# Some Applications of Generalized Alexander Integral Operator 

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#### Abstract

In this paper, we introduce some new subclasses of analytic functions in the open unit disk $\mathcal{U}$ with negative coefficients defined by generalized Alexander integral operator. The aim of the present paper is to determine coefficient inequalities, inclusion relations, neighborhoods, partial sums and integral means properties for functions $f$ belonging to these subclasses.


Keywords: Analytic function; Starlike and convex functions; Alexander integral operator; Neighborhoods; Partial sums; Coefficient inequality; Inclusion relation; Integral means.

## 1. Introduction

Let $\mathcal{A}$ be a class of functions $f$ of the form

$$
\begin{equation*}
f(z)=z+\sum_{k=n+1}^{\infty} a_{k} z^{k} \quad(n \in \mathbb{N}=\{1,2,3, \cdots\}) \tag{1}
\end{equation*}
$$

that are analytic in the open unit $\operatorname{disk} \mathcal{U}=\{z \in \mathbb{C}:|z|<1\}$. Denote by $\mathcal{A}(n)$ the class of functions consisting of functions $f$ of the form

$$
\begin{equation*}
f(z)=z-\sum_{k=n+1}^{\infty} a_{k} z^{k} \quad\left(a_{k} \geq 0, n \in \mathbb{N}\right) \tag{2}
\end{equation*}
$$

which are analytic in $\mathcal{U}$.
Let $\Omega$ be the class of functions $w(z)$ analytic in $\mathcal{U}$ such that $w(0)=0$, $|w(z)|<1$. For the functions $f$ and $g$ in $\mathcal{A}, f$ is said to be subordinate to $g \in \mathcal{U}$ if there exists an analytic function $w(z) \in \Omega$ such that $f(z)=g(w(z))$. This subordination is denoted by $f(z) \prec g(z)$.

Next, following the earlier investigations by Goodman [16], Ruscheweyh [27], Silverman [31] and Altıntaş et al. [3, 4] (see also [5]-[11], [17]-[21]), we define the $(n, \delta)$-neighborhood of a function $f \in \mathcal{A}(n)$ by

$$
\begin{equation*}
\mathcal{N}_{n, \delta}(f)=\left\{g \in \mathcal{A}(n): g(z)=z-\sum_{k=n+1}^{\infty} b_{k} z^{k} \text { and } \sum_{k=n+1}^{\infty} k\left|a_{k}-b_{k}\right| \leq \delta\right\} \tag{3}
\end{equation*}
$$

For $e(z)=z$, we have

$$
\begin{equation*}
\mathcal{N}_{n, \delta}(e)=\left\{g \in \mathcal{A}(n): g(z)=z-\sum_{k=n+1}^{\infty} b_{k} z^{k} \text { and } \sum_{k=n+1}^{\infty} k\left|b_{k}\right| \leq \delta\right\} \tag{4}
\end{equation*}
$$

A function $f \in \mathcal{A}(n)$ is $\alpha$-starlike of complex order $\tau$, denoted by $f \in \mathcal{S}_{n}^{*}(\alpha, \tau)$ if it satisfies the following condition

$$
\Re\left\{1+\frac{1}{\tau}\left(\frac{z f^{\prime}(z)}{f(z)}-1\right)\right\}>\alpha \quad(\tau \in \mathbb{C} \backslash\{0\}, 0<\alpha \leq 1, z \in \mathcal{U})
$$

and a function $f \in \mathcal{A}(n)$ is $\alpha$-convex of complex order $\tau$, denoted by $f \in \mathcal{C}_{n}(\alpha, \tau)$ if it satisfies the following condition

$$
\Re\left\{1+\frac{1}{\tau} \frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right\}>\alpha \quad(\tau \in \mathbb{C} \backslash\{0\}, 0<\alpha \leq 1, z \in \mathcal{U})
$$

For $f \in \mathcal{A}$, the following integral operator defined by Alexander [2]:

$$
\begin{equation*}
A_{-1} f(z)=\int_{0}^{z} \frac{f(s)}{s} d s=z+\sum_{k=n+1}^{\infty} \frac{a_{k}}{k} z^{k} \tag{5}
\end{equation*}
$$

Alexander integral operator was applied for some subclasses of analytic functions in $\mathcal{U}$ by Acu [1], Güney [15] and Kugita et al. [18].

For (5), we consider

$$
\begin{equation*}
A_{-j} f(z)=A_{-j+1}\left(A_{-1} f(z)\right)=z+\sum_{k=n+1}^{\infty} \frac{a_{k}}{k} z^{k} \quad(j \in \mathbb{N}) \tag{6}
\end{equation*}
$$

where $A_{0} f(z)=f(z)$.
From the various definitions of fractional calculus of $f \in \mathcal{A}$ (that is, fractional integrals and fractional derivatives) given in the literature, we would like to recall here the following definitions for fractional calculus which were used by Owa [22], Owa and Srivastava [25] and Owa and Patel [23].

Definition 1.1. The fractional integral of order $r$ for $f \in \mathcal{A}$ is defined by

$$
D_{z}^{-r} f(z)=\frac{1}{\Gamma(r)} \int_{0}^{z} \frac{f(s)}{(z-s)^{1-r}} d s \quad(r>0)
$$

where $f$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z-s)^{r-1}$ is removed by requiring $\log (z-t)$ to be real when $z-s>0$ and $\Gamma$ is the Gamma function.

From Definition 1.1, we know that

$$
\begin{equation*}
D_{z}^{-r} f(z)=\frac{1}{\Gamma(2+r)} z^{1+r}+\sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1+r)} a_{k} z^{k+r} \tag{7}
\end{equation*}
$$

for $r>0$ and $f \in \mathcal{A}$. Further applying the fractional integral for $f \in \mathcal{A}$, we define a new operator $A_{-r} f(z)$ given by

$$
\begin{equation*}
A_{-r} f(z)=\frac{\Gamma\left(\frac{3+r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right)} z^{\frac{1-r}{2}} D_{z}^{-r}\left(z^{\frac{-1-r}{2}} f(z)\right) \tag{8}
\end{equation*}
$$

where $0 \leq r \leq 1$. If $r=0$, then (8) becomes $A_{0} f(z)=f(z)$ and if $r=1$, then (8) leads us that $A_{-1} f(z)$.

With this integral operator, we know

$$
\begin{equation*}
A_{-j-r} f(z)=A_{-j}\left(A_{-r} f(z)\right)=z+\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}} a_{k} z^{k} \tag{9}
\end{equation*}
$$

for $z \in \mathcal{U}$, where $j \in \mathbb{N}$ and $0 \leq r \leq 1 . A_{-j-r} f(z)$ is the generalization of (5).
If $f \in \mathcal{A}(n)$ is given by (2) then we have

$$
\begin{equation*}
\widetilde{A}_{-j-r} f(z)=z-\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}} a_{k} z^{k} \quad(z \in \mathcal{U}) \tag{10}
\end{equation*}
$$

where $j \in \mathbb{N}$ and $0 \leq r \leq 1$.
Finally, by using (10), we investigate the subclasses $\mathcal{M}_{j}(r, \alpha, \tau)$ and $\mathcal{R}_{j}(r, \alpha$, $\tau, \mu)$ of $\mathcal{A}(n)$ consisting of functions $f$ as following:

Definition 1.2. Let $\mathcal{M}_{j}(r, \alpha, \tau)$ denote the subclass of $\mathcal{A}(n)$ consisting of $f$ which satisfy the inequality

$$
\begin{equation*}
\left|\frac{1}{\tau}\left(\frac{z\left[\widetilde{A}_{-j-r} f(z)\right]^{\prime}}{\widetilde{A}_{-j-r} f(z)}-1\right)\right|<\alpha \quad(z \in \mathcal{U}) \tag{11}
\end{equation*}
$$

where $j \in \mathbb{N}, 0 \leq r \leq 1, \tau \in \mathbb{C} \backslash\{0\}$ and $0<\alpha \leq 1$.

Definition 1.3. Let $\mathcal{R}_{j}(r, \alpha, \tau, \mu)$ denote the subclass of $\mathcal{A}(n)$ consisting of $f$ which satisfy the inequality

$$
\begin{equation*}
\left|\frac{1}{\tau}\left[(1-\mu) \frac{\widetilde{A}_{-j-r} f(z)}{z}+\mu\left(\widetilde{A}_{-j-r} f(z)\right)^{\prime}-1\right]\right|<\alpha \quad(z \in \mathcal{U}) \tag{12}
\end{equation*}
$$

where $j \in \mathbb{N}, 0 \leq r \leq 1, \tau \in \mathbb{C} \backslash\{0\}, 0<\alpha \leq 1$ and $0 \leq \mu \leq 1$.

In this paper, we obtain the coefficient inequalities, inclusion relations, neighborhood properties, partial sums and integral means of the subclasses $\mathcal{M}_{j}(r, \alpha$, $\tau)$ and $\mathcal{R}_{j}(r, \alpha, \tau, \mu)$.

## 2. Coefficient Inequalities for Classes $M_{j}(r, \alpha, \tau)$ and $R_{j}(r, \alpha, \tau, \mu)$

Theorem 2.1. Let $f \in \mathcal{A}(n)$. Then $f \in \mathcal{M}_{j}(r, \alpha, \tau)$ if and only if

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} \varphi_{k}(r, \alpha, \tau) a_{k} \leq 1 \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi_{k}(r, \alpha, \tau)=\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right)}\left(\frac{k-1+\alpha|\tau|}{k^{j} \alpha|\tau|}\right) \tag{14}
\end{equation*}
$$

for $j \in \mathbb{N}, 0 \leq r \leq 1, \tau \in \mathbb{C} \backslash\{0\}$ and $0<\alpha \leq 1$.
Proof. Let $f \in \mathcal{A}(n)$. Then, by (11) we can write

$$
\begin{equation*}
\Re\left\{\frac{z\left[\widetilde{A}_{-j-r} f(z)\right]^{\prime}}{\widetilde{A}_{-j-r} f(z)}-1\right\}>-\alpha|\tau| \quad(z \in \mathcal{U}) \tag{15}
\end{equation*}
$$

Using (2) and (10), we have,

$$
\begin{equation*}
\Re\left\{\frac{-\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}}[k-1] a_{k} z^{k}}{z-\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{-r}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}} a_{k} z^{k}}\right\}>-\alpha|\tau| \quad(z \in \mathcal{U}) \tag{16}
\end{equation*}
$$

Since (16) is true for all $z \in \mathcal{U}$, choose values of $z$ on the real axis. Letting $z \rightarrow 1$, through the real values, the inequality (16) yields the desired inequality

$$
\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}}[k-1+\alpha|\tau|] a_{k} \leq \alpha|\tau|
$$

Conversely, supposed that inequality (13) holds true and $|z|=1$. We obtain

$$
\begin{aligned}
\left|\frac{z\left[\widetilde{A}_{-j-r} f(z)\right]^{\prime}}{\widetilde{A}_{-j-r} f(z)}-1\right| & \leq\left|\frac{\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}}[k-1] a_{k} z^{k}}{z-\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}} a_{k} z^{k}}\right| \\
& \leq \frac{\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}}[k-1] a_{k}}{1-\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right) k^{j}} a_{k}} \\
& \leq \alpha|\tau| .
\end{aligned}
$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{M}_{j}(r, \alpha, \tau)$, which establishes the required result.

Theorem 2.2. Let $f \in \mathcal{A}(n)$. Then $f \in \mathcal{R}_{j}(r, \alpha, \tau, \mu)$ if and only if

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} \psi_{k}(r, \alpha, \tau, \mu) a_{k} \leq 1 \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi_{k}(r, \alpha, \tau, \mu)=\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 k+1-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 k+1+r}{2}\right)}\left(\frac{1+\mu(k-1)}{k^{j} \alpha|\tau|}\right) \tag{18}
\end{equation*}
$$

for $j \in \mathbb{N}, 0 \leq r \leq 1, \tau \in \mathbb{C} \backslash\{0\}, 0<\alpha \leq 1$ and $0 \leq \mu \leq 1$.
Proof. We omit the proofs since it is similar to Theorem 2.1.

## 3. Inclusion Relations Involving $N_{n, \delta}(e)$ of the Classes $M_{j}(r, \alpha, \tau)$ and $R_{j}(r, \alpha, \tau, \mu)$

Theorem 3.1. If

$$
\begin{equation*}
\delta=\frac{(n+1)^{j+1} \alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)}(n+\alpha|\tau|)} \quad(|\tau|<1) \tag{19}
\end{equation*}
$$

then $\mathcal{M}_{j}(r, \alpha, \tau) \subset \mathcal{N}_{n, \delta}(e)$.

Proof. Let $f \in \mathcal{M}_{j}(r, \alpha, \tau)$. By Theorem 2.1, we have

$$
\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(n+\alpha|\tau|) \sum_{k=n+1}^{\infty} a_{k} \leq \alpha|\tau|
$$

which implies

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} a_{k} \leq \frac{\alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(n+\alpha|\tau|)} \tag{20}
\end{equation*}
$$

Using (13) and (20), we get

$$
\begin{aligned}
& \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}} \sum_{k=n+1}^{\infty} k a_{k} \\
\leq & \alpha|\tau|+\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(1-\alpha|\tau|) \sum_{k=n+1}^{\infty} a_{k} \\
\leq & \frac{(n+1) \alpha|\tau|}{n+\alpha|\tau|}=\delta .
\end{aligned}
$$

That is,

$$
\sum_{k=n+1}^{\infty} k a_{k} \leq \frac{(n+1) \alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(n+\alpha|\tau|)}=\delta
$$

Thus, by the definition given by (4), $f \in \mathcal{N}_{n, \delta}(e)$, which completes the proof.

Theorem 3.2. If

$$
\begin{equation*}
\delta=\frac{(n+1)^{j+1} \alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)}(1+\mu n)} \quad(|\tau|<1) \tag{21}
\end{equation*}
$$

then $\mathcal{R}_{j}(r, \alpha, \tau, \mu) \subset \mathcal{N}_{n, \delta}(e)$.
Proof. For $f \in \mathcal{R}_{j}(r, \alpha, \tau, \mu)$ and making use of the condition (17), we obtain

$$
\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(1+\mu n) \sum_{k=n+1}^{\infty} a_{k} \leq \alpha|\tau|
$$

so that

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} a_{k} \leq \frac{\alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(1+\mu n)} \tag{22}
\end{equation*}
$$

Thus, using (17) along with (22), we also get

$$
\begin{aligned}
& \mu \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}} \sum_{k=n+1}^{\infty} k a_{k} \\
\leq & \alpha|\tau|+(\mu-1) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}} \sum_{k=n+1}^{\infty} a_{k} \\
\leq & \alpha|\tau|+(\mu-1) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}} \frac{\alpha|\tau|}{(1+\mu n) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}}
\end{aligned}
$$

Hence,

$$
\sum_{k=n+1}^{\infty} k a_{k} \leq \frac{(n+1) \alpha|\tau|}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(1+\mu n)}=\delta
$$

which in view of (4), completes the proof of theorem.

## 4. Neighborhood Properties for the Classes $M_{j}^{\eta}(r, \alpha, \tau)$ <br> and $R_{j}^{\eta}(r, \alpha, \tau, \mu)$

Definition 4.1. For $0 \leq \eta<1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{M}_{j}^{\eta}(r, \alpha, \tau)$ if there exists a function $g \in \mathcal{M}_{j}(r, \alpha, \tau)$ such that

$$
\begin{equation*}
\left|\frac{f(z)}{g(z)}-1\right|<1-\eta \tag{23}
\end{equation*}
$$

For $0 \leq \eta<1$ and $z \in \mathcal{U}$, a function $f \in \mathcal{R}_{j}^{\eta}(r, \alpha, \tau, \mu)$ if there exists a function $g \in \mathcal{R}_{j}(r, \alpha, \tau, \mu)$ such that the inequality (23) holds true.

Theorem 4.2. If $g \in \mathcal{M}_{j}(r, \alpha, \tau)$ and

$$
\begin{equation*}
\eta=1-\frac{\delta(n+\alpha|\tau|) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j+1}}}{(n+\alpha|\tau|) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}-\alpha|\tau|} \tag{24}
\end{equation*}
$$

then $\mathcal{N}_{n, \delta}(g) \subset \mathcal{M}_{j}^{\eta}(r, \alpha, \tau)$.
Proof. Let $f \in \mathcal{N}_{n, \delta}(g)$. Then,

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} k\left|a_{k}-b_{k}\right| \leq \delta \tag{25}
\end{equation*}
$$

which yields the coefficient inequality,

$$
\sum_{k=n+1}^{\infty}\left|a_{k}-b_{k}\right| \leq \frac{\delta}{n+1} \quad(n \in \mathbb{N})
$$

Since $g \in \mathcal{M}_{j}(r, \alpha, \tau)$ by (20), we have

$$
\begin{equation*}
\sum_{k=n+1}^{\infty} b_{k} \leq \frac{\alpha|\tau|}{(n+\alpha|\tau|) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}} \tag{26}
\end{equation*}
$$

and so

$$
\begin{aligned}
\left|\frac{f(z)}{g(z)}-1\right| & <\frac{\sum_{k=n+1}^{\infty}\left|a_{k}-b_{k}\right|}{1-\sum_{k=n+1}^{\infty} b_{k}} \\
& \leq \frac{\delta}{n+1} \frac{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(n+\alpha|\tau|)}{\frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}(n+\alpha|\tau|)-\alpha|\tau|} \\
& =1-\eta .
\end{aligned}
$$

Thus, by definition, $f \in \mathcal{M}_{j}^{\eta}(r, \alpha, \tau)$ for $\eta$ given by (24), which establishes the desired result.

Theorem 4.3. If $g \in \mathcal{R}_{j}(r, \alpha, \tau, \mu)$ and

$$
\begin{equation*}
\eta=1-\frac{\delta(1+\mu n) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j+1}}}{(1+\mu n) \frac{\Gamma\left(\frac{3+r}{2}\right) \Gamma\left(\frac{2 n+3-r}{2}\right)}{\Gamma\left(\frac{3-r}{2}\right) \Gamma\left(\frac{2 n+3+r}{2}\right)(n+1)^{j}}-\alpha|\tau|} \tag{27}
\end{equation*}
$$

then $\mathcal{N}_{n, \delta}(g) \subset \mathcal{R}_{j}^{\eta}(r, \alpha, \tau, \mu)$.
Proof. We omit the proof since it is similar to Theorem 4.2.

## 5. Partial Sums for the Classes $M_{j}(r, \alpha, \tau)$ and $R_{j}(r, \alpha, \tau, \mu)$

Following the earlier works by Silverman [29] and others (see [12, 13, 28, 30]), in this section we investigate the ratio of real parts of functions involving (2) and their sequence of partial sums defined by

$$
\begin{align*}
f_{1}(z) & =z  \tag{28}\\
f_{m}(z) & =z-\sum_{k=n+1}^{m} a_{k} z^{k} \quad(n \in \mathbb{N})
\end{align*}
$$

and determine sharp lower bounds for

$$
\Re\left\{\frac{f(z)}{f_{m}(z)}\right\}, \Re\left\{\frac{f_{m}(z)}{f(z)}\right\}, \Re\left\{\frac{f^{\prime}(z)}{f_{m}^{\prime}(z)}\right\} \text { and } \Re\left\{\frac{f_{m}^{\prime}(z)}{f^{\prime}(z)}\right\} .
$$

Firstly, we obtain partial sums for the class $\mathcal{M}_{j}(r, \alpha, \tau)$.

Theorem 5.1. If $f$ of the form (2) satisfies condition (13), then

$$
\begin{align*}
& \Re\left\{\frac{f(z)}{f_{m}(z)}\right\} \geq 1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)}  \tag{29}\\
& \Re\left\{\frac{f_{m}(z)}{f(z)}\right\} \geq \frac{\varphi_{m+n+1}(r, \alpha, \tau)}{\varphi_{m+n+1}(r, \alpha, \tau)+1} \tag{30}
\end{align*}
$$

where $\varphi_{m+n+1}(r, \alpha, \tau)$ is given by (14). The results are sharp for every $m$, with the extremal function given by

$$
\begin{equation*}
f(z)=z-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)} z^{m+1} \tag{31}
\end{equation*}
$$

Proof. In order to prove (29), it is sufficient to show that

$$
\begin{equation*}
\varphi_{m+n+1}(r, \alpha, \tau)\left[\frac{f(z)}{f_{m}(z)}-\left(1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \prec \frac{1+z}{1-z} \tag{32}
\end{equation*}
$$

for $z \in \mathcal{U}$. We can write

$$
\begin{aligned}
& \varphi_{m+n+1}(r, \alpha, \tau)\left[\frac{f(z)}{f_{m}(z)}-\left(1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \\
= & \varphi_{m+n+1}(r, \alpha, \tau)\left[\frac{1-\sum_{k=n+1}^{\infty} a_{k} z^{k-1}}{1-\sum_{k=n+1}^{m} a_{k} z^{k-1}}-\left(1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \\
= & \frac{1+w(z)}{1+w(z)}
\end{aligned}
$$

Then

$$
w(z)=\frac{-\varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k} z^{k-1}}{2-2 \sum_{k=n+1}^{m} a_{k} z^{k-1}-\varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k} z^{k-1}}
$$

Notice that $w(0)=0$ and

$$
|w(z)| \leq \frac{\varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k}}{2-2 \sum_{k=n+1}^{m} a_{k}-\varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k}}
$$

Now, $|w(z)|<1$ if and only if

$$
2 \varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k} \leq 2-2 \sum_{k=n+1}^{m} a_{k},
$$

which is equivalent to

$$
\begin{equation*}
\sum_{k=n+1}^{m} a_{k}+\varphi_{m+n+1}(r, \alpha, \tau) \sum_{k=m+n+1}^{\infty} a_{k} \leq 1 . \tag{33}
\end{equation*}
$$

In view of (13), this is equivalent to showing that

$$
\begin{equation*}
\sum_{k=n+1}^{m}\left(\varphi_{k}(r, \alpha, \tau)-1\right) a_{k}+\sum_{k=m+n+1}^{\infty}\left(\varphi_{k}(r, \alpha, \tau)-\varphi_{m+n+1}(r, \alpha, \tau)\right) a_{k} \geq 0 . \tag{34}
\end{equation*}
$$

To see that the function $f$ given by (31) gives the sharp result, we observe for $z=r e^{2 \pi i / m}$ that

$$
\frac{f(z)}{f_{m}(z)}=1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)} z^{m} \rightarrow 1-\frac{1}{\varphi_{m+n+1}(r, \alpha, \tau)}
$$

where $r \rightarrow 1^{-}$.
This completes the proof of (29).
The proof of (30) is similar to (29) and will be omitted.
Theorem 5.2. If $f$ of the form (2) satisfies condition (13), then

$$
\begin{align*}
& \Re\left\{\frac{f^{\prime}(z)}{f_{m}^{\prime}(z)}\right\} \geq 1-\frac{m+1}{\varphi_{m+n+1}(r, \alpha, \tau)},  \tag{35}\\
& \Re\left\{\frac{f_{m}^{\prime}(z)}{f^{\prime}(z)}\right\} \geq \frac{\varphi_{m+n+1}(r, \alpha, \tau)}{\varphi_{m+n+1}(r, \alpha, \tau)+m+1} \tag{36}
\end{align*}
$$

where $\varphi_{m+n+1}(r, \alpha, \tau)$ is given by (14). The results are sharp for every $m$, with the extremal function given by (31).

Proof. In order to prove (35), it is sufficient to show that

$$
\begin{equation*}
\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1}\left[\frac{f^{\prime}(z)}{f_{m}^{\prime}(z)}-\left(1-\frac{m+1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \prec \frac{1+z}{1-z}, \tag{37}
\end{equation*}
$$

for $z \in \mathcal{U}$. We can write

$$
\begin{aligned}
& \frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1}\left[\frac{f^{\prime}(z)}{f_{m}^{\prime}(z)}-\left(1-\frac{m+1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \\
= & \frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1}\left[\frac{1-\sum_{k=n+1}^{\infty} k a_{k} z^{k-1}}{1-\sum_{k=n+1}^{m} k a_{k} z^{k-1}}-\left(1-\frac{m+1}{\varphi_{m+n+1}(r, \alpha, \tau)}\right)\right] \\
= & \frac{1+w(z)}{1+w(z)} .
\end{aligned}
$$

Then

$$
w(z)=\frac{-\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k} z^{k-1}}{2-2 \sum_{k=n+1}^{m} k a_{k} z^{k-1}-\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k} z^{k-1}} .
$$

Obviously $w(0)=0$ and

$$
|w(z)| \leq \frac{\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k}}{2-2 \sum_{k=n+1}^{m} k a_{k}-\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k}}
$$

Now, $|w(z)|<1$ if and only if

$$
2 \frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k} \leq 2-2 \sum_{k=n+1}^{m} k a_{k}
$$

which is equivalent to

$$
\begin{equation*}
\sum_{k=n+1}^{m} k a_{k}+\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} \sum_{k=m+n+1}^{\infty} k a_{k} \leq 1 \tag{38}
\end{equation*}
$$

In view of (13), this is equivalent to showing that

$$
\begin{equation*}
\sum_{k=n+1}^{m}\left(\varphi_{k}(r, \alpha, \tau)-k\right) a_{k}+\sum_{k=m+n+1}^{\infty}\left(\varphi_{k}(r, \alpha, \tau)-\frac{\varphi_{m+n+1}(r, \alpha, \tau)}{m+1} k\right) a_{k} \geq 0 \tag{39}
\end{equation*}
$$

Thus, we have completed the proof of (35).
The proof of (36) is similar to (35) and is omitted.

Finally, we get partial sums for the class $\mathcal{R}_{j}(r, \alpha, \tau, \mu)$.
Theorem 5.3. If $f$ of the form (2) satisfies condition (17), then

$$
\begin{align*}
& \Re\left\{\frac{f(z)}{f_{m}(z)}\right\} \geq 1-\frac{1}{\psi_{m+n+1}(r, \alpha, \tau, \mu)}  \tag{40}\\
& \Re\left\{\frac{f_{m}(z)}{f(z)}\right\} \geq \frac{\psi_{m+n+1}(r, \alpha, \tau, \mu)}{\psi_{m+n+1}(r, \alpha, \tau, \mu)+1} \tag{41}
\end{align*}
$$

where $\psi_{m+n+1}(r, \alpha, \tau, \mu)$ is given by (18). The results are sharp for every $m$, with the extremal function given by (31).

Proof. The proof of Theorem 5.3 is similar to Theorem 5.1 and is omitted.

Theorem 5.4. If $f$ of the form (2) satisfies condition (17), then

$$
\begin{align*}
& \Re\left\{\frac{f^{\prime}(z)}{f_{m}^{\prime}(z)}\right\} \geq 1-\frac{m+1}{\psi_{m+n+1}(r, \alpha, \tau, \mu)}  \tag{42}\\
& \Re\left\{\frac{f_{m}^{\prime}(z)}{f^{\prime}(z)}\right\} \geq \frac{\psi_{m+n+1}(r, \alpha, \tau, \mu)}{\psi_{m+n+1}(r, \alpha, \tau, \mu)+m+1} \tag{43}
\end{align*}
$$

where $\psi_{m+n+1}(r, \alpha, \tau, \mu)$ is given by (18). The results are sharp for every $m$, with the extremal function given by (31).

Proof. The proof of Theorem 5.4 is similar to Theorem 5.2 and is omitted.

## 6. Integral Means for the Classes $M_{j}(r, \alpha, \tau)$ and $R_{j}(r, \alpha, \tau, \mu)$

The following subordination result due to Littlewood [19] will be required in our investigation. The integral means of analytic functions was studied in [14], [24] and [26].

Lemma 6.1. If $f$ and $g$ are analytic in $\mathcal{U}$ with $f(z) \prec g(z)$, then

$$
\int_{0}^{2 \pi}\left|f\left(r e^{i \theta}\right)\right|^{\rho} d \theta \leq \int_{0}^{2 \pi}\left|g\left(r e^{i \theta}\right)\right|^{\rho} d \theta
$$

where $\rho>0, z=r e^{i \theta}$ and $0<r<1$.

First, we obtain integral means for the class $\mathcal{M}_{j}(r, \alpha, \tau)$ using Lemma 6.1.

Theorem 6.2. Let $\rho>0$. If $f \in \mathcal{M}_{j}(r, \alpha, \tau)$ is given by (2) and $f_{2}(z)$ is defined by

$$
f_{2}(z)=z-\frac{1}{\varphi_{n+2}(r, \alpha, \tau)} z^{2}
$$

where $\varphi_{m+n+1}(r, \alpha, \tau)$ is defined by (14), then for $z=r e^{i \theta}$ and $0<r<1$, we have

$$
\begin{equation*}
\int_{0}^{2 \pi}|f(z)|^{\rho} d \theta \leq \int_{0}^{2 \pi}\left|f_{2}(z)\right|^{\rho} d \theta \tag{44}
\end{equation*}
$$

Proof. For function $f$ of the form (2) is equivalent to proving that

$$
\int_{0}^{2 \pi}\left|1-\sum_{k=n+1}^{\infty} a_{k} z^{k-1}\right|^{\rho} d \theta \leq \int_{0}^{2 \pi}\left|1-\frac{1}{\varphi_{n+2}(r, \alpha, \tau)} z\right|^{\rho} d \theta
$$

By Lemma 6.1, it suffices to show that

$$
1-\sum_{k=n+1}^{\infty} a_{k} z^{k-1} \prec 1-\frac{1}{\varphi_{n+2}(r, \alpha, \tau)} z
$$

Setting

$$
\begin{equation*}
1-\sum_{k=n+1}^{\infty} a_{k} z^{k-1}=1-\frac{1}{\varphi_{n+2}(r, \alpha, \tau)} w(z) \tag{45}
\end{equation*}
$$

From (45) and (13), we obtain

$$
\begin{aligned}
|w(z)| & =\left|\sum_{k=n+1}^{\infty} \varphi_{n+2}(r, \alpha, \tau) a_{k} z^{k-1}\right| \leq|z| \sum_{k=n+1}^{\infty} \varphi_{n+2}(r, \alpha, \tau) a_{k} \\
& \leq|z| \sum_{k=n+1}^{\infty} \varphi_{m+n+1}(r, \alpha, \tau) a_{k} \leq|z|
\end{aligned}
$$

which completes the proof.

Finally, we have integral means for the class $\mathcal{R}_{j}(r, \alpha, \tau, \mu)$ using Lemma 6.1.

Theorem 6.3. Let $\rho>0$. If $f \in \mathcal{R}_{j}(r, \alpha, \tau, \mu)$ is given by (2) and $f_{2}(z)$ is defined by

$$
f_{2}(z)=z-\frac{1}{\psi_{n+2}(r, \alpha, \tau, \mu)} z^{2}
$$

where $\psi_{m+n+1}(r, \alpha, \tau, \mu)$ is defined by (18), then for $z=r e^{i \theta}$ and $0<r<1$, we have

$$
\begin{equation*}
\int_{0}^{2 \pi}|f(z)|^{\rho} d \theta \leq \int_{0}^{2 \pi}\left|f_{2}(z)\right|^{\rho} d \theta \tag{46}
\end{equation*}
$$

Proof. The proof of Theorem 6.3 is similar to Theorem 6.2 and is omitted.

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