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Optimality Conditions and Saddle Point Criteria for Fractional Interval-Valued Optimization Problem via Convexificator

B. Japamala Rani
Department of Mathematics, School of Science, GITAM-Hyderabad Campus, Hyderabad-502329, India
Department of Mathematics, St. Ann's College for Women-Mehdipatnam, Hyderabad-500028, India
Email: bjapamalarani@gmail.com

Krishna Kummari*
Department of Mathematics, School of Science, GITAM-Hyderabad Campus, Hyderabad-502329, India
Email: krishna.maths@gmail.com

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Abstract. In this work, we use the notion of convexificators to discuss optimality conditions for a fractional interval-valued optimization problem. We illustrate the sufficient optimality conditions established in the paper by the example of a nonconvex fractional interval-valued optimization problem with the help of generalized invex functions. Further, we study saddle point criteria of a Lagrange function defined for a fractional interval-valued optimization problem.

Keywords: Convexificator; Fractional problem; LU optimal solution; Lagrange functions; Saddle point.

1. Introduction

In the last two decades, a large number of research has been devoted for solving fractional programming problems. This follows from the fact that optimization problems with the objective function of ratio of two functions have a wide range

*Corresponding author

of applications in engineering and economics, game theory, and many more (cf. [12, 18, 22]).

Interval-valued optimization problem is used to tackle interval uncertainity that appears in many real world mathematical problems. For example, it is applied to solve the fixed-charge transportation problem [15], chemical engineering problem [17] and municipal solid waste management [20], etc. Interval-valued programming problem was first studied by Ben-Israel and Robers [2]. Wu [21] formulated four kinds of interval-valued optimization problems and discussed optimality conditions. Further, they also established duality results to relate the primal and dual problems. Singh et al. [16] proposed a theoretical and practical solution method for a multiobjective interval-valued programming problem. In the recent past, many mathematicians have shown their interest to study different types of interval-valued programming problems [1, 3, 11, 14, 19].

The notion of convexificators introduced by Demyanov [6] and extended further by Jeyakumar and Luc [10]. Convexificators can be viewed as weaker versions of the notion of subdifferentials as they are in general closed sets unlike the well-known subdifferentials which are convex and compact sets. In literature, a lot of research has been carried out for convexificators regarding its theoretical properties (see, e.g. [5, 7, 8, 11, 13] and the references therein). Recently, making use of these notions, Karush-Kuhn-Tucker necessary optimality conditions for local weak efficient solutions were established by Hejazi and Nobakhtian [9] for a multiobjective fractional programming problem. Also, Hejazi and Nobakhtian [9] gave some constraint qualifications and subsequently they discussed relationship between these constraint qualifications.

In this paper, by using the idea of convexificators, we study optimality conditions for a fractional interval-valued optimization problem. Further, we establish equivalence between the saddle point and LU optimal solution of the fractional interval-valued optimization problem involving generalized invex functions.

2. Preliminaries

In this section, we give a number of basic definitions and lemmas which will be used in the paper. Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. Throughout this paper, we shall be concerned with Banach spaces. Let X^* be topological dual of a given Banach space X with the canonical dual pairing $\langle .,. \rangle$. Let X and Y be Banach spaces and we denote by $L(X,Y)$ the set of continuous linear mappings between X and Y.

Let $f : X \to R \cup \{+\infty\}$, be an extended real-valued function. Then

$$f^-(x,d) = \lim_{t \to 0+} \inf \frac{f(x+td) - f(x)}{t},$$
$$f^+(x,d) = \lim_{t \to 0+} \sup \frac{f(x+td) - f(x)}{t}$$

denote, respectively, the lower and upper Dini directional derivatives of f at

$x \in X$ in the direction of d.

Now, we begin with the definition of convexificator given by Jeyakumar and Luc [10].

Definition 2.1. *A function $f : X \to R \cup \{+\infty\}$ is said to have a convexificator $\partial^* f(x)$ at x if $\partial^* f(x) \subset X^*$ is weak* closed and*

$$f^+(x,d) \geq \inf_{x^* \in \partial^* f(x)} \langle x^*, d\rangle \text{ and } f^-(x,d) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, d\rangle, \ \forall d \in X.$$

Along the lines of Gadhi [8], we now give the definitions of generalized invex functions by using the concept of convexificators. Assume that $f : X \to R$ admits a convexificator $\partial^* f(\bar{x}) \subset L(X, R)$ at $\bar{x} \in X$.

Definition 2.2. *A function $f : X \to R$ is said to be $(\eta, \partial^* f)$-invex at $\bar{x} \in X$ if there exists $\eta : X \times X \to X$ such that,*

$$f(x) - f(\bar{x}) \geq \langle \xi, \eta(x, \bar{x})\rangle, \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X.$$

If strict inequality holds in above definition for $x \neq \bar{x}$, then f is said to be strict $(\eta, \partial^* f)$-invex at $\bar{x}$.

Definition 2.3. *A function $f : X \to R$ is said to be $(\eta, \partial^* f)$-pseudoinvex at $\bar{x} \in X$ if there exists $\eta : X \times X \to X$ such that,*

$$f(x) < f(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x})\rangle < 0, \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X,$$

equivalently

$$\langle \xi, \eta(x, \bar{x})\rangle \geq 0 \Rightarrow f(x) \geq f(\bar{x}), \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X.$$

Definition 2.4. *A function $f : X \to R$ is said to be strict $(\eta, \partial^* f)$-pseudoinvex at $\bar{x} \in X$ if there exists $\eta : X \times X \to X$ such that,*

$$f(x) \leq f(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x})\rangle < 0, \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X,$$

equivalently

$$\langle \xi, \eta(x, \bar{x})\rangle \geq 0 \Rightarrow f(x) > f(\bar{x}), \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X.$$

Definition 2.5. *A function $f : X \to R$ is said to be $(\eta, \partial^* f)$-quasiinvex at $\bar{x} \in X$ if there exists $\eta : X \times X \to X$ such that,*

$$f(x) \leq f(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x}\rangle \leq 0, \text{ for all } \xi \in \partial^* f(\bar{x}) \text{ and } x \in X,$$

equivalently

$$\langle \xi, \eta(x,\bar{x})\rangle > 0 \Rightarrow f(x) > f(\bar{x}),\ \textit{for all}\ \xi \in \partial^* f(\bar{x})\ \textit{and}\ x \in X.$$

In order to proceed further, we need the following fundamental concepts of interval mathematics:

Let $\frac{\mathbb{A}}{\mathbb{B}} = \left[\frac{\alpha_1^L}{\gamma_1^L}, \frac{\alpha_1^U}{\gamma_1^U}\right]$ and $\frac{\mathbb{C}}{\mathbb{D}} = \left[\frac{\alpha_2^L}{\gamma_2^L}, \frac{\alpha_2^U}{\gamma_2^U}\right]$ be two fractional closed intervals with $\frac{\alpha_1^L}{\gamma_1^L} \le \frac{\alpha_1^U}{\gamma_1^U}$ and $\frac{\alpha_2^L}{\gamma_2^L} \le \frac{\alpha_2^U}{\gamma_2^U}$, $\gamma_1^L, \gamma_1^U, \gamma_2^L, \gamma_2^U \neq 0$.

(i) $\frac{\mathbb{A}}{\mathbb{B}} + \frac{\mathbb{C}}{\mathbb{D}} = \left[\frac{\alpha_1^L}{\gamma_1^L} + \frac{\alpha_2^L}{\gamma_2^L}, \frac{\alpha_1^U}{\gamma_1^U} + \frac{\alpha_2^U}{\gamma_2^U}\right]$,

(ii) $\frac{-\mathbb{A}}{\mathbb{B}} = \left[\frac{-\alpha_1^U}{\gamma_1^U}, \frac{-\alpha_1^L}{\gamma_1^L}\right]$,

(iii) $\frac{\mathbb{A}}{\mathbb{B}} - \frac{\mathbb{C}}{\mathbb{D}} = \frac{\mathbb{A}}{\mathbb{B}} + \left(\frac{-\mathbb{C}}{\mathbb{D}}\right) = \left[\frac{\alpha_1^L}{\gamma_1^L} - \frac{\alpha_2^U}{\gamma_2^U}, \frac{\alpha_1^U}{\gamma_1^U} - \frac{\alpha_2^L}{\gamma_2^L}\right]$,

(iv) $\beta\left(\frac{\mathbb{A}}{\mathbb{B}}\right) = \begin{cases} \left[\frac{\alpha_1^L}{\gamma_1^L}, \frac{\alpha_1^U}{\gamma_1^U}\right], & \text{if } \beta \ge 0, \\ \left[\frac{\alpha_1^U}{\gamma_1^U}, \frac{\alpha_1^L}{\gamma_1^L}\right], & \text{if } \beta < 0. \end{cases}$

An order relation $\le_{LU}$ between two intervals $\frac{\mathbb{A}}{\mathbb{B}}$ and $\frac{\mathbb{C}}{\mathbb{D}}$ are defined as

(i) $\frac{\mathbb{A}}{\mathbb{B}} \le_{LU} \frac{\mathbb{C}}{\mathbb{D}}$ iff $\frac{\alpha_1^L}{\gamma_1^L} \le \frac{\alpha_2^L}{\gamma_2^L}$ and $\frac{\alpha_1^U}{\gamma_1^U} \le \frac{\alpha_2^U}{\gamma_2^U}$.

(ii) $\frac{\mathbb{A}}{\mathbb{B}} < \frac{\mathbb{C}}{\mathbb{D}}$ iff $\frac{\mathbb{A}}{\mathbb{B}} \le \frac{\mathbb{C}}{\mathbb{D}}$ and $\frac{\mathbb{A}}{\mathbb{B}} \neq \frac{\mathbb{C}}{\mathbb{D}}$, equivalently

$$\begin{cases} \frac{\alpha_1^L}{\gamma_1^L} < \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} \le \frac{\alpha_2^U}{\gamma_2^U} \end{cases}, \text{ or } \begin{cases} \frac{\alpha_1^L}{\gamma_1^L} \le \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} < \frac{\alpha_2^U}{\gamma_2^U} \end{cases}, \text{ or } \begin{cases} \frac{\alpha_1^L}{\gamma_1^L} < \frac{\alpha_2^L}{\gamma_2^L} \\ \frac{\alpha_1^U}{\gamma_1^U} < \frac{\alpha_2^U}{\gamma_2^U}. \end{cases}$$

Consider the following non-differentiable fractional interval-valued optimization problem:

$$\begin{aligned} &\min \left[\frac{f^L(x), f^U(x)}{g^L(x), g^U(x)}\right] \\ &\text{subject to} \\ &h_i(x) \le 0, i = 1, 2, ..., m, \\ &x \in X, \end{aligned}$$

which further reduces to the problem

$$\min \left[\frac{f^L(x)}{g^U(x)}, \frac{f^U(x)}{g^L(x)}\right]$$
$$\text{subject to}$$
$$h_i(x) \leq 0, i = 1, 2, ..., m,$$
$$x \in X,$$

where $f^L(x),\ f^U(x) \geq 0,\ g^L(x),\ g^U(x) > 0$, and $h_i,\ i = 1, 2, \ldots, m$ are continuous functions on X. Set $f^L = p^L,\ g^U = q^L,\ f^U = p^U,\ g^L = q^U$. Then, the above problem reduces to

$$\text{(NFIVP)} \qquad \min \left[\frac{p^L}{q^L}(x), \frac{p^U}{q^U}(x)\right]$$
$$\text{subject to}$$
$$h_i(x) \leq 0, i = 1, 2, ..., m,$$
$$x \in X.$$

Let $\mathbb{F}$ be the feasible set for the problem (NFIVP).

Definition 2.6. [21] *A feasible point $\bar{x}$ is said to be a LU optimal solution for* (NFIVP) *if and only if there exists no feasible point x such that*

$$\left[\frac{p^L}{q^L}(x), \frac{p^U}{q^U}(x)\right] <_{LU} \left[\frac{p^L}{q^L}(\bar{x}), \frac{p^U}{q^U}(\bar{x})\right].$$

3. Optimality Conditions

For the given feasible solution $\bar{x}$, consider two fractional problems as given below:

$$\text{(FP1)} \qquad \min \phi^L(x) = \frac{p^L}{q^L}(x)$$
$$\text{subject to}$$
$$h_i(x) \leq 0,\ i = 1, 2, \ldots, m,$$
$$\frac{p^U}{q^U}(x) \leq \frac{p^U}{q^U}(\bar{x}),$$
$$x \in X.$$

$$\text{(FP2)} \qquad \min \phi^U(x) = \frac{p^U}{q^U}(x)$$
$$\text{subject to}$$
$$h_i(x) \leq 0,\ i = 1, 2, \ldots, m,$$
$$\frac{p^L}{q^L}(x) \leq \frac{p^L}{q^L}(\bar{x}),$$
$$x \in X.$$

The following result gives the relationship between (NFIVP) and (FP1) and (FP2).

Lemma 3.1. [4] *If $\bar{x}$ is a LU optimal solution for the problem (NFIVP) if and only if $\bar{x}$ is an optimal solution for the problems (FP1) and (FP2).*

Lemma 3.2. [4] $\bar{x}$ *is a LU optimum of the problem (NFIVP) if and only if* $\bar{x}$ *minimizes* $\frac{p^L}{q^L}(x)$ *on the following constraint set*

$$N = \left\{ x \in X | \frac{p^U}{q^U}(x) \leq \frac{p^U}{q^U}(\bar{x}), h_i(x) \leq 0, i = 1, 2, \ldots, m \right\}.$$

Considered the following single-objective fractional problem:

$$\text{(D)} \qquad \min \phi(x) = \frac{p_1}{q_1}(x)$$
$$\text{subject to}$$
$$\ell_i(x) \leq 0, i = 1, 2, ..., m,$$
$$x \in X,$$

where p_1, q_1 and ℓ_i, $i = 1, 2, ..., m$ are continuous functions on X such that $p_1(x) \geq 0$ and $q_1(x) > 0$, for all $x \in X$.

On the lines of Theorem 6 of Gadhi [8], we state the following theorem for the problem (D):

Theorem 3.3. *Suppose that* $\bar{x}$ *is an optimal solution of the problem (D) and a suitable constraint qualification is satisfied at* $\bar{x}$*. Assume that* p_1*,* q_1 *and* ℓ_i*,* $i = 1, 2, ..., m$ *are continuous and admit bounded convexificators* $\partial^* p_1(\bar{x})$*,* $\partial^* q_1(\bar{x})$ *and* $\partial^* \ell_i(\bar{x})$*,* $i = 1, 2, ..., m$ *at* $\bar{x}$ *respectively and that* $\partial^* p_1(\bar{x})$*,* $\partial^* q_1(\bar{x})$ *and* $\partial^* \ell_i(\bar{x})$*,* $i = 1, 2, ..., m$ *are upper semicontinuous at* $\bar{x}$*, then there exist* $\lambda > 0$*, and* $\mu \in R^m_+$ *such that*

$$0 \in \lambda \left(\partial^* p_1(\bar{x}) - \phi(\bar{x}) \partial^* q_1(\bar{x}) \right) + \sum_{i=1}^{m} \mu_i \partial^* \ell_i(\bar{x}), \tag{1}$$

$$\mu_i \ell_i(\bar{x}) = 0, j = 1, 2, ..., m, \tag{2}$$

$$\mu_i \geq 0 \text{ and } \ell_i(\bar{x}) \leq 0, \; i = 1, 2, ..., m. \tag{3}$$

Theorem 3.4. (Karush-Kuhn-Tucker Necessary Optimality Conditions) *Suppose that* $\bar{x}$ *is a LU optimal solution of the problem* (NFIVP) *and a suitable constraint qualification is satisfied at* $\bar{x}$*. Assume that* p^L*,* q^L*,* p^U*,* q^U *and* h_i*,* $i = 1, 2, ..., m$ *are continuous and admit bounded convexificators* $\partial^* p^L(\bar{x})$*,* $\partial^* q^L(\bar{x})$*,* $\partial^* p^U(\bar{x})$*,* $\partial^* q^U(\bar{x})$ *and* $\partial^* h_i(\bar{x})$*,* $i = 1, 2, ..., m$ *at* $\bar{x}$ *respectively and that* $\partial^* p^L(\bar{x})$*,* $\partial^* q^L(\bar{x})$*,* $\partial^* p^L(\bar{x})$*,* $\partial^* q^L(\bar{x})$*, and* $\partial^* h_i(\bar{x})$*,* $i = 1, 2, ..., m$ *are upper semicontinuous at* $\bar{x}$*, then there exist* $\lambda^L > 0$*,* $\lambda^U > 0$ *and* $\mu \in R^m_+$ *such that*

$$\begin{aligned} 0 \in\, & \lambda^L \left(\partial^* p^L(\bar{x}) - \phi^L(\bar{x}) \partial^* q^L(\bar{x}) \right) \\ & + \lambda^U \left(\partial^* p^U(\bar{x}) - \phi^U(\bar{x}) \partial^* q^U(\bar{x}) \right) + \sum_{i=1}^{m} \mu_i \partial^* h_i(\bar{x}), \end{aligned} \tag{4}$$

$$\mu_i h_i(\bar{x}) = 0, j = 1, 2, ..., m, \tag{5}$$

$$\mu_i \geq 0 \text{ and } h_i(\bar{x}) \leq 0, \; i = 1, 2, ..., m. \tag{6}$$

Proof. By assumption, $\bar{x}$ is a LU optimal solution for the problem (NFIVP), and a suitable constraint qualification is satisfied at $\bar{x}$. Since $\bar{x}$ is an LU optimal solution, by Lemma 3.1, $\bar{x}$ is also a optimal solution for the problems (FP1) and (FP2). Hence, by Lemma 3.2, at $\bar{x}$ the minimum value of $\frac{p^L}{q^L}(x)$ is obtained on the constraint set

$$N_L = \left\{ x \in X | \frac{p^U}{q^U}(x) \leq \frac{p^U}{q^U}(\bar{x}), h_i(x) \leq 0, i = 1, 2, ..., m \right\},$$

and the minimum value of $\frac{p^U}{q^U}(x)$ is obtained at $\bar{x}$ on the constraint set

$$N_U = \left\{ x \in X | \frac{p^L}{q^L}(x) \leq \frac{p^L}{q^L}(\bar{x}), h_i(x) \leq 0, i = 1, 2, ..., m \right\}.$$

By Theorem 3.3, it follows that there exist $\lambda^{LL} > 0, \lambda^{LU} > 0, \mu^L \in R^m_+$ and $\lambda^{UL} > 0, \lambda^{UU} > 0, \mu^U \in R^m_+$ such that

$$0 \in \lambda^{LL} \left(\partial^* p^L(\bar{x}) - \phi^L(\bar{x}) \partial^* q^L(\bar{x}) \right)$$
$$+ \lambda^{LU} \left(\partial^* p^U(\bar{x}) - \phi^U(\bar{x}) \partial^* q^U(\bar{x}) \right) + \sum_{i=1}^m \mu_i^L \partial^* h_i(\bar{x}), \tag{7}$$
$$\mu_i^L h_i(\bar{x}) = 0, j = 1, 2, ..., m, \tag{8}$$
$$\mu_i^L \geq 0 \text{ and } h_i(\bar{x}) \leq 0, \ i = 1, 2, ..., m. \tag{9}$$

and

$$0 \in \lambda^{UL} \left(\partial^* p^L(\bar{x}) - \phi^L(\bar{x}) \partial^* q^L(\bar{x}) \right)$$
$$+ \lambda^{UU} \left(\partial^* p^U(\bar{x}) - \phi^U(\bar{x}) \partial^* q^U(\bar{x}) \right) + \sum_{i=1}^m \mu_i^U \partial^* h_i(\bar{x}), \tag{10}$$
$$\mu_i^U h_i(\bar{x}) = 0, j = 1, 2, ..., m, \tag{11}$$
$$\mu_i^U \geq 0 \text{ and } h_i(\bar{x}) \leq 0, \ i = 1, 2, ..., m. \tag{12}$$

From (7) to (12), we have

$$0 \in [\lambda^{LL} + \lambda^{UL}] \left(\partial^* p^L(\bar{x}) - \phi^L(\bar{x}) \partial^* q^L(\bar{x}) \right)$$
$$+ [\lambda^{LU} + \lambda^{UU}] \left(\partial^* p^U(\bar{x}) - \phi^U(\bar{x}) \partial^* q^U(\bar{x}) \right) + \sum_{i=1}^m [\mu_i^L + \mu_i^U] \partial^* h_i(\bar{x}), \tag{13}$$
$$[\mu_i^L + \mu_i^U] h_i(\bar{x}) = 0, j = 1, 2, ..., m, \tag{14}$$
$$[\mu_i^L + \mu_i^U] \geq 0 \text{ and } h_i(\bar{x}) \leq 0, \ i = 1, 2, ..., m. \tag{15}$$

Let us denote $\lambda^{LL} + \lambda^{UL} = \lambda^L, \ \lambda^{LU} + \lambda^{UU} = \lambda^U$ and $\mu^L + \mu^U = \mu$. Thus, from

(13)-(15), it yields

$$0 \in \lambda^L \left(\partial^* p^L(\bar{x}) - \phi^L(\bar{x})\partial^* q^L(\bar{x})\right)$$
$$+\lambda^U \left(\partial^* p^U(\bar{x}) - \phi^U(\bar{x})\partial^* q^U(\bar{x})\right) + \sum_{i=1}^{m} \mu_i \partial^* h_i(\bar{x}),$$
$$\mu_i h_i(\bar{x}) = 0, j = 1, 2, ..., m,$$
$$\mu_i \geq 0 \text{ and } h_i(\bar{x}) \leq 0, \ i = 1, 2, ..., m.$$

This completes the proof. ■

Theorem 3.5. (Sufficient Optimality Conditions) *Suppose that $\bar{x}$ is a feasible solution of* (NFIVP) *and there exist $\lambda^L > 0, \lambda^U > 0, \mu \in R^m_+$ such that* (4)–(6) *are satisfied at $\bar{x}$. Also, assume that*

(i) *$p^L(.) - \phi^L(\bar{x})q^L(.)$ and $p^U(.) - \phi^U(\bar{x})q^U(.)$ are respectively $(\eta, \partial^* p^L - \phi^L(\bar{x})\partial^* q^L)$-invex and $(\eta, \partial^* p^U - \phi^U(\bar{x})\partial^* q^U)$-invex at $\bar{x}$,*

(ii) *$\mu_i h_i$, for $i = 1, 2, ..., m$, is $(\eta, \partial^* h_i(.))$-invex at $\bar{x}$.*

Then $\bar{x}$ is a LU optimal solution for (NFIVP).

Proof. By assumption, (4)-(6) are satisfied at $\bar{x}$ with Lagrange multipliers $\lambda^L > 0, \lambda^U > 0, \mu \in R^m_+$. As it follows from (4), there exist $\xi^L \in \partial^* p^L(\bar{x})$, $\nu^L \in \partial^* q^L(\bar{x})$, $\xi^U \in \partial^* p^U(\bar{x})$, $\nu^U \in \partial^* q^U(\bar{x})$, and $\zeta_i \in \partial^* h_i(\bar{x})$, $i = 1, 2, ..., m$, such that

$$\lambda^L \left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \lambda^U \left[\xi^U - \phi^U(\bar{x})\nu^U\right] + \sum_{i=1}^{m} \mu_i \zeta_i = 0. \tag{16}$$

Suppose contrary to the result, that $\bar{x}$ is not a LU optimal solution for (NFIVP). Hence, by Definition 2.6, there exists a feasible solution x such that

$$\left[\frac{p^L}{q^L}(x), \frac{p^U}{q^U}(x)\right] <_{LU} \left[\frac{p^L}{q^L}(\bar{x}), \frac{p^U}{q^U}(\bar{x})\right]$$

that is $\begin{cases} \dfrac{p^L}{q^L}(x) < \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) \leq \dfrac{p^U}{q^U}(\bar{x}) \end{cases}$, or $\begin{cases} \dfrac{p^L}{q^L}(x) \leq \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) < \dfrac{p^U}{q^U}(\bar{x}) \end{cases}$, or $\begin{cases} \dfrac{p^L}{q^L}(x) < \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) < \dfrac{p^U}{q^U}(\bar{x}) \end{cases}$.

This implies

$$p^L(x) - \phi^L(\bar{x})q^L(x) < p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) \leq p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}),$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) \leq p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}),$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) < p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}).$$

From hypothesis (i), $p^L(.) - \phi^L(\bar{x})q^L(.)$ and $p^U(.) - \phi^U(\bar{x})q^U(.)$ are respectively $(\eta, \partial^* p^L - \phi^L(\bar{x})\partial^* q^L)$-invex and $(\eta, \partial^* p^U - \phi^U(\bar{x})\partial^* q^U)$-invex at $\bar{x}$ and therefore, there exists $\eta : X \times X \to X$ such that

$$\left\langle \left[\xi^L - \phi^L(\bar{x})\nu^L\right], \eta(x,\bar{x}) \right\rangle \leq 0, \text{ for all } \xi^L \in \partial^* p^L(\bar{x}), \text{ and } \nu^L \in \partial^* q^L(\bar{x}),$$

$$\left\langle \left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x,\bar{x}) \right\rangle < 0, \text{ for all } \xi^U \in \partial^* p^U(\bar{x}), \text{ and } \nu^U \in \partial^* q^U(\bar{x}),$$

or

$$\left\langle \left[\xi^L - \phi^L(\bar{x})\nu^L\right], \eta(x,\bar{x}) \right\rangle < 0, \text{ for all } \xi^L \in \partial^* p^L(\bar{x}), \text{ and } \nu^L \in \partial^* q^L(\bar{x}),$$

$$\left\langle \left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x,\bar{x}) \right\rangle \leq 0, \text{ for all } \xi^U \in \partial^* p^U(\bar{x}), \text{ and } \nu^U \in \partial^* q^U(\bar{x}),$$

or

$$\left\langle \left[\xi^L - \phi^L(\bar{x})\nu^L\right], \eta(x,\bar{x}) \right\rangle < 0, \text{ for all } \xi^L \in \partial^* p^L(\bar{x}), \text{ and } \nu^L \in \partial^* q^L(\bar{x}),$$

$$\left\langle \left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x,\bar{x}) \right\rangle < 0, \text{ for all } \xi^U \in \partial^* p^U(\bar{x}), \text{ and } \nu^U \in \partial^* q^U(\bar{x}).$$

From the fact $\lambda^L > 0$, $\lambda^U > 0$ and by above inequalities, we have

$$\left\langle \lambda^L \left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \lambda^U \left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x,\bar{x}) \right\rangle < 0. \tag{17}$$

On the other hand, by using the feasibility of x, $\mu_i \geq 0$, $i = 1, 2, ..., m$ and (5), we obtain

$$\mu_i h_i(x) \leq \mu_i h_i(\bar{x}), i = 1, 2, ..., m,$$

which by hypothesis (ii), we get

$$\langle \mu_i \zeta_i, \eta(x,\bar{x}) \rangle \leq 0, \text{ for all } \zeta_i \in \partial^* h_i(\bar{x}), \ i = 1, 2, ..., m. \tag{18}$$

On adding (17) and (18), we have

$$\left\langle \lambda^L \left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \lambda^U \left[\xi^U - \phi^U(\bar{x})\nu^U\right] + \sum_{i=1}^{m} \mu_i \zeta_i, \eta(x,\bar{x}) \right\rangle < 0,$$

which contradicts (16). Hence, $\bar{x}$ is a LU optimal solution for (NFIVP). ■

In order to illustrate the sufficient optimality conditions established in the Theorem 3.5, we consider the following example:

Example 3.6.

$$\text{(IVP1)} \qquad \min\left[\frac{f_1^L(x), f_1^U(x)}{g_1^L(x), g_1^U(x)}\right]$$
$$= \min\left[\frac{2x^2, x^2+1}{-x+4, -x^2+6}\right]$$
$$\text{subject to}$$
$$h_1(x) = -x+2 \le 0, x \in X = R.$$

Now, we rewrite the considered optimization problem in the following manner

$$\min\left[\frac{2x^2}{-x^2+6}, \frac{x^2+1}{-x+4}\right]$$
$$\text{subject to}$$
$$h_1(x) = -x+2 \le 0, x \in X = R.$$

where $\frac{p_1^L}{q_1^L}(x) = \frac{2x^2}{-x^2+6}$, $\frac{p_1^U}{q_1^U}(x) = \frac{x^2+1}{-x+4}$. The feasible set is $\mathbb{F}_1 = \{x : -x+2 \le 0, x \in S\}$. By simple calculations, for the feasible point $\bar{x} = 2$, we see that $\partial^* p^L(\bar{x}) = \{-8, 8\}$, $\partial^* q^L(\bar{x}) = \{-4, 4\}$, $\partial^* p^U(\bar{x}) = \{-4, 4\}$, $\partial^* q^U(\bar{x}) = \{-1, 1\}$ and $\partial^* h_1(\bar{x}) = \{-1, 1\}$. Also, we see that for the feasible point $\bar{x} = 2$, there exist $\lambda^L > 0, \lambda^U > 0, \mu \in R_+^m$ such that (4)-(6) are satisfied at $\bar{x}$ and it is easy to see that

(i) $p_1^L(.) - \phi_1^L(\bar{x})q_1^L(.)$ and $p_1^U(.) - \phi_1^U(\bar{x})q_1^U(.)$ are respectively $(\eta, \partial^* p_1^L - \phi_1^L(\bar{x})\partial^* q_1^L)$-invex and $(\eta, \partial^* p_1^U - \phi_1^U(\bar{x})\partial^* q_1^U)$-invex at $\bar{x}$,

(ii) μh_1 is $(\eta, \partial^* h_1(.))$-invex at $\bar{x}$.

Therefore, by Theorem 3.5, $\bar{x} = 2$ is a LU optimal solution for (IVP1).

Theorem 3.7. (Sufficient Optimality Conditions) *Suppose that $\bar{x}$ is a feasible solution of* (NFIVP) *and there exist $\lambda^L > 0, \lambda^U > 0, \mu \in R_+^m$ such that* (4)–(6) *are satisfied at $\bar{x}$. Also, assume that*

(i) *$\lambda^L[p^L(.) - \phi^L(\bar{x})q^L(.)] + \lambda^U[p^U(.) - \phi^U(\bar{x})q^U(.)]$ is $(\eta, \lambda^L[\partial^* p^L - \phi^L(\bar{x})\partial^* q^L] + \lambda^U[\partial^* p^U - \phi^U(\bar{x})\partial^* q^U])$-pseudoinvex at $\bar{x}$,*

(ii) *$\mu_i h_i$, for $i = 1, 2, ..., m$, is $(\eta, \partial^* h_i(.))$-quasiinvex at $\bar{x}$.*

Then $\bar{x}$ is a LU optimal solution for (NFIVP).

Proof. By assumption, (4)-(6) are satisfied at $\bar{x}$ with Lagrange multipliers $\lambda^L > 0, \lambda^U > 0, \mu \in R_+^m$. As it follows from (4), there exist $\xi^L \in \partial^* p^L(\bar{x})$, $\nu^L \in \partial^* q^L(\bar{x})$, $\xi^U \in \partial^* p^U(\bar{x})$, $\nu^U \in \partial^* q^U(\bar{x})$, and $\zeta_i \in \partial^* h_i(\bar{x})$, $i = 1, 2, ..., m$, such that

$$\lambda^L\left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \lambda^U\left[\xi^U - \phi^U(\bar{x})\nu^U\right] + \sum_{i=1}^{m} \mu_i \zeta_i = 0. \tag{19}$$

Suppose contrary to the result, that $\bar{x}$ is not a LU optimal solution for (NFIVP). Hence, by Definition 2.6, there exist a feasible solution x such that

$$\left[\frac{p^L}{q^L}(x), \frac{p^U}{q^U}(x)\right] <_{LU} \left[\frac{p^L}{q^L}(\bar{x}), \frac{p^U}{q^U}(\bar{x})\right]$$

that is

$$\left\{\begin{array}{l} \frac{p^L}{q^L}(x) < \frac{p^L}{q^L}(\bar{x}) \\ \frac{p^U}{q^U}(x) \leq \frac{p^U}{q^U}(\bar{x}) \end{array}\right., \text{ or } \left\{\begin{array}{l} \frac{p^L}{q^L}(x) \leq \frac{p^L}{q^L}(\bar{x}) \\ \frac{p^U}{q^U}(x) < \frac{p^U}{q^U}(\bar{x}) \end{array}\right., \text{ or } \left\{\begin{array}{l} \frac{p^L}{q^L}(x) < \frac{p^L}{q^L}(\bar{x}) \\ \frac{p^U}{q^U}(x) < \frac{p^U}{q^U}(\bar{x}) \end{array}\right..$$

This implies

$$p^L(x) - \phi^L(\bar{x})q^L(x) < p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) \leq p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}),$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) \leq p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}),$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) < p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}).$$

From the fact $\lambda^L > 0$, $\lambda^U > 0$ and by above inequalities, we have

$$\begin{aligned} &\lambda^L[p^L(x) - \phi^L(\bar{x})q^L(x)] + \lambda^U[p^U(x) - \phi^U(\bar{x})q^U(x)] \\ &< \lambda^L[p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})] + \lambda^U[p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})]. \end{aligned}$$

From hypothesis (i), $\lambda^L[p^L(.) - \phi^L(\bar{x})q^L(.)] + \lambda^U[p^U(.) - \phi^U(\bar{x})q^U(.)]$ is $(\eta, \lambda^L[\partial^* p^L - \phi^L(\bar{x})\partial^* q^L] + \lambda^U[\partial^* p^U - \phi^U(\bar{x})\partial^* q^U])$-pseudoinvex at $\bar{x}$ and therefore, there exists $\eta : X \times X \to X$ such that

$$\left\langle \lambda^L\left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \lambda^U\left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x,\bar{x})\right\rangle < 0, \tag{20}$$

for all $\xi^L \in \partial^* p^L(\bar{x}), \nu^L \in \partial^* q^L(\bar{x}), \xi^U \in \partial^* p^U(\bar{x})$, and $\nu^U \in \partial^* q^U(\bar{x})$.

On the other hand, by using the feasibility of x, $\mu_i \geq 0$, $i = 1, 2, ..., m$ and (5), we obtain

$$\mu_i h_i(x) \leq \mu_i h_i(\bar{x}), i = 1, 2, ..., m,$$

which by hypothesis (ii), we get

$$\langle \mu_i \zeta_i, \eta(x,\bar{x})\rangle \leq 0, \text{ for all } \zeta_i \in \partial^* h_i(\bar{x}), \ i = 1, 2, ..., m. \tag{21}$$

On adding (20) and (21), we have

$$\left\langle \lambda^L \left[\xi^L - \phi^L(\bar{x})\nu^L \right] + \lambda^U \left[\xi^U - \phi^U(\bar{x})\nu^U \right] + \sum_{i=1}^{m} \mu_i \zeta_i, \eta(x, \bar{x}) \right\rangle < 0,$$

which contradicts (19). Hence, $\bar{x}$ is a LU optimal solution for (NFIVP). ■

4. Lagrangian Type Function and Saddle-point Analysis

In this section, for the feasible point $\bar{x} \in \mathbb{F}$, we define the Lagrangian type function for the primal problem (NFIVP) as follows:

$$\begin{aligned} L(x, \lambda^L, \lambda^U, \mu) = {} & \lambda^L \left(p^L(x) - \phi^L(\bar{x}) q^L(x) \right) + \lambda^U \left(p^U(x) - \phi^U(\bar{x}) q^U(x) \right) \\ & + \sum_{i=1}^{m} \mu_i h_i(x) \end{aligned}$$

where $x \in X$, $\lambda^L \geq 0$, $\lambda^U \geq 0$ and $\mu \in R_+^m$. Now, we define a saddle-point of $L(x, \lambda^L, \lambda^U, \mu)$ and subsequently we discuss its relation to the problem (NFIVP).

Definition 4.1. *A point $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in X \times R_+ \times R_+ \times R_+^m$ is said to be a saddle point for $L(x, \lambda^L, \lambda^U, \mu)$, if*

(i) $L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \mu) \leq L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$, *for all* $\mu \in R_+^m$,
(ii) $L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \leq L(x, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$, *for all* $x \in X$.

Theorem 4.2. *Let $\bar{\lambda}^L > 0$, $\bar{\lambda}^U > 0$ and $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ be a saddle point for $L(x, \lambda^L, \lambda^U, \mu)$. Then $\bar{x}$ is a LU optimal solution to* (NFIVP).

Proof. Suppose contrary to the result, that $\bar{x}$ is not a LU optimal solution for (NFIVP). Hence, by Definition 2.6, there exists a feasible solution x such that

$$\left[\frac{p^L}{q^L}(x), \frac{p^U}{q^U}(x) \right] <_{LU} \left[\frac{p^L}{q^L}(\bar{x}), \frac{p^U}{q^U}(\bar{x}) \right]$$

that is

$$\left\{ \begin{array}{l} \dfrac{p^L}{q^L}(x) < \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) \leq \dfrac{p^U}{q^U}(\bar{x}) \end{array} \right., \text{ or } \left\{ \begin{array}{l} \dfrac{p^L}{q^L}(x) \leq \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) < \dfrac{p^U}{q^U}(\bar{x}) \end{array} \right., \text{ or } \left\{ \begin{array}{l} \dfrac{p^L}{q^L}(x) < \dfrac{p^L}{q^L}(\bar{x}) \\ \dfrac{p^U}{q^U}(x) < \dfrac{p^U}{q^U}(\bar{x}) \end{array} \right.$$

This implies

$$\begin{aligned} p^L(x) - \phi^L(\bar{x}) q^L(x) &< p^L(\bar{x}) - \phi^L(\bar{x}) q^L(\bar{x}), \\ p^U(x) - \phi^U(\bar{x}) q^U(x) &\leq p^U(\bar{x}) - \phi^U(\bar{x}) q^U(\bar{x}), \end{aligned}$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) \leq p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}),$$

or

$$p^L(x) - \phi^L(\bar{x})q^L(x) < p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x}),$$
$$p^U(x) - \phi^U(\bar{x})q^U(x) < p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x}).$$

By above inequalities and from $\lambda^L > 0$, $\lambda^U > 0$, we have

$$\begin{aligned} &\bar{\lambda}^L[p^L(x) - \phi^L(\bar{x})q^L(x)] + \bar{\lambda}^U[p^U(x) - \phi^U(\bar{x})q^U(x)] \\ &< \bar{\lambda}^L[p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})] + \bar{\lambda}^U[p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})]. \end{aligned} \tag{22}$$

Since $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a saddle point for $L(x, \lambda^L, \lambda^U, \mu)$, by Definition 4.1 (i), we get

$$L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \mu) \leq L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}),$$

that is,

$$\sum_{i=1}^{m} \mu_i h_i(\bar{x}) \leq \sum_{i=1}^{m} \bar{\mu}_i h_i(\bar{x}). \tag{23}$$

Taking $(\mu_1, \mu_2, ...\mu_{i-1}, \mu_i, \mu_{i+1}, ..., \mu_m) = (\bar{\mu}_1, \bar{\mu}_2, ...\bar{\mu}_{i-1}, \bar{\mu}_i + 1, \bar{\mu}_{i+1}, ..., \bar{\mu}_m)$ in the above inequality (23), we obtain

$$h_i(\bar{x}) \leq 0, i = 1, 2, ..., m,$$

which shows that $\bar{x}$ is a feasible solution to (NFIVP).

Using $\bar{\mu} \in R^m_+$, above inequality implies

$$\bar{\mu}_i h_i(\bar{x}) \leq 0, i = 1, 2, ..., m. \tag{24}$$

Again taking $\mu_i = 0$, $i = 1, 2, ..., m$, in the inequality (23), we get

$$\bar{\mu}_i h_i(\bar{x}) \geq 0, i = 1, 2, ..., m. \tag{25}$$

From the inequalities (24) and (25), we conclude that

$$\bar{\mu}_i h_i(\bar{x}) = 0, i = 1, 2, ..., m. \tag{26}$$

On other hand, since $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a saddle point for $L(x, \lambda^L, \lambda^U, \mu)$, by Definition 4.1 (ii), we get

$$L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \leq L(x, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}),$$

that is

$$\begin{aligned} &\bar{\lambda}^L \left(p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})\right) + \bar{\lambda}^U \left(p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})\right) + \sum_{i=1}^{m} \bar{\mu}_i h_i(\bar{x}) \\ \leq\ &\bar{\lambda}^L \left(p^L(x) - \phi^L(\bar{x})q^L(x)\right) + \bar{\lambda}^U \left(p^U(x) - \phi^U(\bar{x})q^U(x)\right) + \sum_{i=1}^{m} \bar{\mu}_i h_i(x) \end{aligned}$$

Using the feasibilty of x of the problem (NFIVP) together with $\bar{\mu} \in R_+^m$ and (26), above inequality gives

$$\begin{aligned}&\bar{\lambda}^L \left(p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})\right) + \bar{\lambda}^U \left(p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})\right)\\ \leq\ &\bar{\lambda}^L \left(p^L(x) - \phi^L(\bar{x})q^L(x)\right) + \bar{\lambda}^U \left(p^U(x) - \phi^U(\bar{x})q^U(x)\right).\end{aligned}$$

This contradicts (22). Hence the proof. ■

Theorem 4.3. *Let $\bar{x}$ be a LU optimal solution to* (NFIVP) *and assume that there exist $\bar{\lambda}^L > 0, \bar{\lambda}^U > 0, \bar{\mu} \in R_+^m$ such that* (4)–(6) *are satisfied at $\bar{x}$. Also, assume that*

(i) *$p^L(.) - \phi^L(\bar{x})q^L(.)$ and $p^U(.) - \phi^U(\bar{x})q^U(.)$ are respectively $(\eta, \partial^* p^L - \phi^L(\bar{x})\partial^* q^L)$-invex and $(\eta, \partial^* p^U - \phi^U(\bar{x})\partial^* q^U)$-invex at $\bar{x}$.*

(ii) *$\mu_i h_i$, for $i = 1, 2, ..., m$, is $(\eta, \partial^* h_i(.))$-invex at $\bar{x}$.*

Then $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a saddle point for $L(x, \lambda^L, \lambda^U, \mu)$.

Proof. By assumption, (4)–(6) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda}^L > 0, \bar{\lambda}^U > 0, \bar{\mu} \in R_+^m$. As it follows from (4), there exist $\xi^L \in \partial^* p^L(\bar{x})$, $\nu^L \in \partial^* q^L(\bar{x})$, $\xi^U \in \partial^* p^U(\bar{x})$, $\nu^U \in \partial^* q^U(\bar{x})$, and $\zeta_i \in \partial^* h_i(\bar{x})$, $i = 1, 2, ..., m$, such that

$$\bar{\lambda}^L \left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \bar{\lambda}^U \left[\xi^U - \phi^U(\bar{x})\nu^U\right] + \sum_{i=1}^m \bar{\mu}_i \zeta_i = 0. \tag{27}$$

From the hypothesis (i), $p^L(.) - \phi^L(\bar{x})q^L(.)$ and $p^U(.) - \phi^U(\bar{x})q^U(.)$ are respectively $(\eta, \partial^* p^L - \phi^L(\bar{x})\partial^* q^L)$-invex and $(\eta, \partial^* p^U - \phi^U(\bar{x})\partial^* q^U)$-invex at $\bar{x}$, therefore, there exists $\eta : X \times X \to X$ such that

$$\begin{aligned}&[p^L(x) - \phi^L(\bar{x})q^L(x)] - [p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})]\\ \geq\ &\left\langle \left[\xi^L - \phi^L(\bar{x})\nu^L\right], \eta(x, \bar{x}) \right\rangle, \text{ for all } \xi^L \in \partial^* p^L(\bar{x}), \text{ and } \nu^L \in \partial^* q^L(\bar{x}),\end{aligned}$$

and

$$\begin{aligned}&[p^U(x) - \phi^U(\bar{x})q^U(x)] - [p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})]\\ \geq\ &\left\langle \left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x, \bar{x}) \right\rangle, \text{ for all } \xi^U \in \partial^* p^U(\bar{x}), \text{ and } \nu^U \in \partial^* q^U(\bar{x}).\end{aligned}$$

From the fact $\bar{\lambda}^L > 0$, $\bar{\lambda}^U > 0$ and by above inequalities, we have

$$\begin{aligned}&\bar{\lambda}^L[p^L(x) - \phi^L(\bar{x})q^L(x)] - \bar{\lambda}^L[p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})]\\ \geq\ &\left\langle \bar{\lambda}^L\left[\xi^L - \phi^L(\bar{x})\nu^L\right], \eta(x, \bar{x} \right\rangle, \text{ for all } \xi^L \in \partial^* p^L(\bar{x}), \text{ and } \nu^L \in \partial^* q^L(\bar{x}),\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\bar{\lambda}^U[p^U(x) - \phi^U(\bar{x})q^U(x)] - \bar{\lambda}^U[p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})]\\ \geq\ &\left\langle \bar{\lambda}^U\left[\xi^U - \phi^U(\bar{x})\nu^U\right], \eta(x, \bar{x}) \right\rangle, \text{ for all } \xi^U \in \partial^* p^U(\bar{x}), \text{ and } \nu^U \in \partial^* q^U(\bar{x}).\end{aligned} \tag{29}$$

From hypothesis (ii), we get

$$\sum_{i=1}^{m} \bar{\mu}_i h_i(x) - \sum_{i=1}^{m} \bar{\mu}_i h_i(\bar{x}) \geq \left\langle \sum_{i=1}^{m} \bar{\mu}_i \zeta_i, \eta(x, \bar{x}) \right\rangle, \tag{30}$$

for all $\zeta_i \in \partial^* h_i(\bar{x}),\ i = 1, 2, ..., m$.

On adding (28)-(30), we have

$$\begin{aligned}
&\bar{\lambda}^L[p^L(x) - \phi^L(\bar{x})q^L(x)] + \bar{\lambda}^U[p^U(x) - \phi^U(\bar{x})q^U(x)] + \sum_{i=1}^{m} \bar{\mu}_i h_i(x) \\
&-\left[\bar{\lambda}^L[p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})] + \bar{\lambda}^U[p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})] + \sum_{i=1}^{m} \bar{\mu}_i h_i(\bar{x})\right] \\
\geq &\left\langle \bar{\lambda}^L\left[\xi^L - \phi^L(\bar{x})\nu^L\right] + \bar{\lambda}^U\left[\xi^U - \phi^U(\bar{x})\nu^U\right] + \sum_{i=1}^{m} \bar{\mu}_i \zeta_i,\ \eta(x, \bar{x}) \right\rangle,
\end{aligned}$$

which by (27), yields

$$\begin{aligned}
&\bar{\lambda}^L[p^L(x) - \phi^L(\bar{x})q^L(x)] + \bar{\lambda}^U[p^U(x) - \phi^U(\bar{x})q^U(x)] + \sum_{i=1}^{m} \bar{\mu}_i h_i(x) \\
\geq &\left[\bar{\lambda}^L[p^L(\bar{x}) - \phi^L(\bar{x})q^L(\bar{x})] + \bar{\lambda}^U[p^U(\bar{x}) - \phi^U(\bar{x})q^U(\bar{x})] + \sum_{i=1}^{m} \bar{\mu}_i h_i(\bar{x})\right],
\end{aligned}$$

that is

$$L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \leq L(x, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}). \tag{31}$$

On the other hand, using the feasibility of $\bar{x}$ of the problem (NFIVP) and the fact $\mu \in R_+^m$, we have

$$\mu_i h_i(\bar{x}) \leq 0, i = 1, 2, ..., m, \tag{32}$$

By using (32) and the optimality conditions (5), we get

$$L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \mu) \leq L(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}). \tag{33}$$

By inequalities (31) and (33) we conclude that $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a saddle point for $L(x,\ \lambda^L,\ \lambda^U, \mu)$. Hence the proof. ■

5. Conclusion

In this paper, with the idea of convexificators, we have discussed optimality conditions and saddle point criteria for a nonconvex fractional interval-valued optimization problem. Also, we provided an example to validate the results of sufficient optimality conditions established in this paper. In our opinion, the techniques employed in this paper can be extended for proving the similar results for other classes of fractional programming problems with the functions involving are convexificators. This may be the topic of some of our forthcoming papers.

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