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T^--Rough(T^+-Rough) Ideals in T^--Rough (T^+-Rough) Semigroups Based on Upper (Lower) Approximation Spaces

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Abstract. This paper concerns the behavior of semigroups with set valued mappings. By using the notion of set-valued mapping T, we introduce the notion of a T^--rough(T^+-rough) semigroup on an approximation space and study some of its properties. Also, we define the notion of a T^--rough(T^+-rough) ideal based on an upper(lower)approximation space and several properties are investigated.

Keywords: T-Approximation space; T^--Rough semigroup; T^+-Rough semigroup; T^--Rough ideal; T^+-Rough.

1. Introduction

In 1981, the concept of a rough set was originally proposed by Pawlak [15] as a formal tool for modelling and processing incomplete information in information systems. Since then this subject has been investigated in many papers, and subsequently the algebraic approach to rough sets has been studied by some authors (see [2, 3, 13, 16, 17]). The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in the Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set and the upper approximation is the union of all equivalence classes which have a non-empty intersection with the set. However, equivalence relations are too respective for many applications; for instance, in existing databases, the values of attributes could be either symbolic or real-valued. Rough set theory would have difficulty in handling such values. It is a natural question to ask what happens if we substitute the universe set with an algebraic system.

Some authors have studied the algebraic properties of rough sets. Biswas and Nanda [2] introduced the notion of a rough subgroup. Kuroki in [13], introduced the notion of a rough ideal in a semigroup. Mordeson [14] used covers of the universal set to define an approximation operator on the power set of the given set. Estaji et al. [8] considered connection between a rough set and lattice theory and they introduced the concepts of upper and lower ideals (filters) in a lattice. Also, Estaji et al. in [9] introduced the notion of θ-upper and θ-lower approximations of a fuzzy subset of the lattice. Davvaz [3, 4] concerned a relationship between a rough set and ring theory and considered a ring as a universal set and introduced the notion of a rough ideal and a rough subring with respect to an ideal of a ring [7]. Kazanci et al. [12] introduced the notions of a rough prime (primary) ideal and a rough fuzzy prime (primary) ideal in a ring and gave some properties of such ideals. Rough modules have been investigated by Davvaz et al. in 2006 [5].

Davvaz in 2008 introduced the concept of a T-rough set and a T-rough homomorphism in a group [6], which is a generalization of ordinary homomorphism. Then using the definitions of lower and upper inverses, he introduced the definition of uniform set-valued homomorphism and proved that every set-valued homomorphism is uniform. In [17], the concepts of a set-valued and a strong set-valued homomorphism of a ring are introduced and related properties are investigated. Also, the notions of generalized lower and upper approximation operators, constructed by means of a set-valued mapping, which is a generalization of the notion of lower and upper approximations of a ring, are provided. Also, Hosseini [11, 10] defined the concept of a T-rough semigroup and a T-rough commutative ring by using the definitions of lower and upper approximations.

In [1], Bagirmaz et al. introduced rough semigroup, which extends the notion of a semigroup to include the algebraic structures of a rough set. The concept was introduced in [1] depends on the upper approximation and does not depend on the lower approximation. The main purpose of this paper, is to introduce a T^{-}-rough(T^{+}-rough) semigroup, which extends the notion of a semigroup to include two algebraic structures of T^{-}-rough(T^{+}-rough) sets. Also, we obtain some properties of approximations and these algebraic structures. In fact, in this paper, we define the notion of T^--rough(T^+-rough) semigroups which depends on upper(lower) approximation.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of the work and hence presented in brief. Let U be a non-empty set. It is important to recall that an equivalence relation Θ on a set U is a reflexive, symmetric and transitive binary relation on U. Each equivalence relation Θ on U induces a partition P of U whose classes have form $[x]_\Theta = \{y \in U \mid x\Theta y\}$. Conversely, each partition P induces an equivalence relation Θ on U by setting $x\Theta y \Leftrightarrow x$ and y are in the same set of P.

A pair (U, Θ) where $U \neq \emptyset$ and Θ is an equivalence relation on U is called an *approximation space* [15]. Let (U, Θ) be an approximation space. Given an arbitrary set $X \subseteq U$, it may be impossible to describe a precisely using the equivalence classes of Θ. In this case one may characterize X by pair of approximations:

$$\underline{Apr(X)} = \{x \in U : [x]_\Theta \subseteq X\} \quad \text{and} \quad \overline{Apr(X)} = \{x \in U : [x]_\Theta \cap X \neq \emptyset\}.$$

$\underline{Apr(X)}$ and $\overline{Apr(X)}$ are called *lower rough approximation* and *upper rough approximation* of X, respectively [15]. Given an approximation space (U, Θ) a pair (A, B) in $\mathcal{P}^*(U) \times \mathcal{P}^*(U)$ is called a *rough set* in (U, Θ) if $(A, B) = (\underline{Apr(X)}, \overline{Apr(X)})$ for some $X \in \mathcal{P}^*(U)$. A subset X of U is called *definable* if $\underline{Apr(X)} = \overline{Apr(X)}$.

Definition 2.1. [6] *Let U, W be two nonempty sets and $X \subseteq W$. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping where $\mathcal{P}^*(W)$ denotes the set of all nonempty subsets of W. Then, (U, W, T) is called a T-approximation space. The lower inverse and upper inverse of X under T are defined by*

$$\overline{X} := T^-(X) = \{u \in U \mid T(u) \cap X \neq \emptyset\},$$
$$\underline{X} := T^+(X) = \{u \in U \mid T(u) \subseteq X\}.$$

Also, for any $x \in X$, we set $\overline{x} := \overline{\{x\}}$ and $\underline{x} := \underline{\{x\}}$.

Moreover, we set

$$T[A] = \bigcup_{a \in A} T(a), \ \forall a \subseteq A.$$

For the sake of illustration we consider the following example.

Example 2.2. Let $U = \{x, y, z, t\}$ and $W = \{a, b, c\}$. Consider set-valued function $T : U \longrightarrow \mathcal{P}^*(W)$ defined by

$$T(x) = \{b\}, \ \ T(y) = \{a, c\}, \ \ T(z) = \{b\}, \ \ T(t) = \{a, b, c\}.$$

Then,

$$\begin{array}{ll} T^-(\{a\}) = \{y,t\}; & T^+(\{a\}) = \emptyset; \\ T^-(\{b\}) = \{x,z,t\}; & T^+(\{b\}) = \{x,z\}; \\ T^-(\{c\}) = \{y,t\}; & T^+(\{c\}) = \emptyset; \\ T^-(\{a,b\}) = \{x,z,y,t\}; & T^+(\{a,b\}) = \{x,z\}; \\ T^-(\{a,c\}) = \{y,t\}; & T^+(\{a,c\}) = \{y\}; \\ T^-(\{b,c\}) = \{x,z,y,t\}; & T^+(\{b,c\}) = \{x,z\}; \\ T^-(\{a,b,c\}) = \{x,z,y,t\}; & T^+(\{a,b,c\}) = \{x,z,y,t\}. \end{array}$$

Lemma 2.3. [6] *Let U, W be two non-empty sets and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. If X, Y are two non-empty subsets of Y, then following statements hold:*

(1) $T^-(X \cup Y) = T^-(X) \cup T^-(Y)$;
(2) $T^+(X \cap Y) = T^+(X) \cap T^+(Y)$;
(3) $X \subseteq Y$ *implies* $T^-(X) \subseteq T^-(Y)$;
(4) $X \subseteq Y$ *implies* $T^+(X) \subseteq T^+(Y)$;
(5) $T^+(X) \cup T^+(Y) \subseteq T^+(X \cup Y)$;
(6) $T^-(X \cap Y) \subseteq T^-(X) \cap T^-(Y)$.

The following example shows that for every $X \subseteq W$, it is in general not the case that $X \subseteq T[T^-(X)]$.

Example 2.4. Let $U = \{x,y,z\}$ and $W = \{a,b,c,d\}$ be two sets. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set valued mapping, where $T(x) = \{a,c\}, T(y) = \{a\}$ and $T(z) = \{d\}$. If $X = \{a,b\}$, we have $T^-(X) = \{x,y\}$ and $T[T^-(X)] = \{a,c\}$. Then, $X \nsubseteq T[T^-(X)]$.

3. T^--Rough Semigroup Based on Upper Approximation Space

In this section we introduce the notion of a T^-*-rough semigroup* (T^-*-rough subsemigroup*) on an approximation space and study some of its properties. We also introduce the notion of a T^--rough ideal and then some types of T^--rough ideals on an approximation space are investigated.

Definition 3.1. *Let U, W be two non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. A subset S of W is called a T^--rough semigroup provided the following properties are satisfied:*

(1) *for all* $x, y \in S$, $x * y \in T[\overline{S}]$.
(2) *for all* $x, y, z \in S$, $(x * y) * z = x * (y * z)$.

Let U, W, T be a T-approximation space and $*$ be a binary operation defined on W. Let S be a T^-*-rough semigroup*. We say e is a T*-rough left identity* if for any $y \in S$, we have $e * y = y$. Similarly, e is a T*-rough right identity* of S, if for any $y \in S$, we have $y * e = y$. If e is both a T-rough left and right identity in S, then e is called a T*-rough identity* of S. A T^--rough semigroup is a T^-*-rough monoid*, if it has a T*-rough identity*. In particular, the identity of a T^--rough monoid is unique. T^--rough monoid G is a T^-*-rough group*, if every $x \in G$ has an *inverse* $x^{-1} \in G$ such that $x^{-1} * x = e = x * x^{-1}$, where e is the identity element.

Definition 3.2. *Let (U, W, T) be a T-approximation space and $*$ be a binary operation defined on W.*

1. *A non-empty subset H of a T^--rough semigroup S is called a T^--rough subsemigroup of a T^--rough semigroup S if $H * H \subseteq T[\overline{H}]$.*
2. *A non-empty subset I of a T^--rough semigroup S is called a T^--rough left (resp. right) ideal of S if $S * I \subseteq T[\overline{I}]$ (resp. $I * S \subseteq T[\overline{I}]$).*
3. *A non-empty subset B of a T^--rough semigroup S is called a T^--rough bi-ideal ideal of S if $B * S * B \subseteq T[\overline{B}]$.*
4. *A nonempty subset H of a T^--rough semigroup S is called to a T^--rough prime ideal of S if for all $a, b \in S$ and $a * b \in T[\overline{H}]$, then $a \in H$ or $b \in H$.*

Definition 3.3. *Let U, W be two non-empty sets and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. T is a covering set-valued mapping if for every $X \subseteq \mathcal{P}^*(W)$, there exists $Y \subseteq U$ such that $X \subseteq T[Y]$.*

Proposition 3.4. *Let U, W be two non-empty sets and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. Let X be a non-empty subset of W. The following properties are equivalent:*

1. *T is a covering set-valued mapping;*
2. *for every $x \in W$, $\overline{x} \neq \emptyset$;*
3. *if $X \subseteq \mathcal{P}^*(W)$, then $X \subseteq T[\overline{X}]$.*

Proof. (1)⇒(2) Let $a \in W$ and $\overline{a} = \emptyset$. By definition of $\overline{a}$, there exists no $u \in U$ such that $a \in T(u)$. Hence, T is not a covering set-valued mapping.

(2)⇒(3) Let $x \in X$ and $\overline{x} \neq \emptyset$. Then, there exists $u_x \in U$, such that $x \in T(u_x)$. It follows that $x \in X \cap T(u_x)$. Hence, $X \subseteq T[\overline{X}]$.

(3)⇒(1) Let $X \subseteq \mathcal{P}^*(W)$. Since $X \subseteq T[\overline{X}]$, it follows that there exists $\overline{Y} \subseteq \overline{X} \subseteq U$ such that $X \subseteq T[\overline{Y}]$. Hence, T is a set-valued mapping covering of W. ■

Proposition 3.5. *Let U, W be two non-empty sets and $T : U \longrightarrow \mathcal{P}^*(W)$ be a covering set-valued mapping. If X is a non-empty subset of W, then $T[\underline{X}] \subseteq X$.*

Proof. Let $x \in T[\underline{X}]$. Then there exists $u \in \underline{X}$ such that $x \in T(u)$. So, $T(u) \subseteq X$ and $x \in X$. ■

Proposition 3.6. *Let U, W be two non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a covering set-valued mapping and $S \subseteq W$. Then,*

(1) *If S is a (subsemigroup) semigroup, then S is a T^--rough (subsemigroup) semigroup.*
(2) *If I is a left (right, two sided, bi) ideal of semigroup S, then I is a T^--rough left (right, two sided, bi) ideal of T^--rough semigroup S.*

Proof. (1) Let S be a (subsemigroup) semigroup. Then, $S * S \subseteq S$. Since T is a covering set-valued mapping, by Proposition 3.4, it follows that $S \subseteq T[\overline{S}]$. So, we obtain $S * S \subseteq T[\overline{S}]$ and hence S is a $T-$rough (subsemigroup) semigroup.

(2) Suppose that I is a left (right, two sided, bi) ideal of semigroup S. Since $S * I \subseteq I$ and T is a covering set-valued mapping, by Proposition 3.4, it follows that $I \subseteq T[\overline{I}]$. This yields that $S * I \subseteq T[\overline{I}]$. ■

Proposition 3.7. *Let U and W be two non-empty sets and $*$ be a binary operation defined on W. If any semigroup is a T^--rough semigroup, then $T : U \longrightarrow \mathcal{P}^*(W)$ is a covering set-valued mapping.*

Proof. According to Proposition 3.4 we show that, for every $a \in W$, $\overline{a} \neq \emptyset$. Assume that $a \in W$ and $\overline{a} = \emptyset$. Set $S = \{a^k \mid \quad k = 1, ..., n\}$. Since W is a finite set, it follows that S is a finite subset. Since S is easily seen to be a semigroup, it follows that S is a T^--rough semigroup. On the other hand, we show that S is not a T^--rough semigroup and this is a contradiction. Since $\overline{a} = \emptyset$, it follows that there exists no $u \in U$ such that $a \in T(u)$. Thus, there is no $u \in U$ such that $a * a^n = a \in T(u)$. Consequently, $a * a^n \notin T[\overline{S}]$ and S is not a T^--rough semigroup. ■

Example 3.8. Let $U = \{x, y, z\}$ and $W = \{a, b, c, d\}$ be two sets and $*$ be a binary operation defined on W and W be with the following multiplication table: Let

$*$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

$T : U \longrightarrow \mathcal{P}^*(W)$ be a *set valued mapping* where, $T(x) = \{a\}, T(y) = \{a, b\}$ and $T(z) = \{d\}$. Set $S = \{a, b\}$. Hence, we have $\overline{S} = \{x, y\}$ and so $T[\overline{S}] = \{a, b\}$. So, we conclude that S is a T^--rough semigroup. But T is not a covering set-valued mapping, since $\overline{c} = \emptyset$.

The following examples shows that the covering in Prop. 3.6 can not be removed.

Example 3.9. Let $U = \{x, y, z\}$ and $W = \{a, b, c, d\}$ be two sets, $*$ be a binary operation defined on W and W be with the following multiplication table:

$*$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	a	c	d
d	c	a	c	d

Let $T : U \longrightarrow \mathcal{P}^{*}(W)$ be a set valued mapping where $T(x) = \{a, c\}$, $T(y) = \{a, b\}$ and $T(z) = \{d\}$. So T is a covering set-valued setting. Set $I = \{a, b\}$. Then, I is not an ideal. Since $\overline{I} = \{x, y\}$, it follows that $T[\overline{I}] = \{a, b, c\}$ and I is a T^{-}-rough ideal.

Example 3.10. Let $U = \{x, y, z\}$ and $S = W = \{a, b, c, d, e\}$ be two sets with the following multiplication table: Let $T : U \longrightarrow \mathcal{P}^{*}(W)$ be a set valued mapping

$*$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

where, T(x)={a,c}, $T(y) = \{a\}$ and T(z)={d}. So, S is a semigroup and a T^{-}-rough semigroup. Set $B = \{a, b\}$. Then, B is a bi-ideal of S. Also, $\overline{B} = \{x, y\}$ and $T[\overline{B}] = \{a, c\}$. So, $B * S * B = \{a, b\}$ and $B * S * B \nsubseteq T[\overline{B}]$. Hence, B is not a T^{-}-rough bi-ideal of S.

The following example shows that a T^{-}-rough semigroup is not a semigroup in general.

Example 3.11. Let $T : \mathbb{R} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be a set-valued mapping such that $T(x) = \{x\} \cup \{\infty\}$ for any $x \in \mathbb{R}$. Let $S = \mathbb{Q}$ and $x * y = x/y$ for any $x \in \mathbb{Q}$. We show that $(S, *)$ is a T^{-}-rough semigroup which is not a semigroup. For this purpose, we have $T[\overline{\mathbb{Q}}] = T\{a \in \mathbb{R} : T(a) \cap \mathbb{Q} \neq \varnothing\}$. Then, $T[\overline{\mathbb{Q}}] = T\{a \in \mathbb{R} : a \in \mathbb{Q}\} = \mathbb{Q} \cup \{\infty\}$. Hence, $x * y = x/y \in T[\overline{\mathbb{Q}}]$ for any $x, y \in \mathbb{Q}$. It is equivalent to that S is a T^{-}-rough semigroup, but $a * 0 = a/0 \notin \mathbb{Q}$, so S is not a semigroup.

The following example shows that a T^{-}-rough ideal is not a ideal in general.

Example 3.12. Let $T : \mathbb{R} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be a set-valued mapping such that $T(x) = \mathbb{Q}$ for $x \in \mathbb{R}$. Let $S = \mathbb{Q}$ and $x * y = x.y$ for any $x \in \mathbb{Q}$. Then we show that $I = \mathbb{Z} \subsetneq \mathbb{Q}$ is a T^--rough ideal which is not an ideal. For this purpose, we have $T[\overline{I}] = T[\overline{\mathbb{Z}}] = T\{a \in \mathbb{R} : T(a) \cap \mathbb{Z} \neq \varnothing\}$. Then, $T[\overline{\mathbb{Z}}] = T\{a \in \mathbb{R} : \mathbb{Q} \cap \mathbb{Z} \neq \varnothing\} = \mathbb{R}$. Thus $x * y = x.y \in T[\overline{\mathbb{Z}}]$ for any $x, y \in \mathbb{Z}$. It is equivalent to that I is a T^--rough ideal, but $a/b * c = a.c/b \notin \mathbb{Z}$, so I is not an ideal.

Proposition 3.13. *Let (U, W, T) be a T-approximation space and $*$ be a binary operation defined on W. If H_1 and H_2 are two T^--rough subsemigroups and $T[\overline{H_1 \cap H_2}] = T[\overline{H_1}] \cap T[\overline{H_2}]$, then $H_1 \cap H_2$ is a T^--rough subsemigroup.*

Proof. Suppose that H_1 and H_2 are two T^--rough subsemigroups of a T^--rough semigroup S. Consider $x, y \in H_1 \cap H_2$. Since H_1 and H_2 are two T^--rough subsemigroups, it follows that $x * y \in T[\overline{H_1}]$ and $x * y \in T[\overline{H_2}]$, i.e. $x * y \in T[\overline{H_1}] \cap T[\overline{H_2}]$. Assuming $T[\overline{H_1 \cap H_2}] = T[\overline{H_1}] \cap T[\overline{H_2}]$, we have $x * y \in T[\overline{H_1 \cap H_2}]$. Hence, we conclude that $H_1 \cap H_2$ is a T^--rough subsemigroup. ■

Proposition 3.14. *Let (U, W, T) be a T-approximation space and $*$ be a binary operation defined on W. Let I_1 and I_2 be two T^--rough left (right, two sided) ideals of T^--rough semigroup S. If I_1 and I_2 are two T^--rough left (right, two sided) ideals and $T[\overline{I_1 \cap I_2}] = T[\overline{I_1}] \cap T[\overline{I_2}]$, then $I_1 \cap I_2$ is a T^--rough left (right, two sided) ideal.*

Proof. Let I_1 and I_2 be two $T-$rough left ideals of T^--rough semigroup S. Let $x \in S$ and $y \in I_1 \cap I_2$. Then, $x * y \in T[\overline{I_1}] \cap T[\overline{I_2}]$. Assuming $T[\overline{I_1 \cap I_2}] = T[\overline{I_1}] \cap T[\overline{I_2}]$, we have $x * y \in T[\overline{I_1 \cap I_2}]$. Hence, $I_1 \cap I_2$ is a T^--rough left ideal of S. Similarly, $I_1 \cap I_2$ is a T^--rough right ideal of S. ■

4. T^--Rough Prime Ideals

In this section we introduce the notion of a *T^--rough prime ideal* and and several properties are investigated.

Definition 4.1. *Let U, W be non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued homomorphism. A T^--rough ideal H is called a T^--rough prime ideal of a T^--rough semigroup S if for all $a, b \in S$ and $a * b \in T[\overline{H}]$ implies $a \in H$ or $b \in H$.*

The following example shows that a prime ideal is not a *T^--rough prime ideal* in general.

Example 4.2. Let $U = W = \{a, b, c, d, e\}$ be a non-empty set with the multiplication table (see Tab. 1). Let $T : U \longrightarrow \mathcal{P}^*(W)$ and $T(x) = [x]_R$ and $R = \{E_1, E_2, E_3\}$, where $E_1 = \{a, c\}$, $E_2 = \{b, d\}$ and $E_3 = \{e\}$. Set

$*$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	b	c	a	e
d	d	c	d	d	e
e	c	c	d	d	e

Table 1: multiplication table

$S = \{a, b, c\}$ and $H = \{a\}$. Then, H is a prime ideal of S but H is not a T^{-}-rough prime ideal of the T^{-}-rough semigroup S, since $T[\overline{S}] = \{a, b, c, d\}$ and $T[\overline{H}] = \{a, c\}$ and $b * c = c \in T[\overline{H}]$ but $b \notin H$ and $c \notin H$.

In the following proposition under the stronger conditions we show that a prime ideal is a T^{-}-rough prime ideal.

Proposition 4.3. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^{*}(W)$ be a set-valued mapping. Let $S \subseteq W$ and $T[\overline{H}] = H * H$. If H is a prime ideal of semigroup S, then H is a T^{-}-rough prime ideal of T^{-}-rough semigroup S.*

Proof. Let H be a prime ideal of S and $a * b \in T[\overline{H}]$. Hence, $a * b \in H$, so $a \in H$ or $b \in H$. ■

Proposition 4.4. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^{*}(W)$ be a set-valued mapping. Then, the union of a family of T^{-}-rough prime ideals is a T^{-}-rough prime ideal.*

Proof. Let $\{A_i\}$ be a family of T^{-}-rough prime ideals of S, where i ranges over an arbitrary index set I. Then, by Lemma 2.3, we have $T[\overline{\bigcup_{i \in I} A_i}] = \bigcup_{i \in I} T[\overline{A_i}]$. Thus, if $a, b \in S$ and $a * b \in T[\overline{\bigcup_{i \in I} A_i}]$, we have $a * b \in T[\overline{A_i}]$, for some $i \in I$. Then, $a \in A_i$ or $b \in A_i$ and so $a \in \bigcup_{i \in I} A_i$ or $b \in \bigcup_{i \in I} A_i$. Therefore, $\bigcup_{i \in I} A_i$ is a T^{-}-rough prime ideal of T^{-}-rough semigroup S. ■

In general, the intersection of T^{-}-rough prime ideals is not a T^{-}-rough prime ideal, as is shown in the following proposition:

Proposition 4.5. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^{*}(W)$ be a set-valued mapping. Let $\{A_i \mid i \in I\}$ be a set of T^{-}-rough prime ideals of S and $T[\bigcap_{i \in I} \overline{A_i}] = \bigcap_{i \in I} T[\overline{A_i}]$. Then, $\bigcap_{i \in I} A_i$ is a T^{-}-rough prime ideal of S if and only if it is a T^{-}-rough prime ideal of the union of the given ideals.*

Proof. The sufficiency being obvious, we proceed to prove the necessity. Let $\{A_i \mid i \in I\}$ be a set of T^--rough prime ideals of S.

Let $\bigcap_{i\in I} A_i$ be a T^--rough prime ideal of $\bigcup_{i\in I} A_i$. Let $x, y \in S$ and $x * y \in T[\bigcap_{i\in I} \overline{A_i}]$. If $x \notin \bigcap_{i\in I} T[\overline{A_i}]$ then there exists a $T[\overline{A_j}]$ and $j \in J$ such that $x \notin T[\overline{A_j}]$. Hence, $x \notin T[\overline{A_j}] \cap T[\overline{A_k}] = T[\overline{A_j \cap A_k}]$. But $x * y \in T[\bigcap_{i\in I} \overline{A_i}] \subseteq T[\overline{A_j \cap A_k}]$ and A_j and A_k are T^--rough prime ideals, where $x \in A_j$ and $y \in A_k$. Then, $x, y \in A_j \cup A_k \subseteq \bigcup_{i\in I} A_i$. Since $\bigcap_{i\in I} A_i$ is a T^--rough prime in $\bigcup_{i\in I} A_i$, whence either $x \in \bigcap_{i\in I} A_i$ or $y \in \bigcap_{i\in I} A_i$. Hence, $\bigcap_{i\in I} A_i$ is T^--rough prime ideal of T^--rough semigroup S. ■

5. T^+-Rough Semigroup Based on Lower Approximation Space

In this section we introduce the notion of a T^+*-rough semigroup* (T^+*-rough subsemigroup*) on an approximation space and study some of its properties. We also introduce the notion of a T^+-rough ideal and then some types of T^+-rough ideals on an approximation space are investigated.

Definition 5.1. *Let U, W be two non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. A subset S of W is called a T^+-rough semigroup provided the following properties are satisfied:*

(1) *for all $x, y \in S$, $x * y \in T[\underline{S}]$.*
(2) *for all $x, y, z \in S$, $(x * y) * z = x * (y * z)$.*

Let U, W, T be a T-approximation space and $*$ be a binary operation defined on W. Let S be a T^+*-rough semigroup.* We say e is a T*-rough left identity* if for any $y \in S$, we have $e * y = y$. Similarly, e is a T*-rough right identity* of S, if for any $y \in S$, we have $y * e = y$. If e is both a T-rough left and right identity in S, then e is called a T*-rough identity* of S. A T^+-rough semigroup is a T^+*-rough monoid*, if it has a T*-rough identity.* In particular, the identity of a T^+-rough monoid is unique. T^+-rough monoid G is a T^+*-rough group*, if every $x \in G$ has an *inverse* $x^{-1} \in G$ such that $x^{-1} * x = e = x * x^{-1}$, where e is the identity element.

Definition 5.2. *Let (U, W, T) be a T-approximation space and $*$ be a binary operation defined on W.*

(1) *A non-empty subset H of a T^+-rough semigroup S is called a T^+-rough subsemigroup of a T^+-rough semigroup S if $H * H \subseteq T[\underline{H}]$.*
(2) *A non-empty subset I of a T^+-rough semigroup S is called a T^+-rough left (resp. right) ideal of S if $S * I \subseteq T[\underline{I}]$ (resp. $I * S \subseteq T[\underline{I}]$).*
(3) *A non-empty subset B of a T^+-rough semigroup S is called a T^+-rough bi-ideal ideal of S if $B * S * B \subseteq T[\underline{B}]$.*
(4) *A nonempty subset H of a T^+-rough semigroup S is called to a T^+-rough prime ideal of S if for all $a, b \in S$ and $a * b \in T[\underline{H}]$, then $a \in H$ or $b \in H$.*

Definition 5.3. *Let* U, W *be two non-empty sets and* $T : U \longrightarrow \mathcal{P}^*(W)$ *be a set-valued mapping.* T *is a covering set-valued mapping if for every* $X \subseteq \mathcal{P}^*(W)$, *there exists* $Y \subseteq U$ *such that* $X \subseteq T[Y]$.

Proposition 5.4. *Let* U, W *be two non-empty sets and* $T : U \longrightarrow \mathcal{P}^*(W)$ *be a set-valued mapping. If* X *is a non-empty subset of* W, *then* $T[\underline{X}] \subseteq X$.

Proof. Let $x \in T[\underline{X}]$. Then there exists $u \in \underline{X}$ such that $x \in T(u)$. So, $T(u) \subseteq X$ and $x \in X$. ∎

Proposition 5.5. *Let* U, W *be two non-empty sets and* $*$ *be a binary operation defined on* W. *Let* $T : U \longrightarrow \mathcal{P}^*(W)$ *be a set-valued mapping and* $S \subseteq W$. *Then,*

(1) *If* S *is a* T^{+}*-rough (subsemigroup) semigroup, then* S *is a (subsemigroup) semigroup.*
(2) *If* I *is a* T^{+}*-rough left (right, two sided, bi) ideal of* T^{+}*-rough semigroup* S, *then* I *is a left (right, two sided, bi) ideal of semigroup* S.

Proof. (1) Let S be a T^{+}-rough (subsemigroup) semigroup. Then, $S * S \subseteq T[\underline{S}]$. By Proposition 5.4, $S * S \subseteq S$ and hence S is a (subsemigroup) semigroup.

(2) Suppose that I is a T^{+}-rough left (right, two sided, bi) ideal of T^{+}-rough semigroup S. Since $S * I \subseteq T[\underline{I}]$ and by Proposition 5.4, we have $S * I \subseteq I$ and hence I is a left (right, two sided, bi) ideal of S. ∎

In general, $X \nsubseteq T[\underline{X}]$ for every $X \subseteq \mathcal{P}^*(W)$. In the following proposition under the stronger condition we show that $X \subseteq T[\underline{X}]$.

Proposition 5.6. *Let* U, W *be two non-empty sets and* $T : U \longrightarrow \mathcal{P}^*(W)$ *be a set-valued mapping. Let* X *be a non-empty subset of* W. *Then, the following properties are equivalent:*

(1) $X = \bigcup T(u_x)$, *for some* $u_x \in U$, *for every* $X \subseteq \mathcal{P}^*(W)$.
(2) *if* $X \subseteq \mathcal{P}^*(W)$, $X \subseteq T[\underline{X}]$.

Proof. (1)⇒(2) Let $x \in X$. Since there exists $u_x \in U$ such that $x \in T(u_x)$ and $T(u_x) \subseteq X$, it follows that $x \in T(u_x)$ and $u_x \in \underline{X}$. Hence, $X \subseteq T[\underline{X}]$.

(2)⇒(1) It is clear. ∎

Example 5.7. Let $U = \{x, y, z\}$ and $W = \{a, b, c, d\}$ be two sets and $*$ be a binary operation defined on W and W be with the multiplication table (see Tab. 2): Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a *set valued mapping* where, $T(x) = \{a\}, T(y) = \{a, c\}$ and $T(z) = \{d\}$. Set $S = \{a, b\}$. It is clear that S is a semigroup. We show that S is not a T^{+}-rough semigroup. Since $\underline{S} = \{x\}$, it follows that $T[\underline{S}] = \{a\}$. Consequently, S is not a T^{+}-rough semigroup.

$*$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Table 2: multiplication table

Proposition 5.8. *Let U, W be two non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a surjective set-valued mapping such that range T is the set of all one point subset of W. Then $\underline{x} \neq \emptyset$ for every $w \in W$.*

Proof. We argue by contradiction. Assume that $\underline{x} = \emptyset$. By definition of $\underline{x}$, there exists no $u_x \in U$ such that $T(u_x) = x$. Hence, T is not surjective. ∎

Proposition 5.9. *Let U, W be two non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping such that $X = \bigcup T(u_x)$, for some $u_x \in U$, for every $X \subseteq \mathcal{P}^*(W)$ and $S \subseteq W$. Then,*

(1) *If S is a (subsemigroup) semigroup, then S is a T^+-rough(subsemigroup) semigroup.*

(2) *If I is a left (right, two sided, bi) ideal of semigroup S, then I is a T^+-rough left (right, two sided, bi) ideal of T^+-rough semigroup S.*

Proof. (1) Let S be a (subsemigroup) semigroup. Then, $S * S \subseteq S$. By Proposition 5.6, $S * S \subseteq T[\underline{S}]$ and hence S is a T^+-rough (subsemigroup) semigroup.

(2) Suppose that I is a left (right, two sided, bi) ideal of T^+-rough semigroup S. Since $S * I \subseteq I$ and by Proposition 5.6, we have $S * I \subseteq T[\underline{I}]$ and hence I is a T^+-rough left (right, two sided, bi) ideal of S. ∎

Example 5.10. Let $U = \{x, y, z\}$ and $S = W = \{a, b, c, d, e\}$ be two sets with the following multiplication table:

$*$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set valued mapping where, T(x)={a,c}, $T(y) = \{a\}$ and T(z)={d}. So, S is a semigroup and a T^+-rough semigroup. Set $B = \{a, b\}$. Then, B is a bi-ideal of S. Also, $\underline{B} = \{y\}$ and $T[\underline{B}] = \{a\}$.

So, $B * S * B = \{a, b\}$ and $B * S * B \nsubseteq T[\underline{B}]$. Hence, B is not a T^{+}-rough bi-ideal of S.

Proposition 5.11. *Let (U, W, T) be a T-approximation space and $*$ be a binary operation defined on W and $X = \bigcup T(u_x)$, for some $u_x \in U$, for every $X \subseteq \mathcal{P}^*(W)$. If H_1 and H_2 are two T^{+}-rough subsemigroups, then $H_1 \cap H_2$ is a T^{+}-rough subsemigroup.*

Proof. Suppose that H_1 and H_2 are two T^{+}-rough subsemigroups of a T^{+}-rough semigroup S. Consider $x, y \in H_1 \cap H_2$. Since H_1 and H_2 are two T^{+}-rough subsemigroups, it follows that $x * y \in T[\underline{H_1}]$ and $x * y \in T[\underline{H_2}]$, i.e. $x * y \in T[\underline{H_1}] \cap T[\underline{H_2}] \subseteq H_1 \cap H_2$. Then, by Proposition 5.6, $H_1 \cap H_2 \subseteq T[\underline{H_1 \cap H_2}]$ and $x * y \in T[\underline{H_1 \cap H_2}]$. Hence, we conclude that $H_1 \cap H_2$ is a T^{+}-rough subsemigroup. ∎

Proposition 5.12. *Let (U, W, T) be a T-approximation space $*$ be a binary operation defined on W, $X = \bigcup T(u_x)$, for some $u_x \in U$, for every $X \subseteq \mathcal{P}^*(W)$. Let I_1 and I_2 be two T^{+}-rough left (right, two sided) ideals of T^{+}-rough semigroup S. If I_1 and I_2 are two two T^{+}-rough left (right, two sided) ideals, then $I_1 \cap I_2$ is a T^{+}-rough left (right, two sided) ideal.*

Proof. Let I_1 and I_2 be two $T-$rough left ideals of T^{+}-rough semigroup S. Let $x \in S$ and $y \in I_1 \cap I_2$. Then, $x * y \in T[\underline{I_1}] \cap T[\underline{I_2}] \subseteq I_1 \cap I2$. Then, by Prop. 5.6, $x * y \in T[\underline{I_1 \cap I_2}]$. Hence, $I_1 \cap I_2$ is a T^{+}-rough left ideal of S. Similarly, $I_1 \cap I_2$ is a T^{+}-rough right ideal of S. ∎

6. T^{+}-Rough Prime Ideals

In this section we introduce the notion of a T^{+}*-rough prime ideal* and and several properties are investigated.

Definition 6.1. *Let U, W be non-empty sets and $*$ be a binary operation defined on W. Let $T : U \longrightarrow \mathcal{P}^*(W)$ be set-valued homomorphism. A T^{+}-rough ideal H is called a T^{+}-rough prime ideal of a T^{+}-rough semigroup S if for all $a, b \in S$ and $a * b \in T[\underline{H}]$ implies $a \in H$ or $b \in H$.*

Proposition 6.2. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. Let $S \subseteq W$. If H is a prime ideal of semigroup S, then H is a T^{+}-rough prime ideal of T^{+}-rough semigroup S.*

Proof. Let H be a prime ideal of semigroup S and $a * b \in T[\underline{H}]$. Hence, $a * b \in H$, so $a \in H$ or $b \in H$. ∎

Example 6.3. Let $U = \{x, y, z\}$ and $W = \{a, b, c, d\}$ be two sets and $*$ be a binary operation defined on W and W be with the multiplication table (see Tab. 3). Let $T : U \longrightarrow \mathcal{P}^*(W)$ be a set valued mapping where $T(x) = \{a, c\}$, $T(y) = \{a\}$

$*$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	c	c	d
d	a	a	c	d

Table 3: multiplication table

and $T(z) = \{d\}$. Set $I = \{a, b\}$. Then, I is not an ideal. Since $\underline{I} = \{y\}$, it follows that $T[\underline{I}] = \{a\}$ and I is a T^+-rough ideal.

Proposition 6.4. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping. Let $\{A_i \mid i \in I\}$ be a set of T^+-rough prime ideals of S. Then, $\bigcap_{i \in I} A_i$ is a T^+-rough prime ideal of T^+-rough semigroup S if and only if it is a T^+-rough prime ideal of the union of the given ideals.*

Proof. Let $\{A_i\}$ be a family of T^+-rough prime ideals of S, where i ranges over an arbitrary index set I. If $a, b \in S$ and $a * b \in T[\underline{\bigcap_{i \in I} A_i}]$, it is clear we have $a * b \in \bigcap_{i \in I} T[\underline{A_i}]$ and $a * b \in \bigcap_{i \in I} A_i$. Then, by Proposition 6.2, $\bigcap_{i \in I} A_i$ is a T^+-rough prime ideal of T^+-rough semigroup S. ■

In general, the union of T^+-rough prime ideals is not a T^+-rough prime ideal but under the stronger conditions show that the union of a family of T^+-rough prime ideals is a T^+-rough prime ideal.

Proposition 6.5. *Let U, W be two nonempty sets and $*$ be a binary operation defined on W and $T : U \longrightarrow \mathcal{P}^*(W)$ be a set-valued mapping and $X = \bigcup T(u_x)$, for some $u_x \in U$, for every $X \subseteq \mathcal{P}^*(W)$. Then, the union of a family of T^+-rough prime ideals is a T^+-rough prime ideal.*

Proof. Let $\{A_i\}$ be a family of T^+-rough prime ideals of S, where i ranges over an arbitrary index set I. Thus, if $a, b \in S$ and $a * b \in T[\underline{\bigcup_{i \in I} A_i}]$, we have $a * b \in T[\underline{A_i}]$, for some $i \in I$. Since $X = \bigcup T(u_x)$, for some $\overline{u_x \in U}$, for every $X \subseteq \mathcal{P}^*(W)$. Then, by Proposition 6.2, $\bigcup_{i \in I} A_i$ is a T^+-rough prime ideal of T^+-rough semigroup S. ■

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