

Estimate of Third and Fourth Hankel Determinants for Certain Subclasses of Analytic Functions

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Abstract. In geometric function theory, the estimation of upper bound of the Hankel determinants for various subclasses of analytic functions is an interesting topic of current research. Till now, extensive work has been done on the estimation of second and third Hankel determinants. The present investigation deals with the estimate of fourth Hankel determinant for certain subclasses of analytic functions in the open unit disc $E = \{z : |z| < 1\}$. This work will set the stage in the direction of investigation of fourth Hankel determinant for several other classes.

Keywords: Analytic functions; Univalent functions; Hankel determinant; Coefficient bounds; Functions with bounded boundary rotation.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and further normalized specifically by $f(0) = f'(0) - 1 = 0$. By S , we denote the subclass of A of the form (1) and which are univalent in E .

Let P denote the class of analytic functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

whose real parts are positive in E .

For understanding of the main content, let us recall the following definitions:

$R = \{f : f \in A, \operatorname{Re}(f'(z)) > 0, z \in E\}$, the class of bounded turning functions introduced and studied by MacGregor [17].

$R_1 = \{f : f \in A, \operatorname{Re}\left(\frac{f(z)}{z}\right) > 0, z \in E\}$, the subclass of close-to-star functions studied by MacGregor [18].

$R(\alpha) = \left\{f : f \in A, \operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0, 0 \leq \alpha \leq 1, z \in E\right\}$, the class studied by Murugusundramurthi and Magesh [20]. In particular, $R(1) \equiv R$ and $R(0) \equiv R_1$.

$R'(\alpha) = \{f : f \in A, \operatorname{Re}(f'(z) + \alpha z f''(z)) > 0, \alpha \geq 0, z \in E\}$, the class studied by Sahoo [24]. Particularly, $R'(0) \equiv R$.

For the complex sequence $a_n, a_{n+1}, a_{n+2}, \dots$, the Hankel matrix, named after Herman Hankel (1839-1873), is the infinite matrix whose $(i, j)^{th}$ entry a_{ij} is defined by $a_{ij} = a_{n+i+j-2}$ ($i, j, n \in N$).

The q^{th} Hankel matrix ($q \in N - \{1\}$) is by definition, the following $q \times q$ square sub matrix:

$$\begin{pmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{pmatrix}$$

We observe that the Hankel matrix has constant positive slopping diagonals whose entries also satisfy:

$$a_{ij} = a_{i-1, j+1} \quad (i \in N - \{1\}; j \in N).$$

In 1976, Noonan and Thomas [21] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

In particular cases, $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to $H_2(1) = |a_3 - a_2^2|$ and $H_2(2) = |a_2 a_4 - a_3^2|$.

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [8] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real

and $f \in S$. Also $H_2(2)$ is called the second Hankel determinant. The Hankel determinant in the case $q = 3, n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is called the Third Hankel determinant.

For $f \in S, a_1 = 1$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \quad (2)$$

For any $f \in A$ of the form (1), we can represent the fourth Hankel determinant as

$$H_4(1) = a_7H_3(1) - a_6D_1 + a_5D_2 - a_4D_3, \quad (3)$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$D_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2), \quad (4)$$

$$D_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2), \quad (5)$$

$$D_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2). \quad (6)$$

Hankel determinant has been considered by several authors. For example, Noor [22] determined the rate of growth of $H_q(n)$ as $n \rightarrow \infty$ for the functions given by (1) with bounded boundary. Ehrenborg [7] studied the Hankel determinant of exponential polynomials and in [13], the Hankel transform of an integer sequence is defined and some of its properties have been discussed by Layman. Also Hankel determinant was studied by various authors including Hayman [9], Pommerenke [23], Tang et al. [31] and Vamshee Krishna et al. [32].

Second Hankel determinant for various classes has been extensively studied by various authors including Singh [26, 27], Mehrok and Singh [19], Janteng et al. [10, 12, 11] and many others. Third Hankel determinant has been studied by some of the researchers including Babalola [2], Shanmugam et al. [25], Sudharsan et al. [30], Bucur et al. [5], Altinkaya and Yalcin [1] and Singh and Singh [28, 29].

In this paper, we seek upper bound for the functional $H_{4,1}(f)$ for the functions belonging to the classes $R(\alpha)$ and $R'(\alpha)$. This work will motivate the future researchers to work in this direction.

2. Third and Fourth Hankel Determinants for the Class $R(\alpha)$

Lemma 2.1. [6, 14] If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then for $n, k \in N = \{1, 2, 3, \dots\}$, we have the following inequalities:

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1$$

and

$$|c_n| \leq 2.$$

Lemma 2.2. [20] If $f \in R(\alpha)$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{(1+2\alpha)^2}.$$

Lemma 2.3. [4, 16, 15] If $p \in P$, then

$$\begin{aligned} 2c_2 &= c_1^2 + (4 - c_1^2)x, \\ 4c_3 &= c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \\ 8c_4 &= c_1^4 + (4 - c_1^2)x(c_1^2(x^2 - 3x + 3) + 4x) \\ &\quad - 4(4 - c_1^2)(1 - |x|^2)(c_1(x - 1)\eta + \bar{x}\eta^2 - (1 - |\eta|^2)z), \end{aligned}$$

for some x, z and η satisfying $|x| \leq 1, |z| \leq 1, |\eta| \leq 1$ and $c_1 \in [0, 2]$.

Lemma 2.4. [3] If $p(z) \in P$, then

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1 - \sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

Theorem 2.5. If $f \in R(\alpha)$, then

$$|a_2a_3 - a_4| \leq \begin{cases} 2 & \text{if } \alpha = 0, \\ \frac{2(6\alpha^2 + 3\alpha + 1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)} & \text{if } 0 < \alpha \leq 1. \end{cases}$$

Results are sharp.

Proof. Since $f \in R(\alpha)$, by the definition, we have

$$(1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) = p(z), p(z) \in P. \quad (7)$$

On expanding and equating the coefficients in (7), it yields

$$a_n = \frac{c_{n-1}}{1 + (n - 1)\alpha}. \quad (8)$$

Using Lemma 2.1, (8) gives

$$|a_n| \leq \frac{2}{1 + (n - 1)\alpha}. \quad (9)$$

Again using (8), we obtain

$$|a_2a_3 - a_4| = \left| \frac{c_1c_2}{(1+\alpha)(1+2\alpha)} - \frac{c_3}{(1+3\alpha)} \right|.$$

Substituting for c_2 and c_3 from Lemma 2.3 and letting $c_1 = c$, we get

$$\begin{aligned} |a_2a_3 - a_4| &= \left| \frac{(1+3\alpha-2\alpha^2)c^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} - \frac{\alpha^2cx(4-c^2)}{(1+\alpha)(1+2\alpha)(1+3\alpha)} \right. \\ &\quad \left. + \frac{cx^2(4-c^2)}{4(1+3\alpha)} - \frac{(4-c^2)(1-|x|^2)z}{2(1+3\alpha)} \right|. \end{aligned}$$

Since $|c| = |c_1| \leq 2$, by using Lemma 2.1, we may assume that $c \in [0, 2]$. Then using triangle inequality and $|z| \leq 1$ with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(1+3\alpha-2\alpha^2)c^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{\alpha^2c(4-c^2)\rho}{(1+\alpha)(1+2\alpha)(1+3\alpha)} \\ &\quad + \frac{(4-c^2)}{2(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho^2}{4(1+3\alpha)} \\ &= F(c, \rho). \end{aligned}$$

Then

$$\frac{\partial F}{\partial \rho} = F'(\rho) = \frac{\alpha^2c(4-c^2)}{(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho}{2(1+3\alpha)}.$$

Note that, $F'(\rho) \geq F'(1) > 0$. Then there exists $c^* \in (0, 2)$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise. Then for $c \in (c^*, 2]$, $F(\rho) \leq F(1)$.

But

$$F(c, 1) = \frac{(1+3\alpha-6\alpha^2)c^3 + 16\alpha^2c}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4c-c^3)}{4(1+3\alpha)} = G(c).$$

If $\alpha = 0$, we have $G(c) = c \leq 2$.

Further

$$G'(c) = \frac{3(1+3\alpha-6\alpha^2)c^2 + 16\alpha^2}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4-3c^2)}{4(1+3\alpha)} = 0,$$

which gives

$$c = c_0 = \left(\frac{6\alpha^2 + 3\alpha + 1}{6\alpha^2} \right)^{\frac{1}{2}}.$$

So c_0 is the critical point of $G(c)$. Since $G''(c_0) = \frac{-12\alpha^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}c_0 < 0$, so $G(c)$ has maximum value at c_0 . Hence for $c \in [0, 2]$

$$\max G(c) = G(c_0) = \frac{2(6\alpha^2 + 3\alpha + 1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

The results are sharp for the function

$$f(z) = z + \frac{c_0}{1+\alpha}z^2 + \frac{c_0^2 - 2}{1+2\alpha}z^3 + \frac{c_0(c_0^2 - 3)}{1+3\alpha}z^4 + \dots$$

■

For $\alpha = 1$, Theorem 2.5 gives the following result due to Babalola [2]:

Corollary 2.6. *If $f \in R$, then*

$$|a_2a_3 - a_4| \leq \frac{5\sqrt{5}}{18\sqrt{3}}.$$

Theorem 2.7. *If $f \in R(\alpha)$, then*

$$|a_3 - a_2^2| \leq \frac{2}{1+2\alpha}.$$

Proof. Since $f \in R(\alpha)$, by using Eq. (8), we obtain

$$|a_3 - a_2^2| = \left| \frac{c_2}{(1+2\alpha)} - \frac{c_1^2}{(1+\alpha)^2} \right| = \frac{1}{(1+2\alpha)} \left| c_2 - \frac{2(1+2\alpha)}{(1+\alpha)^2} \cdot \frac{c_1^2}{2} \right|.$$

Using Lemma 2.4, with $0 \leq \sigma = \frac{2(1+2\alpha)}{(1+\alpha)^2} \leq 2$, we have

$$|a_3 - a_2^2| \leq \frac{2}{1+2\alpha}.$$

■

Theorem 2.8. *If $f \in R(\alpha)$, then*

$$|H_3(1)| \leq \begin{cases} \frac{16}{1+2\alpha} \left[\frac{2}{(1+2\alpha)^2} \right] & \text{for } \alpha = 0, \\ \frac{1}{1+4\alpha} + \frac{(6\alpha^2 + 3\alpha + 1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2} & \text{for } 0 < \alpha \leq 1. \end{cases}$$

The bounds are sharp.

Proof. Using Lemma 2.2, Theorems 2.5, 2.7 and inequality (9) in (2), the above result is obvious.

Sharpness follows for the function

$$f(z) = z + \frac{c_0}{1+\alpha}z^2 + \frac{c_0^2 - 2}{1+2\alpha}z^3 + \frac{c_0(c_0^2 - 3)}{1+3\alpha}z^4 + \frac{c_0^4 - 4c_0^2 + 2}{1+4\alpha}z^5 + \dots$$

■

For $\alpha = 1$, Theorem 2.8 gives the following result proved by Babalola [2]:

Corollary 2.9. *If $f \in R$, then*

$$|H_3(1)| \leq 0.7423.$$

Theorem 2.10. *If $f \in R(\alpha)$, then*

$$|H_4(1)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \frac{8}{(1+2\alpha)(1+6\alpha)} \left[\frac{2}{(1+2\alpha)^2} + \frac{1}{1+4\alpha} \right. \\ \left. + \frac{(6\alpha^2+3\alpha+1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2} \right] + \frac{2}{(1+5\alpha)} p(\alpha) \\ + \frac{2}{(1+4\alpha)} q(\alpha) + \frac{2}{(1+3\alpha)} r(\alpha) & \text{for } 0 < \alpha \leq 1, \end{cases} \quad (10)$$

where

$$\begin{aligned} p(\alpha) = 4 & \left[\frac{1}{(1+\alpha)^2(1+5\alpha)} + \frac{1}{(1+3\alpha)(1+2\alpha)^2} \right. \\ & \left. + \frac{1}{(1+\alpha)(1+3\alpha)^2} \right] + \frac{29}{4(1+\alpha)(1+2\alpha)(1+4\alpha)}, \end{aligned} \quad (11)$$

$$\begin{aligned} q(\alpha) = 4 & \left[\frac{63}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} \right. \\ & \left. + \frac{9}{5(1+4\alpha)(1+2\alpha)^2} + \frac{76}{75(1+2\alpha)(1+3\alpha)^2} \right], \end{aligned} \quad (12)$$

$$\begin{aligned} r(\alpha) = 4 & \left[\frac{1}{(1+2\alpha)^2(1+5\alpha)} + \frac{1}{(1+\alpha)(1+3\alpha)(1+5\alpha)} \right. \\ & \left. + \frac{2}{(1+3\alpha)^3} + \frac{1}{(1+\alpha)(1+4\alpha)^2} \right] + \frac{68}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\ & + \frac{1}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}. \end{aligned} \quad (13)$$

Proof. Using (9) in (4), (5) and (6), it gives

$$\begin{aligned} D_1 = & \frac{c_2 c_5}{(1+2\alpha)(1+5\alpha)} - \frac{c_3 c_4}{(1+3\alpha)(1+4\alpha)} - \frac{c_1^2 c_5}{(1+\alpha)^2(1+5\alpha)} \\ & + \frac{c_1 c_2 c_4}{(1+\alpha)(1+2\alpha)(1+4\alpha)} + \frac{c_1 c_3^2}{(1+\alpha)(1+3\alpha)^2} - \frac{c_3 c_2^2}{(1+3\alpha)(1+2\alpha)^2}, \end{aligned} \quad (14)$$

$$\begin{aligned} D_2 = & \frac{c_3 c_5}{(1+3\alpha)(1+5\alpha)} - \frac{c_4^2}{(1+4\alpha)^2} - \frac{c_1 c_2 c_5}{(1+\alpha)(1+2\alpha)(1+5\alpha)} \\ & + \frac{c_1 c_3 c_4}{(1+\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_4 c_2^2}{(1+2\alpha)^2(1+4\alpha)} - \frac{c_2 c_3^2}{(1+2\alpha)(1+3\alpha)^2}, \end{aligned} \quad (15)$$

$$\begin{aligned} D_3 = & \frac{c_1 c_3 c_5}{(1+\alpha)(1+3\alpha)(1+5\alpha)} - \frac{c_1 c_4^2}{(1+\alpha)(1+4\alpha)^2} - \frac{c_2^2 c_5}{(1+2\alpha)^2(1+5\alpha)} \\ & + \frac{2 c_2 c_3 c_4}{(1+2\alpha)(1+3\alpha)(1+4\alpha)} - \frac{c_3^3}{(1+3\alpha)^3}. \end{aligned} \quad (16)$$

On rearranging the terms in (14), (15) and (16), it yields

$$\begin{aligned} D_1 = & \frac{c_5(c_2 - c_1^2)}{(1+\alpha)^2(1+5\alpha)} + \frac{c_3(c_4 - c_2^2)}{(1+3\alpha)(1+2\alpha)^2} - \frac{c_3(c_4 - c_1 c_3)}{(1+\alpha)(1+3\alpha)^2} \\ & - \frac{67 c_4(c_3 - c_1 c_2)}{48(1+\alpha)(1+2\alpha)(1+4\alpha)} + \frac{19 c_2(c_5 - c_1 c_4)}{48(1+\alpha)(1+2\alpha)(1+4\alpha)} \\ & + \frac{c_2 c_5}{48(1+\alpha)(1+2\alpha)(1+4\alpha)}, \end{aligned} \quad (17)$$

$$\begin{aligned} D_2 = & \frac{c_5(c_3 - c_1 c_2)}{(1+\alpha)(1+2\alpha)(1+5\alpha)} - \frac{c_4(c_4 - c_2^2)}{(1+4\alpha)(1+2\alpha)^2} - \frac{c_3(c_5 - c_2 c_3)}{(1+2\alpha)(1+3\alpha)^2} \\ & - \frac{4 c_4(c_4 - c_1 c_3)}{5(1+4\alpha)(1+2\alpha)^2} - \frac{13 c_3(c_5 - c_1 c_4)}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} \\ & + \frac{c_3 c_5}{75(1+2\alpha)(1+3\alpha)^2}, \end{aligned} \quad (18)$$

$$\begin{aligned} D_3 = & \frac{c_5(c_4 - c_2^2)}{(1+2\alpha)^2(1+5\alpha)} - \frac{c_5(c_4 - c_1 c_3)}{(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{c_3(c_6 - c_3^2)}{(1+3\alpha)^3} \\ & - \frac{c_3(c_6 - c_2 c_4)}{(1+3\alpha)^3} + \frac{c_4(c_5 - c_1 c_4)}{(1+\alpha)(1+4\alpha)^2} - \frac{17 c_4(c_5 - c_2 c_3)}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\ & + \frac{c_4 c_5}{4(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}. \end{aligned} \quad (19)$$

Using Lemma 2.1 and applying triangle inequality in (17), (18) and (19), we obtain

$$|D_1| \leq p(\alpha), \quad (20)$$

$$|D_2| \leq q(\alpha), \quad (21)$$

and

$$|D_3| \leq r(\alpha) \quad (22)$$

where $p(\alpha)$, $q(\alpha)$ and $r(\alpha)$ are defined in (11), (12) and (13) respectively.

Hence using Theorem 2.8, (9), (20), (21) and (22) and applying triangle inequality in (3), the result (10) is obvious. ■

On putting $\alpha = 1$ in Theorem 2.10, we obtain the following result:

Corollary 2.11. *Let $f \in R$. Then*

$$|H_4(1)| \leq 0.7973.$$

3. Third and Fourth Hankel Determinants for the Class $R'(\alpha)$

Lemma 3.1. [24] If $f \in R'(\alpha)$, then

$$|a_k| \leq \frac{2}{k(1 + (k - 1)\alpha)}, k \geq 2.$$

Lemma 3.2. [24] If $f \in R'(\alpha)$, then

$$|a_3 - a_2^2| \leq \frac{2}{3(1 + 2\alpha)}.$$

Lemma 3.3. [24] For $0 \leq \alpha \leq \frac{1}{2}$, if $f \in R'(\alpha)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9(1 + 2\alpha)^2}.$$

Theorem 3.4. If $f \in R'(\alpha)$, then

$$|a_2 a_3 - a_4| \leq \frac{(18\alpha^2 + 15\alpha + 5)^{\frac{3}{2}}}{18(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)\sqrt{3(6\alpha^2 + 3\alpha + 1)}}.$$

The result is sharp.

Proof. Since $f \in R'(\alpha)$, by the definition, we have

$$f'(z) + \alpha z f''(z) = p(z), p(z) \in P. \quad (23)$$

On expanding and equating the coefficients in (23), it yields

$$a_n = \frac{c_{n-1}}{n(1 + (n - 1)\alpha)}. \quad (24)$$

Using (24), we obtain

$$|a_2 a_3 - a_4| = \left| \frac{c_1 c_2}{6(1 + \alpha)(1 + 2\alpha)} - \frac{c_3}{4(1 + 3\alpha)} \right|.$$

Substituting for c_2 and c_3 and using Lemma 2.3, we get

$$\begin{aligned} |a_2 a_3 - a_4| &= T(\alpha) \left| (1 + 3\alpha)c_1[c_1^2 + (4 - c_1^2)x] - \frac{3}{4}(1 + \alpha)(1 + 2\alpha)[c_1^3 \right. \\ &\quad \left. + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z] \right| \end{aligned}$$

where

$$T(\alpha) = \frac{1}{12(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

Letting $c_1 = c$ and $|x| = \delta$. Since $|c| = |c_1| \leq 2$, by using Lemma 2.1, we may assume that $c \in [0, 2]$. Then using triangle inequality and $|z| \leq 1$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{T(\alpha)}{4} \left| (1+3\alpha-6\alpha^2)c^3 + 6(1+\alpha)(1+2\alpha)(4-c^2) \right. \\ &\quad \left. + 2(1+3\alpha+6\alpha^2)c(4-c^2)\delta + 3(1+\alpha)(1+2\alpha)(c-2)(4-c^2)\delta^2 \right| \\ &= G(c, \delta). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial G}{\partial \delta} &= G'(\delta) \\ &= \frac{T(\alpha)}{2} [(1+3\alpha+6\alpha^2) - (4-c^2) + 3(1+\alpha)(1+2\alpha)(c-2)(4-c^2)\delta] \\ &> 0. \end{aligned}$$

Note that, $G'(\delta) \geq G'(1) > 0$. Then there exists $c^* \in (0, 2)$ such that $G'(\delta) > 0$ for $c \in (c^*, 2]$ and $G'(\delta) \leq 0$ otherwise. Then for $c \in (c^*, 2]$, $G(\delta) \leq G(1)$. But

$$\begin{aligned} G(c, 1) &= \frac{T(\alpha)}{4} [-4(1+3\alpha+6\alpha^2)c^3 + 4(5+15\alpha+18\alpha^2)c] = G(c), \\ G'(c) &= \frac{T(\alpha)}{4} [-12(1+3\alpha+6\alpha^2)c^2 + 4(5+15\alpha+18\alpha^2)]. \end{aligned}$$

Further $G'(c) = 0$, gives

$$c = d_0 = \left(\frac{18\alpha^2 + 15\alpha + 5}{3(6\alpha^2 + 3\alpha + 1)} \right)^{\frac{1}{2}}.$$

So d_0 is the critical point of $G(c)$. Since $G''(d_0) = \frac{-24T(\alpha)(6\alpha^2+3\alpha+1)d_0}{4} < 0$, so $G(c)$ has maximum value at d_0 . Hence for $c \in [0, 2]$

$$\max G(c) = G(d_0) = \frac{(18\alpha^2 + 15\alpha + 5)^{\frac{3}{2}}}{18(1+\alpha)(1+2\alpha)(1+3\alpha)\sqrt{3(6\alpha^2 + 3\alpha + 1)}}.$$

The result is sharp for the function

$$f(z) = z + \frac{d_0}{2(1+\alpha)}z^2 + \frac{d_0^2 - 2}{3(1+2\alpha)}z^3 + \frac{d_0(d_0^2 - 3)}{4(1+3\alpha)}z^4 + \dots$$

Theorem 3.5. If $f \in R'(\alpha)$, then

$$\begin{aligned} |H_3(1)| &\leq \frac{1}{3(1+2\alpha)} \left[\frac{8}{9(1+2\alpha)^2} + \frac{4}{5(1+4\alpha)} \right. \\ &\quad \left. + \frac{(18\alpha^2 + 15\alpha + 5)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2\sqrt{3(6\alpha^2 + 3\alpha + 1)}} \right]. \end{aligned}$$

Estimate is sharp.

Proof. Using Lemmas 3.1, 3.2, 3.3 and Theorem 3.4 in (2), the above result is obvious.

$$f(z) = z + \frac{d_0}{2(1+\alpha)}z^2 + \frac{d_0^2 - 2}{3(1+2\alpha)}z^3 + \frac{d_0(d_0^2 - 3)}{4(1+3\alpha)}z^4 + \frac{d_0^4 - 4d_0^2 + 2}{5(1+4\alpha)}z^5 + \dots \blacksquare$$

For $\alpha = 0$, the result of Theorem 3.5 justified with that of Corollary 2.9.

Theorem 3.6. *If $f \in R'(\alpha)$, then*

$$\begin{aligned} |H_4(1)| &\leq \frac{2}{21(1+2\alpha)(1+6\alpha)} \left[\frac{8}{9(1+2\alpha)^2} + \frac{4}{5(1+4\alpha)} \right. \\ &\quad \left. + \frac{(18\alpha^2 + 15\alpha + 5)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2 \sqrt{3(6\alpha^2 + 3\alpha + 1)}} \right] \\ &\quad + \frac{1}{3(1+5\alpha)} u(\alpha) + \frac{2}{5(1+4\alpha)} v(\alpha) + \frac{1}{2(1+3\alpha)} w(\alpha), \end{aligned} \quad (25)$$

where

$$\begin{aligned} u(\alpha) &= \frac{1}{6(1+\alpha)^2(1+5\alpha)} + \frac{1}{9(1+3\alpha)(1+2\alpha)^2} + \frac{1}{8(1+\alpha)(1+3\alpha)^2} \\ &\quad + \frac{29}{120(1+\alpha)(1+2\alpha)(1+4\alpha)}, \end{aligned} \quad (26)$$

$$\begin{aligned} v(\alpha) &= \frac{7}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{4}{25(1+4\alpha)(1+2\alpha)^2} \\ &\quad + \frac{19}{225(1+2\alpha)(1+3\alpha)^2}, \end{aligned} \quad (27)$$

$$\begin{aligned} w(\alpha) &= \frac{2}{27(1+2\alpha)^2(1+5\alpha)} + \frac{1}{12(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{1}{8(1+3\alpha)^3} \\ &\quad + \frac{2}{25(1+\alpha)(1+4\alpha)^2} + \frac{17}{240(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\ &\quad + \frac{1}{10800(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}. \end{aligned} \quad (28)$$

Proof. Using (24) in (4), (5) and (6), it gives

$$\begin{aligned} D_1 &= \frac{c_2 c_5}{18(1+2\alpha)(1+5\alpha)} - \frac{c_3 c_4}{20(1+3\alpha)(1+4\alpha)} - \frac{c_1^2 c_5}{25(1+\alpha)^2(1+5\alpha)} \\ &\quad + \frac{c_1 c_2 c_4}{30(1+\alpha)(1+2\alpha)(1+4\alpha)} + \frac{c_1 c_3^2}{32(1+\alpha)(1+3\alpha)^2} \\ &\quad - \frac{c_3 c_2^2}{36(1+3\alpha)(1+2\alpha)^2}, \end{aligned} \quad (29)$$

$$\begin{aligned}
D_2 = & \frac{c_3 c_5}{24(1+3\alpha)(1+5\alpha)} - \frac{c_4^2}{25(1+4\alpha)^2} - \frac{c_1 c_2 c_5}{36(1+\alpha)(1+2\alpha)(1+5\alpha)} \\
& + \frac{c_1 c_3 c_4}{40(1+\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_4 c_2^2}{45(1+2\alpha)^2(1+4\alpha)} \\
& - \frac{c_2 c_3^2}{48(1+2\alpha)(1+3\alpha)^2},
\end{aligned} \tag{30}$$

$$\begin{aligned}
D_3 = & \frac{c_1 c_3 c_5}{48(1+\alpha)(1+3\alpha)(1+5\alpha)} - \frac{c_1 c_4^2}{50(1+\alpha)(1+4\alpha)^2} \\
& - \frac{c_2^2 c_5}{54(1+2\alpha)^2(1+5\alpha)} + \frac{2 c_2 c_3 c_4}{60(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\
& - \frac{c_3^3}{64(1+3\alpha)^3}.
\end{aligned} \tag{31}$$

On rearranging the terms in (29), (30) and (31), it yields

$$\begin{aligned}
D_1 = & \frac{c_5(c_2 - c_1^2)}{24(1+\alpha)^2(1+5\alpha)} + \frac{c_3(c_4 - c_2^2)}{36(1+3\alpha)(1+2\alpha)^2} - \frac{c_3(c_4 - c_1 c_3)}{32(1+\alpha)(1+3\alpha)^2} \\
& - \frac{67 c_4(c_3 - c_1 c_2)}{1440(1+\alpha)(1+2\alpha)(1+4\alpha)} + \frac{19 c_2(c_5 - c_1 c_4)}{1440(1+\alpha)(1+2\alpha)(1+4\alpha)}, \\
& + \frac{c_2 c_5}{1440(1+\alpha)(1+2\alpha)(1+4\alpha)},
\end{aligned} \tag{32}$$

$$\begin{aligned}
D_2 = & \frac{c_5(c_3 - c_1 c_2)}{36(1+\alpha)(1+2\alpha)(1+5\alpha)} - \frac{c_4(c_4 - c_2^2)}{45(1+4\alpha)(1+2\alpha)^2} \\
& - \frac{c_3(c_5 - c_2 c_3)}{48(1+2\alpha)(1+3\alpha)^2} - \frac{4 c_4(c_4 - c_1 c_3)}{225(1+4\alpha)(1+2\alpha)^2} \\
& - \frac{13 c_3(c_5 - c_1 c_4)}{1800(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{c_3 c_5}{3600(1+2\alpha)(1+3\alpha)^2},
\end{aligned} \tag{33}$$

$$\begin{aligned}
D_3 = & \frac{c_5(c_4 - c_2^2)}{54(1+2\alpha)^2(1+5\alpha)} - \frac{c_5(c_4 - c_1 c_3)}{48(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{c_3(c_6 - c_3^2)}{64(1+3\alpha)^3} \\
& - \frac{c_3(c_6 - c_2 c_4)}{64(1+3\alpha)^3} + \frac{c_4(c_5 - c_1 c_4)}{50(1+\alpha)(1+4\alpha)^2} - \frac{17 c_4(c_5 - c_2 c_3)}{960(1+2\alpha)(1+3\alpha)(1+4\alpha)} \\
& + \frac{c_4 c_5}{43200(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.
\end{aligned} \tag{34}$$

Using Lemma 2.1 and applying triangle inequality in (32), (33) and (34), we obtain

$$|D_1| \leq u(\alpha), \tag{35}$$

$$|D_2| \leq v(\alpha), \tag{36}$$

and

$$|D_3| \leq w(\alpha) \tag{37}$$

where $u(\alpha)$, $v(\alpha)$ and $w(\alpha)$ are defined in (26), (27) and (28) respectively.

Hence using Theorem 3.5, Lemma 3.1, (35), (36) and (37) and applying triangle inequality in (3), the result (25) is obvious. ■

For $\alpha = 0$, Theorem 3.6 gives the following result:

Corollary 3.7. *Let $f \in R$. Then*

$$|H_4(1)| \leq 0.7973. \quad (38)$$

The result of Corollary 3.7 agrees with that of Corollary 2.11.

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References

- [1] S. Altinkaya and S. Yalçın, Third Hankel determinant for Bazilevic functions, *Advances in Mathematics: Scientific Journal* **5** (2016) 91–96.
- [2] K.O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, *Inequality Theory and Applications* **6** (2010) 1–7.
- [3] K.O. Babalola and T.O. Opoolla, On the coefficients of certain analytic and univalent functions, In: *Advances in Inequalities for Series*, Ed. by S.S. Dragomir and A. Sofo, Nova Science Publishers, 2006.
- [4] S. Banga and S. Sivaprasad Kumar, The sharp bounds of the second and third Hankel determinants for the class SL^* , *Mathematica Slovaca* **70** (4) (2020) 849–862.
- [5] R. Bucur, D. Breaz, L. Georgescu, Third Hankel determinant for a class of analytic functions with respect to symmetric points, *Acta Universitatis Apulensis* **42** (2015) 79–86.
- [6] C. Carathéodory, Über den variabilitätsbereich der Fourierschen Konstanten von positive harmonischen Funktionen, *Rend. Circ. Mat. Palermo* **32** (1911) 193–217.
- [7] R. Ehrenborg, The Hankel determinant of exponential polynomials, *American Mathematical Monthly* **107** (2000) 557–560.
- [8] M. Fekete and G. Szegő, Eine Bemerkung über ungerade schlichte Funktionen, *J. London Math. Soc.* **8** (1933) 85–89.
- [9] W.K. Hayman, *Multivalent Functions*, Cambridge Tracts in Math. and Math. Phys. **48**, Cambridge University Press, Cambridge, 1958.
- [10] A. Janteng, S.A. Halim, M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Ineq. Pure Appl. Math.* **7** (2) (2006), Art. ID 50, 5 pages.
- [11] A. Janteng, S.A. Halim, M. Darus, Hankel determinant for functions starlike and convex with respect to symmetric points, *J. Quality Measurement and Anal.* **2** (2006) 37–43.
- [12] A. Janteng, S.A. Halim, M. Darus, Hankel determinant for starlike and convex functions, *Int. J. Math. Anal.* **1** (2007) 619–625.
- [13] J.W. Layman, The Hankel transform and some of its properties, *J. of Integer Sequences* **4** (2001) 1–11.
- [14] A.E. Livingston, The coefficients of multivalent close-to-convex functions, *Proc. Amer. Math. Soc.* **21** (1969) 545–552.

- [15] R.J. Libera and E.-J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P , *Proc. Amer. Math. Soc.* **87** (1983) 251–257.
- [16] R.J. Libera and E.-J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* **85** (1982) 225–230.
- [17] T.H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.* **104** (1962) 532–537.
- [18] T.H. MacGregor, The radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.* **14** (1963) 514–520.
- [19] B.S. Mehrok and G. Singh, Estimate of second Hankel determinant for certain classes of analytic functions, *Scientia Magna* **8** (2012) 85–94.
- [20] G. Murugusundramurthi and N. Magesh, Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant, *Bull. Math. Anal. and Appl.* **1** (2009) 85–89.
- [21] J.W. Noonan and D.K. Thomas, On the second Hankel determinant of a really mean p -valent functions, *Trans. Amer. Math. Soc.* **223** (2) (1976) 337–346.
- [22] K.I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, *Rev. Roum. Math. Pures Et Appl.* **28** (8) (1983) 731–739.
- [23] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [24] P. Sahoo, Third Hankel determinant for a class of analytic univalent functions, *Electronic Journal of Mathematical Analysis and Applications* **6** (1) (2018) 322–329.
- [25] G. Shanmugam, B. Adolf Stephen, K.O. Babalola, Third Hankel determinant for α -starlike functions, *Gulf Journal of Mathematics* **2** (2014) 107–113.
- [26] G. Singh, Hankel determinant for a new subclass of analytic functions, *Scientia Magna* **8** (2012) 61–65.
- [27] G. Singh, Hankel determinant for new subclasses of analytic functions with respect to symmetric points, *Int. J. of Modern Math. Sci.* **5** (2013) 67–76.
- [28] G. Singh and G. Singh, Third Hankel determinant for a subclass of alpha convex functions, *Global Journal of Advanced Research* **2** (1) (2015) 221–229.
- [29] G. Singh and G. Singh, On third Hankel determinant for a subclass of analytic functions, *Open Science Journal of Mathematics and Applications* **3** (2015) 172–175.
- [30] T.V. Sudharsan, S.P. Vijayalakshmi, B. Adolf Stephen, Third Hankel determinant for a subclass of analytic univalent functions, *Malaya Journal of Matematik* **2** (2014) 438–444.
- [31] H. Tang, M.K. Aouf, S. Ali, Hankel determinant for transforms of pre-starlike functions of order-alpha, *Southeast Asian Bull. Math.* **40** (2016) 439–449.
- [32] D. Vamshee Krishna, B. Venkateswarlu, T. Ramreddy, Hankel determinant for transforms of pre-starlike functions of order-alpha, *Southeast Asian Bull. Math.* **40** (2) (2016) 131–140.