Southeast Asian Bulletin of Mathematics (2023) 47: 455–461

Southeast Asian
Bulletin of
Mathematics

A Characterization of Finite Dimensional Nilpotent Filippov Algebras $s(A) = 4, 5$

Hamid Darabi
Department of Mathematics, Esfarayen University of Technology, Esfarayen, Iran
Email: darabi@esfarayen.ac.ir

Babak Jabbar Nezhad
Maku, West Azerbaijan, Iran
Email: babak.jab@gmail.com

Received 6 November 2020
Accepted 20 February 2023

Communicated by S. Parvathi

AMS Mathematics Subject Classification(2020): 17B05, 17B30

Abstract. Let A be a nilpotent Filippov (n-Lie) algebra of dimension d and put $s(A) = \binom{d-1}{n} + n - 1 - \dim M(A)$, where $M(A)$ denotes the multiplier of A. The aim of this paper is to classify all nilpotent n-Lie algebras A for which $s(A) = 4$ or 5.

Keywords: n-Lie algebra; Nilpotent n-Lie algebra; Multiplier.

1. Introduction

In 1985, Filippov [10] introduced the concept of *n-Lie (Filippov) algebra*, as an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra when $n = 2$.

The study of n-Lie algebras is important since it has applications in geometry and physics. Beside other problems about n-lie algebras, classification of n-Lie algebras is an important problem in this area. Actually, some cases of n-Lie algebras are classified. For example, Bai et. al [1] classify all n-Lie algebras of dimension $n+2$ over an algebraically closed field of characteristic zero. Some other papers about classification of n-Lie algebras include [2, 5, 13].

In 1986, Kasymov [14] introduced the notion of nilpotency of an n-Lie algebra as follows: An n-Lie algebra A is *nilpotent* if $A^t = 0$ for some positive integer t, where A^i is defined inductively by $A^1 = A$ and $A^{i+1} = [A^i, A, \ldots, A]$. The ideal $A^2 = [A, \ldots, A]$ is called the *derived subalgebra* of A. The *center* of A is defined to be

$$Z(A) = \{x \in A : [x, A, \ldots, A] = 0\}.$$

Let A be an n-Lie algebra over a field Λ with a free presentation

$$0 \to R \to F \to A \to 0,$$

in which F is a free n-Lie algebra. The multiplier of A, denoted by $M(A)$, is defined as

$$M(A) = \frac{R \cap F^2}{[R, F, \ldots, F]}.$$

Notice that the multiplier of an n-Lie algebra is always abelian and any two multiplier of A are isomorphic.

Authors in [7] proved that for an n-Lie algebra A of dimension d, there exists the non-negative integer number $s(A)$ such that $\dim M(A) = \binom{d-1}{n} + n - 1 - s(A)$, where $M(A)$ denotes the Schur multiplier of A. They also classified all nilpotent n-Lie algebras A for which $s(A) = 0, 1, 2$. Moreover, all nilpotent n-Lie algebras A for which $s(A) = 3$ are classified in [8].

Shamsaki et al. [16, 15] classified the nilpotent Lie algebras with $s(A) = 4$ and $s(A) = 5$ as follows.

Theorem 1.1. *Let L be a non-abelian d-dimensional nilpotent Lie algebra. Then*

(i) $s(L) = 4$ *if and only if L is isomorphic to one of the Lie algebras* $L_{5,8} \oplus F(3)$, $L_{4,3} \oplus F(2)$, $L_{5,5} \oplus F(1)$, $L_{5,6}$, $L_{5,7}$, $L_{5,9}$, $L_{6,22}(\epsilon) \oplus F(1)$ *or* $37A$.

(ii) $s(L) = 5$ *if and only if L is isomorphic to one of the Lie algebras* $L_{5,8} \oplus F(4)$, $L_{4,3} \oplus F(3)$, $L_{5,5} \oplus F(2)$, $L_{6,22}(\epsilon) \oplus F(2)$, $L_{6,26} \oplus F(1)$, $L_{6,10}$, $L_{6,23}$, $L_{6,25}$, $L_{6,27}$, $37B$, $37C$ *or* $37D$.

In this paper, we want to investigate nilpotent n-Lie algebras A satisfying $s(A) = 4$ or 5. Moreover, we assume that all n-Lie algebras are finite dimensional, and every non-mentioned bracket is assumed to be zero. In this paper, for nilpotent Lie algebras we choose the same notations used in [3, 11].

Let's have a short review on the concept of special Heisenberg n-Lie algebra which plays an important role in the classification of n-Lie algebras. This n-Lie algebra is introduced in [9]. An n-Lie algebra A is called special Heisenberg if $A^2 = Z(A)$ and $\dim A^2 = 1$. In [9], the first author proves that every special Heisenberg n-Lie algebra has dimension $mn + 1$, where m is a natural number.

They also prove that a special Heisenberg n-Lie algebra of dimension $mn+1$ is given by

$$H(n,m) = \langle x, x_1, \ldots, x_{mn} : [x_{n(i-1)+1}, x_{n(i-1)+2}, \ldots, x_{ni}] = x, i = 1, \ldots, m\rangle.$$

The dimension of Schur multiplier of d-dimensional abelian n-Lie algebra $F(d)$ and special Heisenberg n-Lie algebra are computed in [6, Theorem 3.4], and [9, Theorem 2.3], as follows.

Theorem 1.2. [6, 9] *We have*
(i) $\dim M(F(d)) = \binom{d}{n}$,
(ii) $\dim M(H(n,1)) = n$,
(iii) $\dim M(H(n,m)) = \binom{mn}{n} - 1$, *for all* $m \geq 2$.

We end this section by listing some results that we will use them in the next section.

Theorem 1.3. [6] *Let A and B be two finite dimensional n-Lie algebras. Then*

$$\dim M(A \oplus B) = \dim M(A) + \dim M(B) + \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n},$$

where $a = \dim A/A^2$ *and* $b = \dim B/B^2$.

Theorem 1.4. [9] *Let A be a d-dimensional nilpotent n-Lie algebra and* $\dim A^2 = m \geq 1$. *Then* $\dim M(A) \leq \binom{d-m+1}{n} + (m-2)\binom{d-m}{n-1} + n - m$.

2. Main Results

In this section, we characterizes the structure of all finite dimensional nilpotent n-Lie algebras with $s(A) = 4, 5$.

Lemma 2.1. *Suppose A is a $(n+3)$-dimensional non-abelian nilpotent n-Lie algebra. Then A is isomorphic to $A_{n,n+3,i}$ when $2 \leq i \leq 8$. Furthermore,* $s(A_{n,n+3,2}) = 0$, $s(A_{n,n+3,3}) = s(A_{n,n+3,4}) = \binom{n+2}{n} - n - 1$, $s(A_{n,n+3,5}) = \binom{n+2}{n} - 3n + 1$, $s(A_{n,n+3,6}) = \binom{n+2}{n} - n$, $s(A_{n,n+3,7}) = s(A_{n,n+3,8}) = \binom{n+2}{n} - 2$.

Proof. Using Lemma 2.5 of [5], the dimension of the multipliers of the nilpotent n-Lie algebras of dimension $(n+3)$ are as follows. By Lemma 2.5 of [4], $\dim M(A_{n,n+3,2}) = \binom{n+2}{n} + n - 1$, $\dim M(A_{n,n+3,3}) = \dim M(A_{n,n+3,4}) = 2n$, $\dim M(A_{n,n+3,5}) = 4n - 2$, $\dim M(A_{n,n+3,6}) = 2n - 1$, $\dim M(A_{n,n+3,7}) = \dim M(A_{n,n+3,8}) = n + 1$.

Thus $s(A_{n,n+3,2}) = 0$, $s(A_{n,n+3,3}) = s(A_{n,n+3,4}) = \binom{n+2}{n} - n - 1$, $s(A_{n,n+3,5}) = \binom{n+2}{n} - 3n + 1$, $s(A_{n,n+3,6}) = \binom{n+2}{n} - n$, $s(A_{n,n+3,7}) = s(A_{n,n+3,8}) = \binom{n+2}{n} - 2$. ■

Lemma 2.2. *There is no d-dimensional nilpotent n-Lie algebra A for which $\dim A^2 \geq 3$ and $s(A) = 4, 5$ when $d \geq n + 4$ and $n \geq 3$.*

Proof. Let $\dim A^2 = m$. Since $m \geq 3$ and the bound for the multiplier of A given in Theorem 1.4, is decreasing, we have

$$\binom{d-1}{n} + n - 1 - s(A) \leq \binom{d-2}{n} + \binom{d-3}{n-1} + n - 3.$$

Hence,

$$\binom{d-3}{n-2} \leq s(A) - 2,$$

which does not hold in the cases $d \geq n + 4$, $n \geq 3$ and $s(A) = 4$ or 5. ■

Lemma 2.3. *There is no d-dimensional nilpotent n-Lie algebra A satisfying $\dim A^2 = 2$, $\dim Z(A) = 1$, $d \geq n + 4$, $n \geq 3$ and $s(A) = 4, 5$.*

Proof. By Theorem 3.3 of [8], we have

$$s(A) \geq \binom{d-2}{n-1} - n + 1.$$

Since this inequality does not hold in the cases $d \geq n + 4$, $n \geq 3$ and $s(A) = 4$ or 5. This completes the proof. ■

Lemma 2.4. *There is no d-dimensional nilpotent n-Lie algebra A satisfying $\dim A^2 = 2$, $s(A) = 4, 5$ when*

(i) $\dim Z(A) \geq 3$, $d \geq n + 5$ *and* $n \geq 3$,
(ii) $\dim Z(A) \geq 3$, $d \geq n + 4$ *and* $n \geq 4$,
(iii) $\dim Z(A) = 2$, $A^2 \not\subset Z(A)$, $d \geq n + 5$ *and* $n \geq 3$,
(iv) $\dim Z(A) = 2$, $A^2 \not\subset Z(A)$, $d \geq n + 4$ *and* $n \geq 4$.

Proof. Let A be a d-dimensional nilpotent n-Lie algebra with $\dim A^2 = 2$. If $\dim Z(A) \geq 3$ or $\dim Z(A) = 2$, $A^2 \not\subset Z(A)$ then $s(A) \geq \binom{d-3}{n-2} + 1$ by Theorem 3.3 of [8]. Thus, if $d \geq n + 5$ and $n \geq 3$ then $s(A) \geq 6$ and if $d \geq n + 4$ and $n \geq 4$ then $s(A) \geq 11$. This completes the proof. ■

Lemma 2.5. *Let A be a $(n + 4)$-dimensional nilpotent n-Lie algebra such that $\dim A^2 = 2$ and $A^2 \nsubseteq Z(A)$. Then*

(i) $A \cong A_{n,n+2,2} \oplus F(2)$ *if* $\dim Z(A) = 3$.
(ii) $A \cong A_{n,n+3,3} \oplus F(1)$ *if* $\dim Z(A) = 2$.

Moreover, $s(A_{n,n+2,2} \oplus F(2)) = s(A_{n,n+3,3} \oplus F(1)) = \binom{n+2}{n-1}$.

Proof. (i) Since $A^2 \nsubseteq Z(A)$, we have $\dim A^3 = 1$ and $A^4 = 0$. Thus $A^3 \subseteq Z(A)$. Therefore there exists an $(n + 2)$-dimensional nilpotent n-Lie algebra B such that $A \cong B \oplus F(2)$, $\dim B^2 = 2$ and $\dim Z(B) = 1$. According to Theorem 3.3 of [7], $B \cong A_{n,n+2,2}$.

(ii) Similar to (i), $A^3 \subseteq Z(A)$. Thus there exists an $(n+3)$-dimensional nilpotent n-Lie algebra B such that $A \cong B \oplus F(1)$, $\dim B^2 = 2$ and $\dim Z(B) = 1$. According to the classification of $(n+3)$-dimensional nilpotent n-Lie algebras, $B \cong A_{n,n+3,3}$.

By Theorem 3.3 of [7], $\dim M(A_{n,n+2,2}) = n$. Thus $\dim M(A_{n,n+2,2} \oplus F(2)) = \binom{n+2}{n} + n - 1$ by Theorem 1.3. The proof is complete by definition of $s(A)$. Similarly, by Lemma 2.5 of [4], $\dim M(A_{n,n+3,3}) = 2n$. Thus $\dim M(A_{n,n+3,3} \oplus F(1)) = \binom{n+2}{n} + n - 1$ and $s(A_{n,n+3,3} \oplus F(1)) = \binom{n+2}{n-1}$. ■

Theorem 2.6. *Let A be a nilpotent n-Lie algebra $(n \geq 3)$ of dimension d. Then $s(A) = 4$ if and only if $A \cong H(4,r) \oplus F(d-4r-1)$ for some $r \geq 2$, $A_{4,6,2}$ or $A_{4,7,5}$.*

Proof. If $\dim A^2 = 1$, then by Lemma 3.4 of [8], $A \cong H(4,r) \oplus F(d-4r-1)$ for some $r \geq 2$. If $d \leq n+3$, then by Theorem 3.3 of [7], and Lemma 2.1, we have $A \cong A_{4,6,2}$ or $A_{4,7,5}$. Now, by Lemmas 2.2 and 2.3, we may assume that $\dim A^2 = 2$ and $\dim Z(A) \geq 2$. By Lemma 2.4, there is no nilpotent n-Lie algebra A when $\dim Z(A) \geq 3$ or $\dim Z(A) = 2$, $A^2 \nsubseteq Z(A)$. If $\dim A^2 = 2$ and $A^2 \subseteq Z(A)$, then A is an n-Lie algebra of class two with two-dimensional derived subalgebra. Then, by Theorem 5.2 of [12], the dimension of Schur multiplier A is equal to $\binom{d-2}{n} + j$ when j is $-1, -2, n-2, 2n-2, 3n-k (3 \leq k \leq n+1)$. It is clear that in this case there is no estimated algebra. ■

Theorem 2.7. *Let A be a nilpotent n-Lie algebra $(n \geq 3)$ of dimension d. Then $s(A) = 5$ if and only if A is isomorphic to $A_{5,7,2}$ or $H(5,r) \oplus F(d-5r-1)$, for some $r \geq 2$.*

Proof. If $\dim A^2 = 1$, then by Lemma 3.4 of [8], $A \cong H(5,r) \oplus F(d-5r-1)$ for some $r \geq 2$. If $d \leq n+3$, then by Theorem 3.3 of [7], and Lemma 2.1, we have $A \cong A_{5,7,2}$.

Now, by Lemmas 2.2 and 2.3, we may assume that $\dim A^2 = 2$ and $\dim Z(A) \geq 2$. By Lemma 2.4, in cases $\dim Z(A) \geq 3$ or $\dim Z(A) = 2$, $A^2 \nsubseteq Z(A)$ there is not estimated algebra. If $\dim A^2 = 2$ and $A^2 \subseteq Z(A)$, then A is an n-Lie algebra of class two with two-dimensional derived subalgebra. Then, by Theorem 5.2 of [12], the dimension of Schur multiplier A is equal to $\binom{d-2}{n} + j$ when j is $-1, -2, n-2, 2n-2, 3n-k (3 \leq k \leq n+1)$. It is clear that in this case there is no estimated algebra. ■

The following Theorem can be resulted from Theorems 1.1, 2.6 and 2.7.

Theorem 2.8. *Let A be a non-abelian d-dimensional nilpotent n-Lie algebra. Then*

(i) *$s(A) = 4$ if and only if A is isomorphic to one of the following Lie algebras* $L_{5,8} \oplus F(3)$, $L_{4,3} \oplus F(2)$, $L_{5,5} \oplus F(1)$, $L_{5,6}$, $L_{5,7}$, $L_{5,9}$, $L_{6,22}(\epsilon) \oplus F(1)$, $37A$, $H(4,r) \oplus F(d-4r-1) \quad (r \geq 2)$, $A_{4,6,2}$ *or* $A_{4,7,5}$.

(ii) *$s(A) = 5$ if and only if A is isomorphic to one of the following Lie algebras* $L_{5,8} \oplus F(4)$, $L_{4,3} \oplus F(3)$, $L_{5,5} \oplus F(2)$, $L_{6,22}(\epsilon) \oplus F(2)$, $L_{6,26} \oplus F(1)$, $L_{6,10}$, $L_{6,23}$, $L_{6,25}$, $L_{6,27}$, $37B$, $37C$, $37D$, $H(5,r) \oplus F(d-5r-1) \quad (r \geq 2)$ *or* $A_{5,7,2}$.

We also have listed the obtained n-Lie algebras in this paper in Table 1.

Table 1: All algebras obtained in this paper.

Name	Non-zero multiplication
$A_{n,n+2,2}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}$
$A_{n,n+3,2}$	$[e_1,\dots,e_n] = e_{n+3}$
$A_{n,n+3,3}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+3}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,4}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+3}$
$A_{n,n+3,5}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,6}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}, [e_2,\dots,e_n,e_{n+2}] = e_{n+3}$
$A_{n,n+3,7}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}$, $[e_2,\dots,e_n,e_{n+2}] = [e_1,e_3,\dots,e_{n+1}] = e_{n+3}$
$A_{n,n+3,8}$	$[e_1,\dots,e_n] = e_{n+1}, [e_2,\dots,e_{n+1}] = e_{n+2}, [e_1,e_3,\dots,e_{n+1}] = e_{n+3}$
$L_{4,3}$	$[e_1,e_2] = e_3,\ [e_1,e_3] = e_4$
$L_{5,5}$	$[e_1,e_2] = e_3,\ [e_1,e_3] = [e_2,e_4] = e_5$
$L_{5,6}$	$[e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_1,e_4] = [e_2,e_3] = e_5$
$L_{5,7}$	$[e_1,e_2] = e_3, [e_1,e_3] = e_4, [e_1,e_4] = e_5$
$L_{5,8}$	$[e_1,e_2] = e_4,\ [e_1,e_3] = e_5$
$L_{5,9}$	$[e_1,e_2] = e_3,\ [e_1,e_3] = e_4,\ [e_2,e_3] = e_5$
$L_{6,10}$	$[e_1,e_2] = e_3, [e_1,e_3] = [e_4,e_5] = e_6$
$L_{6,23}$	$[e_1,e_2] = e_3, [e_1,e_3] = [e_2,e_4] = e_5, [e_1,e_4] = e_6$
$L_{6,25}$	$[e_1,e_2] = e_3, [e_1,e_3] = e_5, [e_1,e_4] = e_6$
$L_{6,26}$	$[e_1,e_2] = e_4, [e_1,e_3] = e_5, [e_2,e_4] = e_6$
$L_{6,27}$	$[e_1,e_2] = e_3, [e_1,e_3] = e_5, [e_2,e_4] = e_6$
$L_{6,22}(\epsilon)$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_1,e_3] = e_6, [e_2,e_4] = \epsilon e_6$
$37A$	$[e_1,e_2] = e_5, [e_2,e_3] = e_6, [e_2,e_4] = e_7$
$37B$	$[e_1,e_2] = e_5, [e_2,e_3] = e_6, [e_3,e_4] = e_7$
$37C$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_2,e_3] = e_6, [e_2,e_4] = e_7$
$37D$	$[e_1,e_2] = [e_3,e_4] = e_5, [e_1,e_3] = e_6, [e_2,e_4] = e_7$

References

[1] R. Bai, G. Song, Y. Zhang, On classification of n-Lie algebras, *Front. Math. China* **6** (4) (2011) 581–606.

[2] R. Bai, X. Wang, W. Xiao, H. An, The structure of low dimensional n-Lie algebras over a field of characteristic 2, *Linear Algebra Appl.* **428** (2008) 1912–1920.

[3] S. Cicalo, W.A. de Graaf, C. Schneider, Six-dimensional nilpotent Lie algebras, *Linear Algebra Appl.* **436** (2012) 163–189.

[4] H. Darabi, M. Eshrati, H. Arabyani, Characterizing nilpotent n-Lie algebras by their multiplier, *Results Math.* **76** (2021), 11 pages.

[5] H. Darabi, M. Imanparast, On classification of 9-dimensional nilpotent 3-ary algebras of class two, *Bull. Iranian Math. Soc.* **47** (3) (2021) 929–937.

[6] H. Darabi, F. Saeedi, On the Schur multiplier of n-Lie algebras, *J. Lie Theory* **27** (2017) 271–281.

[7] H. Darabi, F. Saeedi, M. Eshrati, A characterization of finite dimensional nilpotent Filippov algebras, *J. Geom. Phys.* **101** (2016) 100–107.

[8] H. Darabi, F. Saeedi, M. Eshrati, Capable n-Lie algebras and the classification of nilpotent n-Lie algebras with $s(A) = 3$, *J. Geom. Phys.* **110** (2016) 25–29.

[9] M. Eshrati, F. Saeedi, H. Darabi, On the multiplier of nilpotent n-Lie algebras, *J. Algebra* **450** (2016) 162–172.

[10] V.T. Filippov, n-Lie algebras, *Sib. Math. Zh.* **26**(6) (1985) 126–140.

[11] M.P. Gong, *Classification of Nilpotent Lie Algebras of Dimension 7*, Ph.D. Thesis, University of Waterloo, Waterloo, Canada, 1998.

[12] Z. Hoseini, F. Saeedi, H. Darabi, Characterization of capable nilpotent n-Lie algebras of class two by their Schur multipliers, *Rendiconti del Circolo Matematico di Palermo Series 2* **68** (2019) 541–555.

[13] M. Jamshidi, F. Saeedi, H. Darabi, Classification of $(n+6)$-dimensional nilpotent n-Lie algebras of class two with $n \geq 4$, *Arab J. Math. Sci.* **7** (2) (2021) 139–150.

[14] S.M. Kasymov, Theory of n-Lie algebras, *Algebra Logika* **26** (3) (1987) 277–297.

[15] P. Niroomand, A. Shamsaki, On characterizing nilpotent Lie algebra by their multiplier, $s(L) = 6, 7$, *Results Math.* **77** (2022), 24 pages.

[16] A. Shamsaki, P. Niroomand, On characterizing nilpotent Lie algebras by their multiplier, $s(L) = 4$, *Rendiconti del Circolo Matematico di Palermo Series 2* **69** (2020) 259–272.