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Certain Chebyshev Fractional Type Integral Inequalities Involving Saigo-Maeda Operator

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Abstract. In this present investigation, we establish some new fractional integral inequalities using Marchiev-Saigo-Maeda(MSM) fractional integral operator for synchronous function. The results are obtained by considering weighted form of the Chebyshev functional and are general in nature which can be reduces to the results from Saigo's, Riemann-Liouville and Erdlyi-Kober fractional integral operator.

Keywords: Chebyshevs functional; Integral inequalities; Marichev-Saigo-Maeda fractional integral operators.

1. Introduction

Fractional integral inequalities are one of the most useful and powerful tools in the development of theory of fractional calculus. These inequalities have wide applications in probability and statistical problems, transform theory, numerical quadrature and mostly in establishing uniqueness of solutions in fractional boundary value problems (see [1, 2]). Let the functional

$$\begin{aligned} &T(h,g)\\ =&\frac{1}{(c-a)}\int_a^c h(x)g(x)dx-\Big(\frac{1}{(c-a)}\int_a^c h(x)dx\Big)\Big(\frac{1}{(c-a)}\int_a^c g(x)dx\Big), \end{aligned}$$

where h and g are two integrable functions and synchronous on $[a, c]$, i.e.

$$(h(x) - h(y))(g(x) - g(y)) \geq 0,$$

for any $x, y \in [a, c]$. Then the Chebychev integral inequality [5] is given by $T(h, g) \geq 0$.

Under different assumptions (Gr̈uss inequality, Chebyshev inequalities etc.), Chebyshev functionals are playing an important role to give a lower bound or an upper bound for $T(h, g)$, in the theory of approximations. Such type of works may be found in the recent papers (see [3, 4, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25]). In the present paper our work is based on weighted form of Chebyshev functional [5].

$$\begin{aligned} &T(h, g, p) \\ &= \int_a^c p(x)dx - \int_a^c h(x)g(x)p(x)dx - \int_a^c h(x)p(x)dx \int_a^c g(x)p(x)dx, \end{aligned} \tag{1}$$

where h and g are two integrable functions on $[a, c]$ and $p(x)$ is a positive and integrable function on $[a, c]$.

Dragomir [9] introduced the following inequality:

$$2|T(h, g, p)| \leq ||h'||_r ||g'||s \left[\iint_a^c |x - y| p(x)p(y)dxdy \right],$$

where h and g are two differentiable functions and $f' \in L_r(a, c), g' \in L_s(a, c)$, $r^{-1} + s^{-1} = 1, r > 1$.

Many researchers have developed several inequalities related to this functional (1), (see [7, 19]). Here, we develop certain integral inequalities involving Marchiev-Saigo-Maeda (MSM) fractional integral operator related to the weighted Chebyshev's functional. Also we establish some other results as a special case of main result.

2. Some Definitions and Preliminaries

In this section, we present some useful definitions and preliminaries which are helpful in further investigation.

Definition 2.1. *A real valued function $g(t) (t > 0)$ is said to be in the space $C_\nu (\nu \in R)$, if there exists a real number $q > \nu$ such that $g(t) = t^q \phi(t)$, where $\phi(t) \in C(0, \infty)$.*

Definition 2.2. *Let $\eta, \eta', \zeta, \zeta', \delta \in C$ and $Re(\delta) > 0$. Then a MSM fractional integral,*

$$I_x^{\eta, \eta', \zeta, \zeta', \delta} f(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\delta-1} t^{-\eta'} F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{t}{x}, 1-\frac{x}{t}\Big) f(t)dt, \tag{2}$$

where $F_3(-)$ is in kernel is the Horn function.

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!},$$

and $(\lambda)_n$ is the Pochhammer symbol

$$(\lambda)_n = \lambda(\lambda+1)...(\lambda+n-1), \quad (\lambda)_0 = 1.$$

Definition 2.3. *Let $\eta, \eta', \zeta, \zeta', \delta \in C$ and $Re(\delta) > \max[0, Re(\eta, \eta', \zeta, \zeta')] > 0$. Then the following inequality holds* [12]

$$F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{t}{x}, 1-\frac{x}{t}\Big) > 0,$$

provided $-1 < 1-\frac{t}{x} < 0$ and $0 < 1-\frac{x}{t} < \frac{1}{2}$ also, if $f(x) > 0$, then

$$I_x^{\eta,\eta',\zeta,\zeta',\delta} f(x) > 0.$$

Definition 2.4. *Let $\eta, \eta', \zeta, \zeta', \delta \in C$ and $Re(\delta) > 0, Re(\rho) > \max[0, Re(\eta+\eta'+\zeta-\delta), Re(\eta'-\zeta')]$. Then*

$$\begin{aligned} &I_x^{\eta,\eta',\zeta,\zeta',\delta}(x^{\rho-1}) \\ =&x^{\rho-\eta-\eta'+\delta-1}\frac{\Gamma(\rho)\Gamma(\rho+\delta-\eta-\eta'-\zeta)\Gamma(\rho+\zeta'-\eta')}{\Gamma(\rho+\delta-\eta-\eta')\Gamma(\rho+\delta-\eta'-\zeta)\Gamma(\rho+\zeta')}. \end{aligned} \tag{3}$$

3. Main Result

In this section, we establish certain integral inequality involving MSM operator. We obtain our results related to weighted Chebychev's functional.

Theorem 3.1. *Let f and g be two synchronous functions on $[0, \infty)$ and let p be a positive function. If $h' \in L_r([0,\infty)), g' \in L_s([0,\infty)), r^{-1}+s^{-1} = 1, r > 1$, then*

$$\begin{aligned} &2\Big|I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\} \\ &-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\}\Big| \\ \leq\ & \frac{x^{-2\eta}}{\Gamma^2(\delta)}||h'||_r||g'||_s \iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta, \eta', \zeta, \zeta', \delta, \\ &1-\frac{t}{x}, 1-\frac{x}{t}\Big) \times F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{u}{x}, 1-\frac{x}{u}\Big)|t-u|p(t)p(u)dtdu \\ \leq\ & ||h'||_r||g'||_s x(I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\})^2. \end{aligned} \tag{4}$$

Proof. Let h and g be two synchronous functions. Then, by using Definition 2.1, for all $t, u \in (0, x), x \geq 0$, we define

$$\mathcal{G}(t,u) = (h(t) - h(u))(g(t) - g(u)), \tag{5}$$

and consider

$$H(x,t) = \frac{x^{-\eta}}{\Gamma(\delta)} t^{-\eta'}(x-t)^{\delta-1} F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{t}{x}, 1-\frac{x}{t}\Big). \tag{6}$$

We see that in the above series, each term is positive because of the conditions given in Theorem 3.1, and hence the function $H(x,t)$ remains positive, for all $t \in (0,x)(x > 0)$.

Now multiplying both sides of (5) by $H(x,t)p(t)$(where $H(x,t)$ is given by (6) and integrating with respect to t from 0 to x and using (2), we get

$$\begin{aligned} &\frac{x^{-\eta}}{\Gamma(\delta)} \int_0^x t^{-\eta'}(x-t)^{\delta-1} F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{t}{x}, 1-\frac{x}{t}\Big) p(t)\mathcal{G}(t,u)dt \\ &= I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\} - h(u) I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\} \\ &\quad - g(u) I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\} + h(u)g(u) I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}. \end{aligned} \tag{7}$$

Multiplying (7) by $H(x,u)p(u)$ and integrating with respect to u from 0 to x

$$\begin{aligned} &\frac{x^{-2\eta}}{\Gamma^2(\delta)} \iint_0^x t^{-\eta'} u^{-\eta'} (x-t)^{\delta-1}(x-u)^{\delta-1} F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{t}{x}, 1-\frac{x}{t}\Big) \\ &\times F_3\Big(\eta, \eta', \zeta, \zeta', \delta, 1-\frac{u}{x}, 1-\frac{x}{u}\Big) p(t)p(u)\mathcal{G}(t,u)dtdu \\ &= 2\Big(I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\} I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\} \\ &\quad - I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\} \times I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\}\Big), \end{aligned} \tag{8}$$

by (5) we may write

$$\mathcal{G}(t,u) = \iint_t^u h'(y)g'(z)dydz.$$

By the help of Hölder's inequality for the Double integral

$$\Big| \iint_t^u h(y)g(z)dydz \Big| \leq \Big| \iint_t^u |h(y)|^r dydz \Big|^{r^{-1}} \Big| \iint_t^u |g(z)|^s dydz \Big|^{s^{-1}},$$

for $r^{-1} + s^{-1} = 1$, we get

$$\Big|\mathcal{G}(t,u)\Big| \leq \Big| \iint_t^u |h'(y)|^r dydz \Big|^{r^{-1}} \Big| \iint_t^\rho |g'(z)|^s dydz \Big|^{s^{-1}}. \tag{9}$$

We know

$$\Big| \iint_t^u |h'(y)|^r dydz \Big|^{r^{-1}} = |t-u|^{r^{-1}} \Big| \int_t^u |h'(y)|^r dy \Big|^{r^{-1}},$$

and

$$\Big|\iint_t^u |g'(z)|^s dydz\Big|^{s^{-1}} = |t-u|^{s^{-1}}\Big|\int_t^u |g'(z)|^s dz\Big|^{s^{-1}}.$$

Then (9) reduces to

$$\Big|\mathcal{G}(t,u)\Big| \leq |t-u|\Big|\int_t^u |h'(y)|^r dy\Big|^{r^{-1}}\Big|\int_t^u |g'(z)|^s dz\Big|^{s^{-1}}, \tag{10}$$

by Eq. (8)

$$\begin{aligned}
&\frac{x^{-2\eta}}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\Big|dtdu\\
\leq\ &\frac{x^{-2\eta}}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\alpha'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\\
&\times|t-u|\int_t^u |h'(y)|^r dy\Big|^{r^{-1}}\Big|\int_t^u |g'(z)|^s dz\Big|^{s^{-1}}dtdu.
\end{aligned}\tag{11}$$

Now by applying again Hölder's inequality on the right hand side of (11), we obtain

$$\begin{aligned}
&\frac{x^{-2\eta}}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\Big|dtdu\\
\leq\ &\Big[\frac{x^{-r\eta}}{\Gamma^r(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|\int_t^u |h'(y)|^r dy\Big|dtdu\Big]^{r^{-1}}\\
&\times\Big[\frac{x^{-s\eta}}{\Gamma^s(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|\int_t^u |g'(z)|^s dz\Big|dtdu\Big]^{s^{-1}}.
\end{aligned}$$

We know that

$$\Big|\int_t^u |h(y)|^p dy\Big| \leq ||h||_p^p. \tag{12}$$

Then

$$\begin{aligned}
&\frac{x^{-2\eta}}{\Gamma^2(\delta)}\iint_0^x t^{-\alpha'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\Big|dtdu\\
\leq\ &\Big[\frac{x^{-r\eta}||h'||_r^r}{\Gamma^r(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu\Big]^{r^{-1}}\\
&\times\Big[\frac{x^{-s\eta}||g'||_s^s}{\Gamma^s(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu\Big]^{s^{-1}}. \qquad (13)
\end{aligned}$$

Since $r^{-1}+s^{-1}=1$, the above inequality (13) becomes

$$\begin{aligned}
&\frac{x^{-2\eta}}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\Big|dtdu\\
\leq\ &\frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\eta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu. \qquad (14)
\end{aligned}$$

By using (8), above inequality gives

$$\begin{aligned}
&2\Big|I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\}\\
&-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\}\Big|\\
\leq\ &\frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},\\
&1-\frac{x}{t}\Big)\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu.
\end{aligned}$$

This shows the left side of the inequality (4).

To prove the right-hand side of the inequality (4), we observe that $0\leq t\leq x, 0\leq u\leq x$, therefore

$$0\leq|t-u|\leq x$$

with the help of (14), we have

$$\frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu$$
$$\leq \frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\times x\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)dtdu$$
$$= ||h'||_r||g'||_s(I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\})^2.$$

The proof is completed. ∎

Theorem 3.2. *Let h and g be two synchronous functions on $[0,\infty)$ and p a positive function. If $h'\in L_r([0,\infty)), g'\in L_s([0,\infty)), r^{-1}+s^{-1}=1, r>1$, then*

$$\Big|I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)h(x)g(x)\}$$
$$+I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\}$$
$$-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)g(x)\}$$
$$-I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\}\Big|$$
$$\leq \frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}||h'||_r||g'||_s\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)|t-u|p(t)p(u)dtdu$$
$$\leq ||h'||_r||g'||_s xI_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)\}. \tag{15}$$

Proof. To prove the above theorem multiplying (7) by

$$\frac{x^{-\lambda}}{\Gamma(\nu)}u^{-\lambda'}(x-u)^{\nu-1}F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(u),$$
$$(u\in(0,x); x>0)$$

integrating with respect to u from 0 to x, we get

$$\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t})$$
$$\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\mathcal{G}(t,u)dtdu$$
$$= I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)h(x)g(x)\}+I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)\}$$
$$\times I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)g(x)\}$$
$$-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)h(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)g(x)\}-I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)h(x)\}$$
$$\times I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)g(x)\}. \tag{16}$$

By using (10), then (16) becomes

$$\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},$$
$$1-\frac{x}{t})\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\ \Big|dtdu$$
$$\leq\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},$$
$$1-\frac{x}{t})\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)$$
$$\times|t-u|\Big|\int_t^u|h'(y)|^r dy\Big|^{r^{-1}}\Big|\int_t^u|g'(z)|^s dz\Big|^{s^{-1}}dtdu. \tag{17}$$

Applying Hölder's inequality on the right hand side of (17), we get

$$\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t})$$
$$\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\ \Big|dtd\rho$$
$$\leq\Big[\frac{x^{-r\eta-r\lambda}}{\Gamma^r(\delta)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t})$$
$$\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|\Big|\int_t^u|h'(y)|^r dy\Big|dtdu\Big]^{r^{-1}}$$
$$\times\Big[\frac{x^{-s\alpha-s\eta}}{\Gamma^s(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t})$$
$$\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|\Big|\int_t^u|g'(z)|^s dz\Big|dtdu\Big]^{s^{-1}}. \tag{18}$$

By using (12), inequality (18) becomes

$$\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\lambda'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\lambda',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t})$$
$$\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\Big|\mathcal{G}(t,u)\ \Big|dtdu$$
$$\leq\frac{x^{-\eta-\lambda}}{\Gamma(\delta)\Gamma(\nu)}||h'||_r||g'||_s\iint_0^x t^{-\alpha'}u^{-\lambda'}(x-t)^{\delta-1}(x-\rho)^{\nu-1}F_3\Big(\eta,\lambda',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t})\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times|t-u|dtdu.$$

This shows the left side of the inequality (15).

To prove the right-hand side of the inequality (15), we observe that $0\leq t\leq x, 0\leq u\leq x$, therefore

$$0\leq|t-u|\leq x, \tag{19}$$

$$\frac{x^{-\eta-\lambda}||h'||_r||g'||_s}{\Gamma(\delta)\Gamma(\nu)}\iint_0^x t^{-\eta'}u^{-\lambda'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)\times\Big|\mathcal{G}(t,u)\Big|dtdu$$
$$\leq \frac{x^{-\eta-\lambda}||h'||_r||g'||_s}{\Gamma(\delta)\Gamma(\nu)}\times x\iint_0^x t^{-\eta'}u^{-\nu'}(x-t)^{\delta-1}(x-u)^{\nu-1}F_3\Big(\lambda,\lambda',\mu,\mu',\nu,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\lambda,\lambda',\mu,\mu',\nu,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)dtdu$$
$$\leq ||h'||_r||g'||_s x I_x^{\eta,\eta',\zeta,\zeta',\delta}\{p(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{p(x)\}. \tag{20}$$

This completes the proof. ∎

Remark 3.3. For $\lambda=\eta,\lambda'=\eta',\mu=\zeta,\mu'=\zeta',\nu=\delta$. Then Theorem 3.2 reduces to Theorem 3.1.

4. Some Results and Special Cases

In this section we consider some results of Theorems 3.1 and 3.2 by choosing suitably the function $p(t)$. Let us set $p(t)=t^{\tau}(\tau\in[0,\infty),t\in(0,\infty)$. Then by using (3), Theorems 3.1 and 3.2 yield the following results.

Corollary 4.1. *Let h and g be two synchronous functions on $[0,\infty)$. If $h'\in L_r([0,\infty)),g'\in L_s([0,\infty)),r^{-1}+s^{-1}=1,r>1$ then*

$$2\Big|\frac{\Gamma(\tau+1)\Gamma(\tau+1+\delta-\eta-\eta'-\zeta)\Gamma(\tau+1+\zeta'-\eta')}{\Gamma(\tau+1+\delta-\eta-\eta')\Gamma(\tau+1+\delta-\eta'-\zeta)\Gamma(\tau+1+\zeta')}$$
$$\times x^{\tau+1-\eta-\eta'+\delta-1}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^{\tau}h(x)g(x)\}$$
$$-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^{\tau}h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^{\tau}g(x)\}\Big|$$
$$\leq \frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'+\tau}\rho^{-\eta'+\tau}(x-t)^{\delta-1}(x-\rho)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)\times|t-u|dtdu$$
$$\leq ||h'||_r||g'||_s\frac{\Gamma^2(\tau+1)\Gamma^2(\tau+1+\delta-\eta-\eta'-\zeta)\Gamma^2(\tau+1+\zeta'-\eta')}{\Gamma^2(\tau+1+\delta-\eta-\eta')\Gamma^2(\tau+1+\delta-\eta'-\zeta)\Gamma^2(\tau+1+\zeta')}$$
$$\times x^{1+2\tau-2\eta-2\eta'+2\delta}, \tag{21}$$

for $\tau>0,x>0,\zeta'>-1,min(\delta-\eta-\eta',\zeta'-\eta')>-1,\delta-\eta'>max(1-\zeta,1-\eta)$.

Corollary 4.2. *Let h and g be two synchronous functions on $[0,\infty)$. If $h'\in$*

$L_r([0,\infty)), g' \in L_s([0,\infty)), r^{-1}+s^{-1}=1, r>1,$ *then*

$$\Big|\frac{\Gamma(\tau+1)\Gamma(\tau+1+\delta-\eta-\eta'-\zeta)\Gamma(\tau+1+\zeta'-\eta')}{\Gamma(\tau+1+\delta-\eta-\eta')\Gamma(\tau+1+\delta-\eta'-\zeta)\Gamma(\tau+1+\zeta')}x^{\tau+1-\eta-\eta'+\delta-1}$$
$$\times I_x^{\lambda,\lambda',\mu,\mu',\nu}\{x^\tau h(x)g(x)\}$$
$$+\frac{\Gamma(\tau+1)\Gamma(\tau+1+\nu-\lambda-\lambda'-\mu)\Gamma(\tau+1+\mu'-\lambda')}{\Gamma(\tau+1+\nu-\lambda-\lambda')\Gamma(\tau+1+\nu-\lambda'-\mu)\Gamma(\tau+1+\mu')}$$
$$\times x^{\tau+1-\lambda-\lambda'+\nu-1}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^\tau h(x)g(x)\}$$
$$-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^\tau f(x)\}I_x^{\lambda,\lambda',\mu,\mu',\nu}\{x^\tau g(x)\}$$
$$-I_x^{\lambda,\lambda',\mu,\mu',\nu}\{x^\tau h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{x^\tau g(x)\}\Big|$$
$$\leq \frac{x^{-\eta-\lambda}||h'||_r||g'||_s}{\Gamma(\delta)\Gamma(\nu)}\times\iint_0^x t^{-\eta'+\tau}u^{-\lambda'+\tau}(x-t)^{\delta-1}(x-u)^{\nu-1}$$
$$\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big)F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)dtdu$$
$$\leq ||h'||_r||g'||_s\frac{\Gamma(\tau+1)\Gamma(\tau+1+\delta-\eta-\eta'-\beta)\Gamma(\tau+1+\zeta'-\eta')}{\Gamma(\tau+1+\delta-\eta-\eta')\Gamma(\tau+1+\delta-\eta'-\zeta)\Gamma(\tau+1+\zeta')}x^{\tau+1-\eta-\eta'+\delta-1}$$
$$\times\frac{\Gamma(\tau+1)\Gamma(\tau+1+\nu-\lambda-\lambda'-\mu)\Gamma(\tau+1+\mu'-\lambda')}{\Gamma(\tau+1+\mu-\lambda-\lambda')\Gamma(\tau+1+\nu-\lambda'-\mu)\Gamma(\tau+1+\mu')}x^{\tau+1-\lambda-\lambda'+\nu-1} \tag{22}$$

for all $x>0, \tau\geq 0, \zeta'>-1, \mu'>-1, 1+\delta>\eta+\eta'+\zeta, 1+\zeta'>\eta', \delta>0, \min(\delta-\eta-\eta'-\zeta, \zeta'-\eta', \nu-\lambda-\lambda'-\mu, \mu'-\lambda')>-1, \delta-\eta'>\max(1-\zeta,1-\eta), 1+\nu>\lambda+\lambda'+\mu, 1+\mu'>\lambda', \nu>0, \nu-\lambda'>\max(1-\mu,1-\lambda)$.

Further put $\tau=0$ in Corollaries 4.1 and 4.2 or $p(t)=1$ in Theorems 3.1 and 3.2. Then we get the following corollary.

Corollary 4.3. *Let h and g be two synchronous functions on $[0,\infty)$. If $h'\in L_r([0,\infty)), g'\in L_s([0,\infty)), r^{-1}+s^{-1}=1, r>1$. Then*

$$2\Big|\frac{\Gamma(1+\delta-\eta-\eta'-\zeta)\Gamma(1+\zeta'-\eta')}{\Gamma(1+\delta-\eta-\eta')\Gamma(1+\delta-\eta'-\zeta)\Gamma(1+\zeta')}$$
$$\times x^{1-\eta-\eta'+\delta-1}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{f(x)g(x)\}-I_x^{\eta,\eta',\zeta,\zeta',\delta}\{h(x)\}I_x^{\eta,\eta',\zeta,\zeta',\delta}\{g(x)\}\Big|$$
$$\leq \frac{x^{-2\eta}||h'||_r||g'||_s}{\Gamma^2(\delta)}\iint_0^x t^{-\eta'}u^{-\eta'}(x-t)^{\delta-1}(x-u)^{\delta-1}F_3\Big(\eta,\eta',\zeta,\zeta',\delta,$$
$$1-\frac{t}{x},1-\frac{x}{t}\Big)\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)\times|t-u|dtdu$$
$$\leq ||h'||_r||g'||_s\frac{\Gamma^2(1+\delta-\eta-\eta'-\zeta)\Gamma^2(1+\zeta'-\eta')}{\Gamma^2(1+\delta-\eta-\eta')\Gamma^2(1+\delta-\eta'-\zeta)\Gamma^2(1+\zeta')}x^{s-\eta-\eta'+\delta}, \tag{23}$$

for all $x>0, \tau\geq 0, \zeta'>-1, min(\delta-\eta-\eta'-\zeta, \zeta'-\eta')>-1, \delta-\eta'>max(1-\zeta,1-\eta)$.

Corollary 4.4. *Let h and g be two synchronous functions on $[0,\infty)$. If $h' \in L_r([0,\infty)), g' \in L_s([0,\infty)), r^{-1}+s^{-1}=1, r>1$, then*

$$\begin{aligned}
&\Big|\frac{\Gamma(1+\delta-\eta-\eta'-\zeta)\Gamma(1+\zeta'-\eta')}{\Gamma(1+\delta-\eta-\eta')\Gamma(1+\delta-\eta'-\zeta)\Gamma(1+\zeta')}x^{1-\eta-\eta'+\delta-1}\\
&\times I_x^{\lambda,\lambda',\mu,\mu',\nu}\{x^\tau h(x)g(x)\} + \frac{\Gamma(1+\nu-\lambda-\lambda'-\mu)\Gamma(1+\mu'-\lambda')}{\Gamma(1+\mu-\lambda-\lambda')\Gamma(1+\nu-\lambda'-\mu)\Gamma(1+\mu')}\\
&\times x^{1-\lambda-\lambda'+\nu-1} I_x^{\eta,\eta',\zeta,\zeta',\delta}\{h(x)g(x)\} - I_x^{\eta,\eta',\zeta,\zeta',\delta}\{f(x)\} I_x^{\lambda,\lambda',\mu,\mu',\nu}\{g(x)\}\\
&-I_x^{\lambda,\lambda',\mu,\mu',\nu}\{h(x)\} I_x^{\eta,\eta',\zeta,\zeta',\delta}\{g(x)\}\Big|\\
\leq\ & \frac{x^{-\eta-\lambda}||h'||_r||g'||_s}{\Gamma(\delta)\Gamma(\nu)} \iint_0^x t^{-\eta'}\rho^{-\nu'}(x-t)^{\delta-1}(x-u)^{\nu-1}\\
&\times F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{t}{x},1-\frac{x}{t}\Big) F_3\Big(\eta,\eta',\zeta,\zeta',\delta,1-\frac{u}{x},1-\frac{x}{u}\Big)p(t)p(u)dtdu\\
\leq\ & ||h'||_r||g'||_s \frac{\Gamma(1)\Gamma(1+\delta-\eta-\eta'-\zeta)\Gamma(1+\zeta'-\eta')}{\Gamma(1+\delta-\eta-\eta')\Gamma(1+\eta-\eta'-\zeta)\Gamma(1+\zeta')}x^{1-\eta-\eta'+\delta-1}\\
&\times\frac{\Gamma(1)\Gamma(1+\nu-\lambda-\lambda'-\mu)\Gamma(1+\mu'-\lambda')}{\Gamma(1+\mu-\lambda-\lambda')\Gamma(1+\nu-\lambda'-\mu)\Gamma(1+\mu')}x^{1-\lambda-\lambda'+\nu-1}. \qquad (24)
\end{aligned}$$

Now we can see that the operator (2) would reduce immediately to the well known Saigo, Erdelyi-Kober and Riemann-Liouville type fractional integral operators respectively given by the following relationships

$$I^{\eta,0,\zeta,\zeta',\delta}h(x) = I_x^{\delta,\eta-\delta,-\zeta} = \frac{x^{-\eta}}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}\, {}_2F_1\Big(\eta,\zeta,\delta,1-\frac{t}{x}\Big)h(t)dt, \quad (25)$$

$$I^{\delta,0,\zeta,\zeta',\delta}h(x) = I_x^{\delta,-\zeta} = \frac{x^{-\delta+\zeta}}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}\, t^{-\zeta}h(t)dt, \quad (26)$$

$$I_x^{0,0,\zeta,\zeta',\delta}h(x) = I_x^{\delta} = \frac{1}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}\, h(t)dt. \quad (27)$$

Now If we consider $\eta'=0$ (and $\lambda'=0$ additionally for Theorem 3.2 and use of (25), Theorems 3.1 and 3.2, provide, respectively, the known fractional integral inequalities due to Purohit and Raina [19]. Again, if we take $\eta'=0, \eta=0$ (and $\lambda'=0, \lambda=0$ additionally for Theorem 3.2) and use of (27), Theorems 3.1 and 3.2, correspond to the known integral inequalities due to Dahmani et al. [7].

5. Conclusion

Here, we have discussed some inequalities involving Marchiev-Saigo-Maeda (MSM) fractional integral operator by taking into account of synchronous function. Generalization of these inequalities are obtained by using weighted form of the Chebyshev functional. The key results reduces to the results from Saigo's, Riemann-Liouville and Erdlyi-Kober fractional integral operator. We conclude this study that our results are general in nature and other important integral inequalities can be derived from our key results as the particular cases.

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