, E_1)$, $\overline{\overline{(A, E_1)}} \subseteq f^{-1}\left(\overline{\overline{f(A, E_1)}}\right)$. Hence, $f\left(\overline{\overline{(A, E_1)}}\right) \subseteq \overline{\overline{f(A, E_1)}}$.

(4)⇒(5) Let (B, E_2) be any binary soft subset over V_1, V_2. Therefore, $f^{-1}(B, E_2)$ is a binary soft subset over U_1, U_2. From (4), $f\left(\overline{\overline{f^{-1}(B, E_2)}}\right) \subseteq \overline{\overline{f\left(f^{-1}(B, E_2)\right)}} \subseteq \overline{\overline{(B, E_2)}}$. Hence, $\overline{\overline{f^{-1}(B, E_2)}} \subseteq f^{-1}\overline{\overline{(B, E_2)}}$.

(5)⇒(6) Let (B, E_2) be any binary soft subset over V_1, V_2. We know that $(B, E_2)^{\circledcirc} = \left(\overline{\overline{(B, E_2)'}}\right)'$. Therefore, $f^{-1}(B, E_2)^{\circledcirc} = f^{-1}\left(\overline{\overline{(B, E_2)'}}\right)' = \left(f^{-1}\left(\overline{\overline{(B, E_2)'}}\right)\right)' \subseteq \left(\overline{\overline{f^{-1}(B, E_2)'}}\right)' = \left(\overline{\overline{(f^{-1}(B, E_2))'}}\right)' = \left(f^{-1}(B, E_2)\right)^{\circledcirc}$, hence (6).

(6)⇒(1) Let $(H, E_2) \in \tau_{b2}$. We know that $\left(f^{-1}(H, E_2)\right)^{\circledcirc} \subseteq f^{-1}(H, E_2)$. Now, $f^{-1}(H, E_2) = f^{-1}\left((H, E_2)^{\circledcirc}\right) \subseteq \left(f^{-1}(H, E_2)\right)^{\circledcirc}$, from (6). Therefore, $f^{-1}(H, E_2) = \left(f^{-1}(H, E_2)\right)^{\circledcirc}$. Hence, $f^{-1}(H, E_2) \in \tau_{b1}$. ■

Theorem 4.6. *$f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is a binary soft continuous function if and only if for each $e \in E$, $f_e : (U_i, \tau_e) \to (V_i, \tau_{p(e)})$, $i = 1, 2$, is continuous.*

Proof. Take $i = 1$. Let $A \in \tau_{p(e)}$. Then obviously, $A \subseteq V_1$. Therefore, there exists $B \subseteq V_2$ such that $(A, B) = G(e')$ for some $e' \in p(e)$, which implies $(G, E') \in \tau_{b2}$. Since f is binary soft continuous, $f^{-1}(G, E') \in \tau_{b1}$. Therefore, $f_e^{-1}(A, B) = f^{-1}(G)(e')$ with $p(e) = e'$ such that $f_e^{-1}(A) \subseteq U_1$, $f_e^{-1}(B) \subseteq U_2$ and $f^{-1}(A) \in \tau_e$. Hence, $f_e : (U_1, \tau_e) \to (V_1, \tau_{p(e)})$ is continuous. Similarly, for $i = 2$.

Conversely, let f_e be continuous for each $e \in E$. Let $(G, E') \in \tau_{b2}$. Therefore, $G(e') = (A, B)$, where $A \subseteq V_1$, $B \subseteq V_2$. Since f_e is continuous for each $e \in E$, f_e is continuous for those $e' \in p(e)$. Therefore, for an open set A in $(V_1, \tau_{p(e)})$, there exists $f_e^{-1}(A) \subseteq U_1$ such that $f_e^{-1}(A) \in \tau_e$. Similarly, for an open set B in $(V_2, \tau_{p(e)})$, there exists $f_e^{-1}(B) \subseteq U_2$ such that $f_e^{-1}(B) \in \tau_e$. Therefore, $f_e^{-1}(A, B) \subseteq (U_1, U_2)$, where $p(e) = e'$ and $f^{-1}(G, E') \in \tau_{b1}$. Hence, f is binary soft continuous. ■

Definition 4.7. *Let (U_1, U_2, τ_{b1}, E) and $(V_1, V_2, \tau_{b2}, E')$ be two binary soft topological spaces. Then $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is said to be*

(1) *a binary soft open map, if for each $(G, E) \in \tau_{b1}$, $f(G, E) \in \tau_{b2}$.*
(2) *a binary soft closed map, if for each binary soft closed set (F, E) in (U_1, U_2, τ_{b1}, E), $f(G, E)$ is binary soft closed in $(V_1, V_2, \tau_{b2}, E')$.*

Proposition 4.8. *$f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is binary soft open (respectively binary soft closed), then for each $e \in E$, $f_e : (U_i, \tau_e) \to (V_i, \tau_{p(e)})$, $i = 1, 2$, is open (respectively closed).*

Theorem 4.9. *$f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is binary soft open if and only if for any binary soft subset (G, E) over U_1, U_2, $f\left((G, E)^{\circledcirc}\right) \subseteq (f(G, E))^{\circledcirc}$.*

Proof. Let f be binary soft open, and (G,E) be any binary soft subset over U_1, U_2. We know that $(G,E)^{\circledcirc} \in \tau_{b1}$. Therefore, $f\left((G,E)^{\circledcirc}\right) \in \tau_{b2}$. Now, $(G,E)^{\circledcirc} \subseteq (G,E)$, implies $f\left((G,E)^{\circledcirc}\right) \subseteq (f(G,E))$. Hence, $f\left((G,E)^{\circledcirc}\right) \subseteq (f(G,E))^{\circledcirc}$.

Conversely, let $(G,E) \in \tau_{b1}$. Therefore, $(G,E) = (G,E)^{\circledcirc}$. Now, $f(G,E) = f\left((G,E)^{\circledcirc}\right) \subseteq (f(G,E))^{\circledcirc} \subseteq f(G,E)$. ■

Theorem 4.10. $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ *is binary soft closed if and only if for any binary soft subset* (G,E) *over* U_1, U_2, $\overline{\overline{f(G,E)}} \subseteq f\overline{\overline{(G,E)}}$.

Proof. Let f be binary soft closed, and (G,E) be any binary soft subset over U_1, U_2. We know that $\overline{\overline{(G,E)}}$ is binary soft closed in (U_1, U_2, τ_{b1}, E). Therefore, $f\overline{\overline{(G,E)}}$ is binary soft closed in $(V_1, V_2, \tau_{b2}, E')$. Now, $(G,E) \subseteq \overline{\overline{(G,E)}}$, implies $f(G,E) \subseteq f\overline{\overline{(G,E)}}$. Hence, $\overline{\overline{f(G,E)}} \subseteq f\overline{\overline{(G,E)}}$.

Conversely, let (G,E) be binary soft closed in (U_1, U_2, τ_{b1}, E). Therefore, $(G,E) = \overline{\overline{(G,E)}}$. Now, $\overline{\overline{f(G,E)}} \subseteq f\overline{\overline{(G,E)}} = f(G,E) \subseteq \overline{\overline{f(G,E)}}$. ■

Definition 4.11. *A binary soft function* $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ *is said to be a binary soft homeomorphism, if* f *is a binary soft bijection, binary soft continuous and* f^{-1} *is also binary soft continuous.*

If there exists a binary soft homeomorphism between two binary soft topological spaces, then they are said to be binary soft homeomorphic.

Theorem 4.12. *If* $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ *is binary soft bijection, then the following statements are equivalent:*

(1) f *is a binary soft homeomorphism.*
(2) f *is binary soft open and binary soft continuous.*
(3) f *is binary soft closed and binary soft continuous.*

Remark 4.13. The concepts of binary soft continuous, binary soft closed and binary soft open are independent.

Example 4.14. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $E = \{1, 2\}$ and $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $E' = \{i, ii\}$ with functions $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$, $p : E \to E'$ defined as:

$u_1(a_1) = x_2$, $u_1(a_2) = x_1$; $u_2(b_1) = y_1$, $u_2(b_2) = y_2$; $p(1) = i$, $p(2) = ii$.

Let $\tau_{b1} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_2\}))\}, \{(1, (\{a_2\}, \{b_2\})), (2, (\emptyset, \{b_1\}))\}\}$ and $\tau_{b2} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E'}}, \{(i, (\{x_2\}, \{y_1\})), (ii, (\{x_1, x_2\}, \{y_2\}))\}\}$.

Then $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is binary soft continuous, but f is neither binary soft open nor binary soft closed.

Example 4.15. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $E = \{1, 2\}$ and $V_1 = \{x_1, x_2\}$, $V_2 = \{y\}$, $E' = \{i, ii\}$ with functions $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$, $p : E \to E'$ defined as:

$u_1(a_1) = x_1$, $u_1(a_2) = x_2$; $u_2(b_1) = u_2(b_2) = y$; $p(1) = i$, $p(2) = ii$.

Let $\tau_{b1} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_1, a_2\}, \{b_2\}))\}, \{(1, (\{a_2\}, \{b_2\})), (2, (\emptyset, \{b_1\}))\}\}$ and $\tau_{b2} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E'}}, \{(i, (\{x_1\}, \{y\})), (ii, (\{x_1, x_2\}, \{y\}))\}, \{(i, (\{x_2\}, \{y\})), (ii, (\emptyset, \{y\}))\}, \{(i, (\emptyset, \{y\})), (ii, (\emptyset, \{y\}))\}\}$.

Then $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is binary soft open, but f is neither binary soft continuous nor binary soft closed.

Example 4.16. Let $U_1 = \{a_1, a_2\}$, $U_2 = \{b_1, b_2\}$, $E = \{1, 2\}$ and $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $E' = \{i, ii\}$ with functions $u_1 : U_1 \to V_1$, $u_2 : U_2 \to V_2$, $p : E \to E'$ defined as:

$u_1(a_1) = u_1(a_2) = x_1$; $u_2(b_1) = y_1$, $u_2(b_2) = y_1$; $p(1) = i$, $p(2) = ii$.

Let $\tau_{b1} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E}}, \{(1, (\{a_1\}, \{b_1\})), (2, (\{a_2\}, \emptyset))\}\}$ and $\tau_{b2} = \{\widetilde{\widetilde{\emptyset}}, \widetilde{\widetilde{E'}}, \{(i, (\{x_2\}, \{y_2\})), (ii, (\{x_2\}, \emptyset))\}, \{(i, (\{x_2\}, \emptyset)), (ii, (\{x_2\}, \emptyset))\}\}$.

Then $f : (U_1, U_2, \tau_{b1}, E) \to (V_1, V_2, \tau_{b2}, E')$ is binary soft closed, but f is neither binary soft continuous nor binary soft open.

5. Applications

The concept of binary soft continuity is helpful to study the notions of compactness and connectedness in binary soft topological spaces. Also, binary soft homeomorphisms are applied to verify the topological properties of binary soft topological spaces.

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