Southeast Asian Bulletin of Mathematics (2023) 47: 479–494

Southeast Asian
Bulletin of
Mathematics

Some Variants of the Rothberger Property Using Generalized Open Sets*

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Received 26 July 2021
Accepted 9 December 2021

Communicated by W.S. Cheung

AMS Mathematics Subject Classification(2020): 54D20, 54A10, 54C08, 54D10

Abstract. We define and study new weak versions of the classical star-Rothberger covering property using α-open and θ-open sets of a topological space. We discuss their relations with some known weak version of the Rothberger property. It is also proved that for an extramally disconnected S-paracompact-T_2 space the properties: Rothberger [25, 28], semi-Rothberger [26], α-Rothberger [14], θ-Rothberger [14] are equivalent. Moreover, for an extremely disconnected space the θ-Rothberger property coincides with the almost Rothberger [29] property.

Keywords: Star-selection principles; Star-Rothberger and Rothberger property; θ-continuity; S-paracompact space.

1. Introduction

Throughout this paper a space X or $(X, \mathcal{T})$, means a topological space. A space X is called Menger [9, 22] if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{B}_k : k \in \mathbb{N})$ such that for every $k \in \mathbb{N}$, $\mathcal{B}_k$ is a nite subset of $\mathcal{A}_k$ such that $\bigcup_{k\in\mathbb{N}} \mathcal{B}_k$ is a cover of X. Then evidently every Menger space is Lindelöf. In 1938, Rothberger [25, 28] introduced the Rothberger property: for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there is a sequence $(A_k : k \in \mathbb{N})$

*The rst and second author acknowledges the fellowship grant of University Grant Commission, India and Council of Scientific & Industrial Research, India, respectively.

such that for each $k \in \mathbb{N}$, A_k is a member of $\mathcal{A}_k$, and $\{A_k : k \in \mathbb{N}\}$ is an open cover of X. The Rothberger property is stronger than the Menger property. In 1996, Scheepers [28] associated the selection principles to the Ramsey and game theories and after this paper selection principles became a more attractive area of topology. In 1999 Kočinac [10] generalized the Rothberger property using the star operator and defined the star-Rothberger property. For detailed study related to weak version of the Menger and Rothberger properties, we refer to [30, 13, 11, 16, 12, 15].

In 2019, Kočinac [14] studied α-Rothberger and θ-Rothberger properties using α-open, θ-open sets, respectively. Continuing this thread we further study the α-Rothberger and θ-Rothberger properties and also introduces their star-version.

The paper is organized as follows. Section 2 contains some known results used in the paper. Section 3 contains the equivalence of the Rothberger, semi-Rothberger, α-Rothberger, θ-Rothberger properties together with characterization of the α-Rothberger property. In Section 4 we introduce the star-α-Rothberger and star θ-Rothberger properties using α-open and θ-open sets and also discuss their relations to some known covering properties. We further, investigate the behaviour of the star α-Rothberger and star θ-Rothberger properties under the various types of maps. In Section 5, we define the strongly star α-Rothberger, strongly star θ-Rothberger properties and show that these are not hereditary properties but the strongly star α-Rothberger property is preserved under open-F_σ subspaces.

2. Preliminiaries

For a subset A of a space X, $Cl(A)$ or $\overline{A}$ and $Int(A)$ denotes the closure and interior of A, respectively. The generalizations of open sets, θ-open, α-open, semi-open sets in X will be used for the definitions of variations of the Rothberger property:

A subset A of a space X is said to be:

(1) θ-open if for each $x \in A$ there is an open set $B \subset X$ such that $x \in B \subset Cl(B) \subset A$ [31];

(2) α-open if $A \subset Int(Cl(Int(A)))$, or equivalently, if $A = B \setminus N$, where B is open and N is a nowhere dense set in X [23], or equivalently, if there is an open set B such that $B \subset A \subset Int(Cl(B))$. The complement of α-open set is α-closed. Equivalently, a set A is α-closed in X if $Cl(Int(Cl(A))) \subseteq A$.

(3) semi-open if there exists an open set $B \subset X$ such that $B \subset A \subset Cl(B)$, or equivalently, if $A \subset Cl(Int(A))$ [17]. $SO(X)$ denotes the set of all semi-open sets. The complement of a semi-open sets in X is called semi-closed. $sCl(A)$ denotes the semi-closure of $A \subset X$, that is $sCl(A)$ is the intersection of all semi-closed sets containing A. The set A is semi-closed if and only if $sCl(A) = A$.

Clearly, we have:

$$\theta\text{-open} \Rightarrow \text{open} \Rightarrow \alpha\text{-open} \Rightarrow \text{semi-open}.$$

A space X is called semi-regular [4] if for each $x \in X$ and for each semi-closed set U not containing x, there exist disjoint semi-open subsets A and B of X such that $x \in A$ and $U \subset B$.

Lemma 2.1. [4] *For a space X the following statements are equivalent:*

(1) *X is semi-regular;*
(2) *For each $x \in X$ and $A \in SO(X)$ such that $x \in A$, there is a $B \in SO(X)$ such that $x \in B \subset sCl(B) \subset A$.*

Recall that, a space X is called extremally disconnected if the closure of each open set in X is open.

Lemma 2.2. [24] *If X is extremally disconnected, then $sCl(A) = Cl(A)$ for all $A \in SO(X)$.*

A space X is called S-paracompact [1] if for each open cover of X has a locally finite semi-open refinement.

Lemma 2.3. [1] *A S-paracompact-T_2 space X is semi-regular.*

Lemma 2.4. [8] *Let A be a subset of X. Then $Int(Cl(A)) \subset sCl(A)$.*

Jankovic [8], proved that a space X is extremally disconnected if and only if for all $A \in SO(X)$, $A \subset Int(Cl(A))$.

A map $f : X \to Y$ from a space X to a space Y is called:

(1) α-continuous [21] (α-irresolute [20]) if the preimage of any open (α-open) subset of Y is α-open in X.
(2) α-open (strongly α-open) if the image of any α-open subset of X is α-open (open) in Y.
(3) θ-continuous [6, 7] (resp., strongly θ-continuous [19]) if for each $x \in X$ and each open set B of Y containing $f(x)$ there exists an open set A of X containing x such that $f(Cl(A)) \subset Cl(B)$ (resp., $f(Cl(A)) \subset B$).

3. The θ-Rothberger Spaces and α-Rothberger Spaces

In this section we will show that for the class of an extremally disconnected S-paracompact-T_2 spaces the Rothberger [25, 28], semi-Rothberger [26], θ-Rothberger [14], α-Rothberger [14] properties are equivalent. Moreover, we prove that for an extremally disconnected spaces the θ-Rothberger [14] property is equivalent to the almost Rothberger [29] property. Also a characterization of the α-Rothberger property is given.

A space X is called semi-Rothberger [26], in short s-Rothberger (resp., θ-Rothberger [14], α-Rothberger [14]) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of semi-open (resp., θ-open, α-open) covers of X there is a sequence $(A_k : k \in \mathbb{N})$, where $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} A_k = X$.

Evidently, the following implications hold:

$$\text{s-Rothberger} \Rightarrow \alpha\text{-Rothberger} \Rightarrow \text{Rothberger} \Rightarrow \theta\text{-Rothberger}.$$

A space has semi-Rothberger property means it is a semi-Rothberger space and so on throughout the paper.

Remark 3.1. We observe that for the class of regular spaces, open set is equivalent to θ-open set. Therefore, for the class of regular spaces the Rothberger property is equivalent to the θ-Rothberger property.

Next, we show that the Rothberger property is not equivalent to the θ-Rothberger property.

Example 3.2. There is a θ-Rothberger space X, which is not Rothberger.

Consider $U = \{u_\alpha : \alpha < \omega_1\}$, $V = \{v_i : i \in \omega\}$ and $W = \{< u_\alpha, v_i >: \alpha < \omega_1, i \in \omega\}$, where ω, ω_1 are the rst innite cardinal and the rst uncountable cardinal, respectively. Let $X = W \cup U \cup \{x\}$ where x is not a member of $W \cup U$. We topologize X as follows: for $u_\alpha \in U$ for each $\alpha < \omega_1$ the basic neighborhood takes of the form $A_{u_\alpha}(i) = \{u_\alpha\} \cup \{< u_\alpha, v_j >: j \geq i\}$, $i \in \omega$, the basic neighborhood of a point x takes of the form $A_x(\alpha) = \{x\} \cup \bigcup\{< u_\beta, v_i >: \beta > \alpha, i \in \omega\}$, $\alpha < \omega_1$ and each point of W is isolated. From the construction of topology on X, the subset $\{u_\alpha : \alpha < \omega_1\}$ of X is an uncountable discrete closed set of X. Thus X is not a Lindelöf space. Hence X is not Rothberger, because every Rothberger space is Lindelöf. We will show X is a θ-Rothberger space. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. For fixed $k = 1$, there exists an $A_1 \in \mathcal{A}_1$ such that $x \in A_1$. As A_1 is θ-open, there is an open set A'_1 with $x \in A'_1 \subset \overline{A'_1} \subset A_1$. Again by the construction of topology on X, there is a $\beta < \omega_1$ such that $A_x(\beta) \subseteq A'_1$, $\overline{A_x(\beta)} \subseteq \overline{A'_1}$, this implies that the set $\{u_\alpha : \alpha > \beta\} \cup \{x\} \cup \{< u_\alpha, v_i >: \alpha > \beta, i \in \omega\} \subseteq \overline{A'_1} \subset A_1$. Moreover the subset $Y = \bigcup_{\alpha\leq\beta}(u_\alpha \cup \{< u_\alpha, v_i >: i \in \omega\})$ is countable. We can enumerate Y as $\{y_k : k \in \mathbb{N}\}$. Thus we can find $A_{k+1} \in \mathcal{A}_{k+1}$ such that $y_k \in A_{k+1}$, for each $k \in \mathbb{N}$. Thus we have a sequence $(A_k : k \in \mathbb{N})$ with $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} A_k = X$. Hence X is θ-Rothberger.

Example 3.3. There is a Rothberger space which is not α-Rothberger.

Let X be an uncountable set and A is a xed nite subset of X. Then $\mathcal{T} = \{\phi, A, X\}$ form a topology on X. Clearly the space X is Rothberger. We note that sets of the forms $A \cup \{p\}$, $p \in X \setminus A$, are α-open. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of X, where $\mathcal{A}_k = \{A \cup \{p\} : p \in X \setminus A\}$ for each $k \in \mathbb{N}$. Hence the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ witness of X is not an α-Rothberger space because α-open cover $\mathcal{A}_k$ does not have a countable subcover.

The Rothberger property is not equivalent to the α-Rothberger property. However, for the class of an extremally disconnected S-paracompact-T_2 spaces these properties are equivalent.

Theorem 3.4. *Let X be a Rothberger extremally disconnected S-paracompact-T_2 space. Then X is semi-Rothberger.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of semi-open covers of X. For each $x \in X$ there exists a $B_{k,x} \in SO(X)$ such that $x \in B_{k,x} \subset sCl(B_{k,x}) \subset A_k$ for some $A_k \in \mathcal{A}_k$, because S-paracompact-T_2 space is semi-regular. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of semi-open covers of X, where $\mathcal{B}_k = \{B_{k,x} : x \in X\}$. As X is extremally disconnected, $B \subset Int(Cl(B))$ for each $B \in SO(X)$. Let $\mathcal{C}_k = \{Int(Cl(B)) : B \in \mathcal{B}_k\}$. Then $(\mathcal{C}_k : k \in \mathbb{N})$ is a sequence of open covers of X. Since X is Rothberger space, there exists a sequence $(C_k : k \in \mathbb{N})$, where $C_k \in \mathcal{C}_k$ for each $k \in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} C_k = X$. Using Lemma 2.4, for each C_k there exists an $A_{C_k} \in \mathcal{A}_k$ such that $C_k \subset A_{C_k}$. Hence we can construct a sequence $(A_{C_k} : k \in \mathbb{N})$ with $C_k \subset A_{C_k}$ for each k, such that $\bigcup_{k\in\mathbb{N}} A_{C_k} = X$. ■

We can not drop the condition of extremal disconnectedness in Theorem 3.3. Consider the real line $\mathbb{R}$. Since the real line $\mathbb{R}$ is T_2-paracompact, therefore $\mathbb{R}$ is S-paracompact-T_2. But the real line $\mathbb{R}$ is not semi-Rothberger since it is not semi-Menger [27]. On the other hand, the real line $\mathbb{R}$ is a Rothberger space.

Al-zoubi [1], has shown that an extremally disconnected S-paracompact-T_2 space is regular. For the regular spaces, Rothberger property is equivalent to the θ-Rothberger property. We have the following corollary:

Corollary 3.5. *For an extremally disconnected S-paracompact-T_2 space X, the following statements are equivalent:*

(1) *X is semi-Rothberger;*
(2) *X is α-Rothberger;*
(3) *X is Rothberger;*
(4) *X is θ-Rothberger.*

In the next theorem, a characterization of the α-Rothberger spaces is given.

Theorem 3.6. *For a space X, the following statements are equivalent:*

(1) *X is α-Rothberger;*
(2) *For each non-empty subset Y of X and each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-open sets in X such that $\overline{Y} \subset \cup\mathcal{A}_k$ for each $k \in \mathbb{N}$, there exists a sequence $(A_k : k \in \mathbb{N})$, $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, with $Y \subset \bigcup_{k\in\mathbb{N}} A_k$.*

Proof. (2) $\Rightarrow$ (1) Obvious. (1) $\Rightarrow$ (2). Let Y be a non-empty subset of X and $(\mathcal{A}_k : k \in \mathbb{N})$ a sequence of α-open sets in X such that $\overline{Y} \subset \cup\mathcal{A}_k$ for each $k \in \mathbb{N}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of α-open covers of X, where $\mathcal{B}_k = \mathcal{A}_k \cup \{X \setminus \overline{Y}\}$ for each $k \in \mathbb{N}$. As X is α-Rothberger, there exists a sequence $(B_k : k \in \mathbb{N})$, where $B_k \in \mathcal{B}_k$ for each $k \in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} B_k = X$. Then clearly, $Y \subset \bigcup_{k\in\mathbb{N}} B_k \setminus \{X \setminus \overline{Y}\}$. ■

Now, we will show that for an extremally disconnected spaces the θ-Rothberger property is equivalent to the well known almost-Rothberger property.

A space X is called almost Rothberger [29] if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of X there exists a sequence $(A_k : k \in \mathbb{N})$, $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} \overline{A_k} = X$.

Clearly, the almost Rothberger property implies the θ-Rothberger property.

Theorem 3.7. *For an extremally disconnected space X, the following statements are equivalent:*

(1) *X is θ-Rothberger;*

(2) *For each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of θ-open covers of X there is a sequence $(A_k : k \in \mathbb{N})$, where for each $k \in \mathbb{N}$, $A_k \in \mathcal{A}_k$ such that $\bigcup_{k\in\mathbb{N}} \overline{A_k} = X$;*

(3) *X is almost-Rothberger.*

Proof. (1)$\Rightarrow$(2). Obvious. (2)$\Rightarrow$ (1). Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. For each $A_k \in \mathcal{A}_k$ and for each $x \in A_k$ there is an open set B_x such that $x \in B_x \subset \overline{B_x} \subset A_k$. As X is an extremally disconnected, so $\overline{B_x}$ is θ-open. Put $\mathcal{B}_k = \{\overline{B_x} : x \in A_k\}$. Then $(\mathcal{C}_k : k \in \mathbb{N})$ is a sequence of θ-open covers of X, where $\mathcal{C}_k = \bigcup_{A_k\in\mathcal{A}_k} \mathcal{B}_k$ for each $k \in \mathbb{N}$. By the assumption there exists a sequence $(C_k : k \in \mathbb{N})$, $C_k \in \mathcal{C}_k$ such that $\bigcup_{k\in\mathbb{N}} \overline{C_k} = X$. From the above construction, for each C_k there is an $A_k \in \mathcal{A}_k$ such that $\overline{C_k} \subset A_k$. The sequence $(A_k : k \in \mathbb{N})$ such that $\overline{C_k} \subset A_k$ for each $k \in \mathbb{N}$ is witness for the sequence $(\mathcal{A}_k : k \in \mathbb{N})$ means X is θ-Rothberger.

(2)$\Rightarrow$ (3). Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of X. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of θ-open covers of X, where for each $k \in \mathbb{N}$, $\mathcal{B}_k = \{\overline{A} : A \in \mathcal{A}_k\}$. So, there is a sequence $(B_k : k \in \mathbb{N})$, $B_k \in \mathcal{B}_k$, such that $\bigcup_{k\in\mathbb{N}} \overline{B_k} = X$. Then the sequence $(A_k : k \in \mathbb{N})$ where $\overline{A_k} = B_k$ is witness for the sequence $(\mathcal{A}_k : k \in \mathbb{N})$. That means the space X is an almost Rothberger.

(3)$\Rightarrow$(2). Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. Then for each $x \in X$ and for each $k \in \mathbb{N}$, there exists an open set $B_{k,x}$ such that $x \in B_{k,x} \subset \overline{B_{k,x}} \subset A_k$ for some $A_k \in \mathcal{A}_k$. Put $\mathcal{B}_k = \{B_{k,x} : x \in X\}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of open covers of X. Since X is an almost Rothberger space, there exists a sequence $(B_k : k \in \mathbb{N})$ with $B_k \in \mathcal{B}_k$ such that $\bigcup_{k\in\mathbb{N}} \overline{B_k} = X$. For each B_k there is an $A_k \in \mathcal{A}_k$, such that $\overline{B_k} \subset A_k$. Then we have a sequence $(A_k : k \in \mathbb{N})$, where $\overline{B_k} \subset A_k$, for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} \overline{A_k} = X$. ■

Caldas et al. [2] introduced an α-closure operator, denoted by $Cl_\alpha(A)$. For a subset A of a space X, $Cl_\alpha(A)$ is intersection of all α-closed sets containing A. We define a class of space using the α-closure operator.

Definition 3.8. *A space X is called almost α-Rothberger if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-open covers of X, there is a sequence $(A_k : k \in \mathbb{N})$, where $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$.*

Clearly, each α-Rothberger space is almost α-Rothberger.

Theorem 3.9. *If a space X has a dense subset which is α-Rothberger in X, then X is almost α-Rothberger.*

Proof. Let D be an α-Rothberger dense subset of X and $(\mathcal{A}_k : k \in \mathbb{N})$ a sequence of α-open covers of X. Since D is α-Rothberger in X, for each $k \in \mathbb{N}$, there exists an $A_k \in \mathcal{A}_k$, such that $D \subset \bigcup_{k\in\mathbb{N}} A_k \subset \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$. Since D is dense in X, so X is only α-closed set containing D. Then, we have $Cl_\alpha(D) = Cl(D)$. Hence $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$, because $Cl_\alpha(\bigcup_{k\in\mathbb{N}} A_k) = \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$. ■

Corollary 3.10. *Every seperable space is almost α-Rothberger.*

In Example 3.2, the space X is not α-Rothberger. On the other hand, X is seperable space. Hence X is an almost α-Rothberger space.

Recall that, a subset A of a space X is α-regular open, if $\mathrm{Int}_\alpha(\mathrm{Cl}_\alpha(A)) = A$.

Theorem 3.11. *A space X is an almost α-Rothberger if and only if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-regular open covers of X, there exists a sequence $(A_k : k \in \mathbb{N})$, $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$.*

Proof. The forward part is obvious.

Conversely, let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of X. Then for each $k \in \mathbb{N}$, $\mathcal{B}_k = \{Int_\alpha(Cl_\alpha(A_k)) : A_k \in \mathcal{A}_k\}$ is a α-regular open cover of X. From the assumption, we have a sequence $(Int_\alpha(Cl_\alpha(A_k)) : k \in \mathbb{N})$, where for each $k \in \mathbb{N}$, $Int_\alpha(Cl_\alpha(A_k))$ is a member of $\mathcal{B}_k$ such that $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(Int_\alpha(Cl_\alpha(A_k)))$. We note that $Cl_\alpha(Int_\alpha(Cl_\alpha(A_k))) \subseteq Cl_\alpha(A_k)$. Therefore, $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(A_k)$. Hence X is an almost α-Rothberger space. ■

Theorem 3.12. *An α-continuous image of an almost α-Rothberger space is almost-Rothberger.*

Proof. Consider an α-continuous map $f : X \to Y$ from an almost α-Rothberger space X onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of Y. From the α-continuity of f, $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of α-open covers of X, where $\mathcal{B}_k = (f^{-1}(A) : A \in \mathcal{A}_k)$ for each $k \in \mathbb{N}$. Since X is an almost α-Rothberger, there exists a sequence $(f^{-1}(A_k) : k \in \mathbb{N})$, where $f^{-1}(A_k) \in \mathcal{B}_k$ such that

$X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(f^{-1}(A_k))$. Again from the α-continuity of f, $Y \subset \bigcup_{k\in\mathbb{N}} Cl(A_k)$. Thus Y is an almost Rothberger space. ■

Since each continuous map is α-continuous, we have the following corollary:

Corollary 3.13. *The continuous image of an almost α-Rothberger space is almost Rothberger.*

Remark 3.14. Velicko [31] introduced the θ-closure operator, it is denoted by $Cl_\theta(A)$. For a subset A of a space X, $Cl_\theta(A) = \{x \in X$: for each neighbourhood U of x, $Cl(U) \cap A \neq \phi\}$. It is interesting to dene and investigate the following two classes of spaces. A space X is said to be almost θ-Rothberger (resp., weakly θ-Rothberger) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of θ-open covers of X there is a sequence $(A_k : k \in \mathbb{N})$, where for each k, $A_k \in \mathcal{A}_k$ such that $X = \bigcup_{k\in\mathbb{N}} Cl_\theta(A_k)$ (resp., $X = Cl_\theta(\bigcup_{k\in\mathbb{N}} A_k)$. Observe that θ-Rothberger space is almost θ-Rothberger, almost θ-Rothberger space is weakly θ-Rothberger.

4. The Star α-Rothberger Spaces and Star θ-Rothberger Spaces

In this section we generalize the star semi-Rothberger [26], (star-Rothberger [10]) properties and introduce the star α-Rothberger (star θ-Rothberger) properties, respectively. We prove that for an extremally disconnected S-paracompact-T_2 spaces the star-Rothberger [10], star semi-Rothberger [26], star θ-Rothberger, star α-Rothberger properties are equivalent. We also characterize the star α-Rothberger property as well as see behavior of these properties under the various types of maps.

A space X is called star semi-Rothberger in short SsR [26], (resp., star-Rothberger or SR [10]) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of semi-open (resp., open) covers of X there is a sequence $(A_k : k \in \mathbb{N})$ for each k, $A_k \in \mathcal{A}_k$ such that $\bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k) = X$, where $St(A_k, \mathcal{A}_k) = \bigcup\{A \in \mathcal{A}_k : A \cap A_k \neq \phi\}$.

In the similar way, we define two classes of spaces.

Definition 4.1. *A space X is called star α-Rothberger in short $S_\alpha R$ (resp., star θ-Rothberger or $S_\theta R$) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-open (resp., θ-open) covers of X there is a sequence $(A_k : k \in \mathbb{N})$, $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k) = X$.*

Then we have the following implications:

$$\mathrm{SsR} \Rightarrow \mathrm{S_\alpha R} \Rightarrow \mathrm{SR} \Rightarrow \mathrm{S_\theta R}.$$

Presently, the authors do not know whether the reverse implications are true. But note that the α-Rothberger space is star α-Rothberger, but converse need not be true.

Example 4.2. There is a star α-Rothberger space which is not α-Rothberger.

Let X be an uncountable space with the topology $\mathcal{T} = \{X, \phi, \{a\}\}$, where a is a fixed point of X. Note that each subset of X containing point a is α-open. Then $\mathcal{A} = \{\{a, x\} : x \in X\}$ is an α-open cover of X, but $\mathcal{A}$ does not contain a countable subcover. Thus the space X cannot be α-Rothberger. On the other hand, for any α-open cover $\mathcal{A}$ of X, we have $St(\{a\}, \mathcal{A}) = X$. Hence X is a star α-Rothberger space.

Next, a characterization of the $\mathrm{S}_\alpha\mathrm{R}$ property is given.

Theorem 4.3. *For a space X, the following statements are equivalent:*

(1) *X is star α-Rothberger;*

(2) *For each non-empty subset Y of X and each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-open sets in X such that $\overline{Y} \subset \cup\mathcal{A}_k$ for each $k \in \mathbb{N}$, there is a sequence $(A_k : k \in \mathbb{N})$, $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $Y \subset \bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k)$.*

Proof. (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2). Let Y be a non-empty subset of X and $(\mathcal{A}_k : k \in \mathbb{N})$ is a sequence of α-open sets of X such that for each $k \in \mathbb{N}$, $\overline{Y} \subset \cup\mathcal{A}_k$. Let $\mathcal{B}_k = \mathcal{A}_k \cup \{X \setminus \overline{Y}\}$ for each $k \in \mathbb{N}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of α-open covers of X. Since X is a star α-Rothberger space, there exists a sequence $(B_k : k \in \mathbb{N})$, $B_k \in \mathcal{B}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k) = X$. Moreover, for each $y \in Y$, $y \in \bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k) \setminus \{X \setminus \overline{Y}\}$. Hence we can find a sequence $\{A_k : k \in \mathbb{N}\}$, where $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $Y \subset \bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k)$. ∎

Theorem 4.4. *A space X is star θ-Rothberger if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of closed covers of X there exists a sequence $(A_k : k \in \mathbb{N})$, where $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k) = X$.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of X. For each $x \in X$, there is a $A_{x,k} \in \mathcal{A}_k$ and open set $B_{x,k}$ such that $x \in B_{x,k} \subset \overline{B_{x,k}} \subset A_{x,k}$. Then for each fixed $k \in \mathbb{N}$, $\mathcal{B}_k = \{\overline{B_{x,k}} : x \in X\}$ is a cover of X by closed sets. Then by the assumption, for each $k \in \mathbb{N}$, there is a $B_k \in \mathcal{B}_k$ such that $\bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k) = X$. Form the construction, for each $B_k \in \mathcal{B}_k$, there is an $A_k \in \mathcal{A}_k$ such that $B_k \subset A_k$. Hence the sequence $(A_k : k \in \mathbb{N})$ where $B_k \subset A_k$ for each $k \in \mathbb{N}$, is witness of the sequence $(\mathcal{A}_k : k \in \mathbb{N})$. That means X is star θ-Rothberger space. ∎

Now, we show that for the class of extremally disconnected S-paracompact-T_2 spaces the star-Rothberger [10] and the star semi-Rothberger [26] properties are equivalent.

Theorem 4.5. *Let X be an extremally disconnected S-paracompact-T_2 space. If X is star-Rothberger then X is also star semi-Rothberger.*

Proof. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of semi-open covers of X. Then for each $x \in X$ there exists a $B_{k,x} \in SO(X)$ such that $x \in B_{k,x} \subset sCl(B_{k,x}) \subset A$ for some $A \in \mathcal{A}_k$, being a S-paracompact-T_2 space is semi-regular. Put $\mathcal{B}_k = \{B_{k,x} : x \in X\}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of semi-open covers of X. Since the space X is an extremally disconnected, $B \subset Int(Cl(B))$ for every $B \in SO(X)$. Let $\mathcal{C}_k = \{Int(Cl(B)) : B \in \mathcal{B}_k\}$. Then $(\mathcal{C}_k : k \in \mathbb{N})$ is a sequence of open covers of X. Since X is a star-Rothberger space, there exists a sequence $(C_k : k \in \mathbb{N})$, $C_k \in \mathcal{C}_k$ for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} St(C_k, \mathcal{C}_k) = X$. By Lemma 2.4, for each $C_k \in \mathcal{C}_k$ there is an $A_{C_k} \in \mathcal{A}_k$ such that $C_k \subset A_{C_k}$. Hence we can construct a sequence $(A_k : k \in \mathbb{N})$ with $A_k \in \mathcal{A}_k$ containing C_k for each $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k) = X$. ∎

Since an extremally disconnected S-paracompact-T_2 spaces are regular [1] and in regular spaces, θ-open sets coincide with open sets, we have the following corollary:

Corollary 4.6. *For an extremally disconnected S-paracompact-T_2 space X, the following statements are equivalent:*

(1) *X is star semi-Rothberger;*
(2) *X is star α-Rothberger;*
(3) *X is star-Rothberger;*
(4) *X is star θ-Rothberger.*

Let $\mathcal{T}_\alpha$ and $\mathcal{T}_\theta$ be the family of all α-open and θ-open sets of a space $(X, \mathcal{T})$, respectively. Then $\mathcal{T}_\alpha$ and $\mathcal{T}_\theta$ is also forms topologies on X. Moreover $\mathcal{T} \subset \mathcal{T}_\alpha$ and $\mathcal{T}_\theta \subset \mathcal{T}$ for details, see [23, 18].

We have some interesting results related to the star α-Rothberger and star θ-Rothberger properties.

Theorem 4.7. *A space $(X, \mathcal{T})$ is star α-Rothberger (resp., star θ-Rothberger) if and only if $(X, \mathcal{T}_\alpha)$, (resp., $(X, \mathcal{T}_\theta)$) is star-Rothberger.*

Proof. The proof is straight forward and thus omitted. ∎

Theorem 4.8. *The star α-Rothbergerness is a semi-topological property.*

Proof. Let $(X, \mathcal{T})$ be a star α-Rothberger space and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ a semi-homeomorphism from the space $(X, \mathcal{T})$ onto a space $(Y, \mathcal{T}')$. Then $f : (X, \mathcal{T}_\alpha) \to (Y, \mathcal{T}'_\alpha)$ is a homeomorphism [3]. Since $(X, \mathcal{T})$ is star α-Rothberger, so $(X, \mathcal{T}_\alpha)$ is star-Rothberger and star-Rothberger is a topological property. So $(Y, \mathcal{T}'_\alpha)$ is star-Rothberger. Hence $(Y, \mathcal{T}')$ is star α-Rothberger. ∎

Now we will discuss behavior of the star α-Rothberger and the star θ-Rothberger spaces under the various types of maps.

Theorem 4.9. *An α-continuous image of a star α-Rothberger space is star-Rothberger.*

Proof. Let X be a star α-Rothberger space and $f : X \to Y$ be an α-continuous map from X onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of open covers of Y. Set $\mathcal{B}_k = \{f^{-1}(A_k) : A_k \in \mathcal{A}_k\}$, for each $k \in \mathbb{N}$. Since f is an α-continuous map, $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of α-open covers of X. As space X is an star α-Rothberger, there exists a $B_k \in \mathcal{B}_k$ such that $X = \bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k)$. Let $A_{B_k} = f(B_k)$, $k \in \mathbb{N}$. Then the sequence $(A_{B_k} : k \in \mathbb{N})$ witness for the sequence $(\mathcal{A}_k : k \in \mathbb{N})$. That means Y is a star-Rothberger. ■

Similarly, we have the following theorem.

Theorem 4.10. *An α-irresolute image of a star α-Rothberger space is star α-Rothberger.*

Theorem 4.11. *Let $(X, \mathcal{T})$ be a star α-Rothberger space. Then the following statements hold:*

(1) *$(X, \mathcal{T})$ is an α-continuous image of a star-Rothberger space;*
(2) *$(X, \mathcal{T})$ is an α-open preimage of a star-Rothberger space.*

Proof. The space $(X, \mathcal{T}_\alpha)$ is star-Rothberger because $(X, \mathcal{T})$ is a star α-Rothberger space. Then the identity map $1_X : (X, \mathcal{T}_\alpha) \to (X, \mathcal{T})$ is an α-continuous. On the other hand, $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}_\alpha)$ is an α-open map. ■

Theorem 4.12. *A strongly θ-continuous image of a star θ-Rothberger space X is star-Rothberger.*

Proof. Let $f : X \to Y$ be a strongly θ-continuous map from a star θ-Rothberger space X onto a space Y. Consider a sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of open covers of Y. For $x \in X$, $f(x) \in A_k$ for some $A_k \in \mathcal{A}_k$ for each $k \in \mathbb{N}$. By the strongly θ-continuity of f, there exists an open set $B_{x,k}$ of x such that $f(Cl(B_{x,k})) \subset A_k$. That is $f^{-1}(A_k)$ is θ-open set. Put $\mathcal{B}_k := \{f^{-1}(A) : A \in \mathcal{A}_k\}$. Then $(\mathcal{B}_k : k \in \mathbb{N})$ is a sequence of θ-open covers of X. Since X is a star θ-Rothberger space, there exists $B_k \in \mathcal{B}_k$ for each $k \in \mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k)$. For each $k \in \mathbb{N}$, we can choose $A_k \in \mathcal{A}_k$ such that $B_k = f^{-1}(A_k)$. Then we have

$$Y = f(X) = f(\textstyle\bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k)) = \bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k).$$

Hence, Y is star-Rothberger. ■

Theorem 4.13. *An θ-continuous image of a star θ-Rothberger space X is star θ-Rothberger.*

Proof. Let $f : X \to Y$ be an θ-continuous map from a star θ-Rothberger space X onto a space Y. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of θ-open covers of Y. From the θ-continuity of map f, $\mathcal{B}_k = \{f^{-1}(A) : A \in \mathcal{A}_k\}$ is an θ-open cover of X for each $k \in \mathbb{N}$. Using the fact that X is a star θ-Rothberger space, there exists a $B_k \in \mathcal{B}_k$ for each $k \in \mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k)$. For each $k \in \mathbb{N}$ and for $B_k \in \mathcal{B}_k$, we may choose $A_k \in \mathcal{A}_k$ such that $B_k = f^{-1}(A_k)$. We have

$$Y = f(X) = f(\bigcup_{k\in\mathbb{N}} St(B_k, \mathcal{B}_k)) = \bigcup_{k\in\mathbb{N}} St(A_k, \mathcal{A}_k).$$

Hence, Y is star θ-Rothberger. ■

Since continuity implies θ-continuity, we have the following corollary:

Corollary 4.14. *A continuous image of a star θ-Rothberger space is star θ-Rothberger.*

We end this section with the following remark:

It is interesting to use the α-closure operator [2], and the θ-closure operator [31] to define and explore the following two classes of spaces. A space X is said to be almost star α-Rothberger in short AS$_\alpha$R (resp., almost star θ-Rothberger, in short AS$_\theta$R) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of α-open (resp., θ-open) covers of X there exists a sequence $(A_k : k \in \mathbb{N})$ such that for each k, A_k is a member of $\mathcal{A}_k$ such that $X = \bigcup_{k\in\mathbb{N}} Cl_\alpha(St(A_k, \mathcal{A}_k))$ (resp., $X = \bigcup_{k\in\mathbb{N}} Cl_\theta(St(A_k, \mathcal{A}_k))$. Observe that AS$_\alpha$R and AS$_\theta$R spaces are generalizations of S$_\alpha$R and S$_\theta$R spaces, respectively.

5. The Strongly Star α-Rothberger Spaces and Strongly Star θ-Rothberger Spaces

In this section we introduce new strongly star-selection principles using the α-open and θ-open sets. We further study their hereditary properties and the relation with other known spaces.

A space X is called strongly star semi-Rothberger in short SSsR [26], (resp., strongly star-Rothberger or SSR [10]) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of semi-open (resp., open) covers of X there exists a sequence $(x_k : k \in \mathbb{N})$ of elements of X such that $\bigcup_{k\in\mathbb{N}} St(x_k, \mathcal{A}_k) = X$.

Definition 5.1. *A space X is called strongly star θ-Rothberger in short SS$_\theta$R (resp., strongly star α-Rothberger or SS$_\alpha$R) if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of θ-open (resp., α-open) covers of X there is a sequence $(x_k : k \in \mathbb{N})$ of elements of X such that $\bigcup_{k\in\mathbb{N}} St(x_k, \mathcal{A}_k) = X$*

Clearly, the following implications hold:

$$\text{SSsR} \Rightarrow \text{SS}_\alpha\text{R} \Rightarrow \text{SSR} \Rightarrow \text{SS}_\theta\text{R}.$$

By the similar argument as in Remark 1, for the class of regular spaces the $\mathrm{SS}_\theta\mathrm{R}$ property is equivalent to the SSR property. Here also, the authors do not know whether the reverse implications are true. But we have a class of spaces for which these properties are equivalents.

Theorem 5.2. *For an extremally disconnected S-paracompact-T_2 space X, the SSR property implies the SSsR property.*

Proof. The proof is almost similar to that of Theorem 4.5 and thus omitted. ■

The extremally disconnected S-paracompact-T_2 space is regular [1]. Then we have the following corollary:

Corollary 5.3. *For an extremally disconnected S-paracompact-T_2 space X, the following statements are equivalent:*

(1) *X is SSsR;*
(2) *X is $SS_\alpha R$;*
(3) *X is SSR;*
(4) *X is $SS_\theta R$.*

Theorem 5.4. *A space X is $SS_\theta R$ if for each sequence $(\mathcal{A}_k : k \in \mathbb{N})$ of closed covers of X, there exists a sequence $(x_k : k \in \mathbb{N})$ of elements of X such that $\bigcup_{k\in\mathbb{N}} St(x_k, \mathcal{A}_k) = X$.*

Proof. In the proof, we use the almost similar facts in Theorem 4.4. Thus the proof is omitted. ■

Next we will show that $\mathrm{SS}_\alpha\mathrm{R}$ property need not be hereditary.

Example 5.5. A subspace of an $\mathrm{SS}_\alpha\mathrm{R}$ space need not be $\mathrm{SS}_\alpha\mathrm{R}$.

Let x_0 be a fixed point of an uncountable set X. The family $\mathcal{T} = \{A \subset X : x_0 \notin A\} \cup \{A \subset X : X \setminus A \text{ is finite set}\}$ of subsets of X forms a topology on X, for more details see [5, Example 1.1.8]. It is easy to check that the space X is $\mathrm{SS}_\alpha\mathrm{R}$. Consider the subspace $Y = X \setminus \{x_0\}$. The one point subsets $\{x\}, x \in Y$ are α-open. Then the α-open cover $\mathcal{A} = \{\{x\} : x \in Y\}$ of Y has no countable subcover. Hence the subspace Y cannot be a $\mathrm{SS}_\alpha\mathrm{R}$ space.

Remark 5.6. Note that in Example 5.5, Y is an α-open (open) subset of X. So, the α-open (open) subspace of a $\mathrm{SS}_\alpha\mathrm{R}$ space need not be $\mathrm{SS}_\alpha\mathrm{R}$.

Remark 5.7. Let X be a space of Example 5.5. It is also easy to check that X is a $\mathrm{SS}_\theta\mathrm{R}$ space and $Y = X \setminus \{x_0\}$ is an open subspace of X, which is not $\mathrm{SS}_\theta\mathrm{R}$. It means $\mathrm{SS}_\theta\mathrm{R}$ is not a hereditary property.

Theorem 5.8. *The $SS_\alpha R$ property is preserved by open F_σ-subspaces.*

Proof. Let X be a SS_αR space and $Y = \bigcup_{k\in\mathbb{N}} M_k$ an open F_σ-subspace of X, where each M_k is closed subset of X. Without loss of generality, we may assume $M_k \subseteq M_{k+1}$ for all $k \in \mathbb{N}$. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of Y. Since Y is open in X, so each α-open subset of Y is α-open in X. Then $\mathcal{B}_k = \mathcal{A}_k \cup \{X \setminus M_k\}$ is an α-open cover of X for each $k \in \mathbb{N}$. Since X is SS_αR space, there is a sequence $(x_k : k \in \mathbb{N})$ of elements of X such that $\bigcup_{k\in\mathbb{N}} St(x_k, \mathcal{B}_k) = X$. Let $C = Y \cap \{x_k : k \in \mathbb{N}\}$. Then for every $y \in Y$, there exists $k \in \mathbb{N}$ such that $y \in St(x_k, \mathcal{B}_k)$. Hence $y \in St(C, \mathcal{A}_k)$. This means Y is strongly star α-Rothberger. ■

Corollary 5.9. *A clopen subspace of an $SS_\alpha R$ space is $SS_\alpha R$.*

Recall that, for a continuous real valued function $f : X \to \mathbb{R}$ from a space X to $\mathbb{R}$. The set of the form $f^{-1}(\mathbb{R} \setminus \{0\})$ is called cozero-set in a space X. The cozero-set is an open F_σ-set. Then we have the following corollary:

Corollary 5.10. *A cozero-set of an $SS_\alpha R$ space is $SS_\alpha R$.*

Theorem 5.11. *An α-continuous image of an $SS_\alpha R$ space is SSR.*

Proof. The proof is almost similar to the proof of Theorem 4.9, with some suitable adjustments, thus omitted. ■

Theorem 5.12. *An θ-continuous image of an $SS_\theta R$ space X is $SS_\theta R$.*

Proof. The proof is almost similar to the proof of Theorem 4.13, with some minor changes, thus ommited. ■

Corollary 5.13. *The continuous image of an $SS_\theta R$ space is $SS_\theta R$.*

Theorem 5.14. *For a space $(X, \mathcal{T})$ the following statements are equivalent:*

(1) *$(X, \mathcal{T})$ is $SS_\alpha R$;*
(2) *$(X, \mathcal{T})$ admits a strongly α-open bijection onto an SSR space $(Y, \mathcal{T}')$.*

Proof. (1)⇒(2) Since $(X, \mathcal{T})$ is SS_αR space, $(X, \mathcal{T}_\alpha)$ is SSR space. The identity map $I_X : (X, \mathcal{T}) \to (X, \mathcal{T}_\alpha)$ is a strongly α-open bijection.

(2)⇒ (1) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a strongly α-open bijection from a space $(X, \mathcal{T})$ onto a SSR space $(Y, \mathcal{T}')$. Let $(\mathcal{A}_k : k \in \mathbb{N})$ be a sequence of α-open covers of $(X, \mathcal{T})$. Then $\mathcal{B}_k = (f(A_k) : A_k \in \mathcal{A}_k)$ is a sequence of open covers of Y. Choose $y_k \in Y$ such that $Y = \bigcup_{k\in\mathbb{N}} St(y_k, \mathcal{B}_k)$. Evidently, then $X = \bigcup_{k\in\mathbb{N}} St(x_k, \mathcal{A}_k)$, where $x_k = f(y_k)$. ■

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