

## Evaluation of Some Gröbner-Hofreiter-Type Integrals Using Hypergeometric Approach with Applications

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Received 15 February 2021

Accepted 22 December 2021

Communicated by H.M. Srivastava

**AMS Mathematics Subject Classification(2020):** 33E20, 33B15, 33B10, 33C05, 33E50

**Abstract.** Gröbner-Hofreiter-type integrals were evaluated by the use of applicable contour integrals and Cauchy's residue theorem. In this paper, we obtain the solutions of Gröbner-Hofreiter-type integrals and other associated integrals with suitable convergence conditions by using hypergeometric approach. Some applications of Gröbner-Hofreiter-type integrals are also obtained in the form of Weber-Anger-type functions.

**Keywords:** Hypergeometric function; Gröbner-Hofreiter integral; Weber function; Anger function; Srivastava-Daoust double hypergeometric function.

### 1. Introduction and Preliminaries

In this paper, we shall use the following notations:

$$\mathbb{N} := \{1, 2, 3, \dots\}; \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, \dots\}.$$

The symbols  $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$  and  $\mathbb{R}^-$  denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

The Pochhammer symbol  $(\alpha)_p (\alpha, p \in \mathbb{C})$  is defined in [13, p. 22 Eq.(1), p. 32, Q.N.(8) and Q.N.(9)] (see also [17, p. 23, Eq.(22) and Eq.(23)]).

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A natural generalization of the Gaussian hypergeometric series  ${}_2F_1[\alpha, \beta; \gamma; z]$  is accomplished by introducing any arbitrary number of numerator and denominator parameters (see [17, p. 42, Eq.(1)]).

In an earlier paper [15], Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2, p. 150] by means of the double hypergeometric series (see also [15, p. 199] and [16]).

$$\begin{aligned} F(x, y) &:= F_{C: D; D'}^{A: B; B'} \left( \begin{matrix} [(a_A) : \vartheta, \varphi] : [(b_B) : \psi]; [(b'_{B'}) : \psi'] ; \\ [(c_C) : \delta, \varepsilon] : [(d_D) : \eta]; [(d'_{D'}) : \eta'] ; \end{matrix} \begin{matrix} x, y \\ \end{matrix} \right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_{m\vartheta_j+n\varphi_j} \prod_{j=1}^B (b_j)_{m\psi_j} \prod_{j=1}^{B'} (b'_j)_{n\psi'_j}}{\prod_{j=1}^C (c_j)_{m\delta_j+n\varepsilon_j} \prod_{j=1}^D (d_j)_{m\eta_j} \prod_{j=1}^{D'} (d'_j)_{n\eta'_j}} \frac{x^m}{m!} \frac{y^n}{n!} \end{aligned} \quad (1)$$

where the coefficients

$$\begin{cases} \vartheta_1, \dots, \vartheta_A; \varphi_1, \dots, \varphi_A; \psi_1, \dots, \psi_B; \psi'_1, \dots, \psi'_{B'}; \delta_1, \dots, \delta_C; \\ \varepsilon_1, \dots, \varepsilon_C; \eta_1, \dots, \eta_D; \eta'_1, \dots, \eta'_{D'} \end{cases} \quad (2)$$

are real and positive.

Indeed it is easy to observe that when  $y \rightarrow 0$ ,  $F(x, y)$  reduces to the generalized hypergeometric series  ${}_p\psi_q^*$  introduced by Wright [19, 20] and when the positive real coefficients in Eq. (2) are all taken as unity, it would equal

$$F_{C: D; D'}^{A: B; B'} \left[ \begin{matrix} (a_A) : (b_B); (b'_{B'}); \\ (c_C) : (d_D); (d'_{D'}); \end{matrix} \begin{matrix} x, y \\ \end{matrix} \right], \quad (3)$$

where  $F_{C: D; D'}^{A: B; B'}[x, y]$  denotes Kampé de Fériet's double hypergeometric function in the contracted notation of Burchnall and Chaundy [4, p. 112] in preference, for the sake of generality and elegance, to the one used by Kampé de Fériet himself [2, p. 150].

$$E_1 = \left( \mu_1^{1+\sum_{j=1}^D \eta_j - \sum_{j=1}^B \psi_j} \right) \frac{\prod_{j=1}^C (\mu_1 \delta_j + \mu_2 \varepsilon_j)^{\delta_j} \prod_{j=1}^D (\eta_j)^{\eta_j}}{\prod_{j=1}^A (\mu_1 \vartheta_j + \mu_2 \varphi_j)^{\vartheta_j} \prod_{j=1}^B (\psi_j)^{\psi_j}}, \quad (4)$$

$$E_2 = \left( \mu_2^{1+\sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{B'} \psi'_j} \right) \frac{\prod_{j=1}^C (\mu_1 \delta_j + \mu_2 \varepsilon_j)^{\varepsilon_j} \prod_{j=1}^{D'} (\eta'_j)^{\eta'_j}}{\prod_{j=1}^A (\mu_1 \vartheta_j + \mu_2 \varphi_j)^{\varphi_j} \prod_{j=1}^{B'} (\psi'_j)^{\psi'_j}}, \quad (5)$$

$$\Delta_1 = 1 + \sum_{j=1}^C \delta_j + \sum_{j=1}^D \eta_j - \sum_{j=1}^A \vartheta_j - \sum_{j=1}^B \psi_j, \quad (6)$$

$$\Delta_2 = 1 + \sum_{j=1}^C \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^A \varphi_j - \sum_{j=1}^{B'} \psi'_j. \quad (7)$$

*Case I.* The double power series in Eq. (1) converges for all complex values of  $x$  and  $y$  when  $\Delta_1 > 0$  and  $\Delta_2 > 0$ .

*Case II.* The double power series in Eq. (1) is convergent when  $\Delta_1 = 0$ ,  $\Delta_2 = 0$ ,  $|x| < \rho_1$ ,  $|y| < \rho_2$  where

$$\rho_1 = \min_{\mu_1, \mu_2 > 0} (E_1), \quad \rho_2 = \min_{\mu_1, \mu_2 > 0} (E_2).$$

*Case III.* The double power series in Eq. (1) would diverge except when, trivially,  $x = y = 0$  when  $\Delta_1 < 0$  and  $\Delta_2 < 0$ .

Hypergeometric forms of some elementary functions [17, p. 44, Eq.(8), Eq.(9) and Eq.(10)], see also Prudnikov [11, p. 489, Eq.(7.3.5.1)]

$$(1-z)^{-a} = {}_1F_0 \left[ \begin{matrix} a \\ -; \end{matrix} \middle| z \right], \quad |z| < 1, a \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad (8)$$

When  $|z| = 1$  and  $z \neq 1$ , then

$${}_1F_0 \left[ \begin{matrix} a \\ -; \end{matrix} \middle| z \right] \quad (9)$$

is well defined when  $\Re(a) < 1$ .

The present article is organized as follows. In section 2 we have mentioned Gröbner-Hofreiter-type integrals. In section 3, we have given the proof of Gröbner-Hofreiter-type integrals. In Section 4, we have obtained other associated integrals as special cases of main integrals and section 5 is related to the applications in Weber-Anger-type functions.

## 2. Main Integrals

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense, are tacitly excluded and the values of parameters and arguments are adjusted in such a way that Gamma functions in the right hand sides are meaningful and well defined. Then

### Integral 1

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta e^{i\gamma t} dt \\ &= \frac{\pi e^{-i(\frac{\pi}{2}\alpha)} \Gamma(1+\beta)}{2^{\alpha+\beta} \Gamma(\frac{2+\alpha+\beta+\gamma}{2}) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2} \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} \middle| -1 \right], \end{aligned} \quad (10)$$

where  $\Re(\alpha) > -1$ ,  $\Re(\beta) > -1$ ,  $\Re(\alpha + \beta) > -2$ , and  $\frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

**Integral 2**

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta \cos(\gamma t) dt \\ &= \frac{\pi \cos\left(\frac{\pi}{2}\alpha\right) \Gamma(1+\beta)}{2^{\alpha+\beta} \Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right) \Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (11)$$

where  $\alpha, \beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (11) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2$ .

**Integral 3**

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta \sin(\gamma t) dt \\ &= -\frac{\pi \sin\left(\frac{\pi}{2}\alpha\right) \Gamma(1+\beta)}{2^{\alpha+\beta} \Gamma\left(\frac{2+\alpha+\beta+\gamma}{2}\right) \Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (12)$$

where  $\alpha, \beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (12) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2$ .

**Integral 4**

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^\alpha (\cos t)^\beta e^{i\gamma t} dt \\ &= \frac{e^{i(\alpha+2\beta+2\gamma-1)\frac{\pi}{2}} \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right) \Gamma(1+\alpha)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ & \quad - \frac{e^{-i(2\alpha+\beta+\gamma+1)\frac{\pi}{2}} \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right) \Gamma(1+\beta)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (13)$$

where  $\Re(\alpha) > -1, \Re(\beta) > -1, \Re(\alpha + \beta) > -2$  and  $\frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

**Integral 5**

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^\alpha (\cos t)^\beta \cos(\gamma t) dt \\ &= \frac{\cos\left\{(\alpha+2\beta+2\gamma-1)\frac{\pi}{2}\right\} \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right) \Gamma(1+\alpha)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma\left(\frac{2+\alpha-\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ & \quad - \frac{\cos\left\{(2\alpha+\beta+\gamma+1)\frac{\pi}{2}\right\} \Gamma\left(\frac{2-\alpha-\beta-\gamma}{2}\right) \Gamma(1+\beta)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma\left(\frac{2-\alpha+\beta-\gamma}{2}\right)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (14)$$

where  $\alpha, \beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (14) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) >$

$-2, (\alpha + \beta + \gamma) \neq 2, 4, 6, \dots$

**Integral 6**

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^{\alpha} (\cos t)^{\beta} \sin(\gamma t) dt \\ &= \frac{\sin \{(\alpha+2\beta+2\gamma-1)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\alpha)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2+\alpha-\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ &+ \frac{\sin \{(2\alpha+\beta+\gamma+1)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (15)$$

where  $\alpha, \beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (15) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2, (\alpha + \beta + \gamma) \neq 2, 4, 6, \dots$

**Integral 7**

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} (\sin t)^{\alpha} (\cos t)^{\beta} e^{i\gamma t} dt \\ &= \frac{e^{-i(\alpha-1)\frac{\pi}{2}} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\alpha)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2+\alpha-\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ &- \frac{e^{i(1+\beta+\gamma)\frac{\pi}{2}} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (16)$$

where  $\Re(\alpha) > -1, \Re(\beta) > -1, \Re(\alpha + \beta) > -2, \frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

**Integral 8**

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} (\sin t)^{\alpha} (\cos t)^{\beta} \cos(\gamma t) dt \\ &= \frac{\cos \{(\alpha-1)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\alpha)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2+\alpha-\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ &- \frac{\cos \{(1+\beta+\gamma)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^{\alpha+\beta} (\alpha+\beta+\gamma) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (17)$$

where  $\alpha, \beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (17) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2, (\alpha + \beta + \gamma) \neq 2, 4, 6, \dots$

### Integral 9

$$\begin{aligned}
& \int_0^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta \sin(\gamma t) dt \\
&= -\frac{\sin \left\{ (\alpha - 1) \frac{\pi}{2} \right\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1 + \alpha)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma(\frac{2+\alpha-\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2+\alpha-\beta-\gamma}{2}; \end{matrix} -1 \right] \\
&\quad - \frac{\sin \left\{ (1 + \beta + \gamma) \frac{\pi}{2} \right\} \Gamma(\frac{2-\alpha-\beta-\gamma}{2}) \Gamma(1 + \beta)}{2^{\alpha+\beta} (\alpha + \beta + \gamma) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \tag{18}
\end{aligned}$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of Eq. (18) is well defined and  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2, (\alpha + \beta + \gamma) \neq 2, 4, 6, \dots$

### 3. Proof of Gröbner-Hofreiter-Type Integrals

*Proof of integral (10).* Let

$$\begin{aligned}
I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta e^{i\gamma t} dt, \\
I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{e^{it} - e^{-it}}{2i} \right)^\alpha \left( \frac{e^{it} + e^{-it}}{2} \right)^\beta e^{i\gamma t} dt \\
&= \frac{1}{(i)^\alpha 2^{\alpha+\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ita} (1 - e^{-2it})^\alpha e^{it\beta} (1 + e^{-2it})^\beta e^{i\gamma t} dt \\
&= \frac{1}{(i)^\alpha 2^{\alpha+\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it(\alpha+\beta+\gamma)} {}_1F_0 \left[ \begin{matrix} -\alpha; \\ -; \end{matrix} e^{-2it} \right] {}_1F_0 \left[ \begin{matrix} -\beta; \\ -; \end{matrix} -e^{-2it} \right] dt.
\end{aligned}$$

Since  $\pm e^{-2it} \neq 1$ , but  $|\pm e^{-2it}| = 1$ , therefore  $\Re(\alpha) > -1, \Re(\beta) > -1$ .

Therefore

$$\begin{aligned}
I &= \frac{1}{(i)^\alpha 2^{\alpha+\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it(\alpha+\beta+\gamma)} \sum_{n=0}^{\infty} \frac{(-\alpha)_n e^{-2itn}}{n!} \sum_{m=0}^{\infty} \frac{(-\beta)_m (-1)^m e^{-2itm}}{m!} dt \\
&= \frac{1}{(i)^\alpha 2^{\alpha+\beta}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\alpha)_n (-\beta)_m (-1)^m}{n! m!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\alpha+\beta+\gamma-2n-2m)t} dt \\
&= \frac{1}{(i)^\alpha 2^{\alpha+\beta}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\alpha)_n (-\beta)_m (-1)^m}{n! m! (\alpha + \beta + \gamma - 2n - 2m)} \\
&\quad \times 2 \left[ \frac{e^{i(\alpha+\beta+\gamma-2n-2m)\frac{\pi}{2}} - e^{-i(\alpha+\beta+\gamma-2n-2m)\frac{\pi}{2}}}{2i} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(i)^\alpha 2^{\alpha+\beta-1}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\alpha)_n (-\beta)_m (-1)^m}{n! m! (\alpha+\beta+\gamma-2n-2m)} \sin \left\{ (\alpha+\beta+\gamma-2n-2m) \frac{\pi}{2} \right\} \\
&= \frac{\sin \left\{ (\alpha+\beta+\gamma) \frac{\pi}{2} \right\}}{(i)^\alpha 2^{\alpha+\beta-1}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\alpha)_n (-1)^n (-\beta)_m (-1)^{2m} \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n \left(\frac{-\alpha-\beta-\gamma+2n}{2}\right)_m}{n! m! (\alpha+\beta+\gamma) \left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n \left(\frac{2-\alpha-\beta-\gamma+2n}{2}\right)_m} \\
&= \frac{\sin \left\{ (\alpha+\beta+\gamma) \frac{\pi}{2} \right\}}{(i)^\alpha 2^{\alpha+\beta-1} (\alpha+\beta+\gamma)} \sum_{n=0}^{\infty} \frac{(-\alpha)_n (-1)^n \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n n!} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(-\beta)_m \left(\frac{-\alpha-\beta-\gamma+2n}{2}\right)_m}{\left(\frac{2-\alpha-\beta-\gamma+2n}{2}\right)_m m!} \\
&= \frac{\sin \left\{ (\alpha+\beta+\gamma) \frac{\pi}{2} \right\}}{(i)^\alpha 2^{\alpha+\beta-1} (\alpha+\beta+\gamma)} \sum_{n=0}^{\infty} \frac{(-\alpha)_n (-1)^n \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n n!} \\
&\quad \times {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\alpha-\beta-\gamma+2n}{2}; & 1 \\ \frac{2-\alpha-\beta-\gamma+2n}{2}; & \end{matrix} \right], \tag{19}
\end{aligned}$$

when  $\Re(\alpha) > -1, \Re(\beta) > -1$  and  $\frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ , then using Gauss classical summation theorem in Eq. (19), we get

$$\begin{aligned}
I &= \frac{\sin(\alpha+\beta+\gamma) \frac{\pi}{2}}{(i)^\alpha 2^{\alpha+\beta-1} (\alpha+\beta+\gamma)} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(-\alpha)_n (-1)^n \left(\frac{-\alpha-\beta-\gamma}{2}\right)_n \Gamma(\frac{2-\alpha-\beta-\gamma+2n}{2}) \Gamma(1+\beta)}{\left(\frac{2-\alpha-\beta-\gamma}{2}\right)_n n! \Gamma(\frac{2-\alpha-\beta-\gamma+2n+2\beta}{2})}. \tag{20}
\end{aligned}$$

On further simplification, we get the required result (10). ■

*Proof of Integral (11) and (12).* From Eq. (10) we have

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta e^{i\gamma t} dt \\
&= \frac{\pi e^{-i(\frac{\pi}{2}\alpha)} \Gamma(1+\beta)}{2^{\alpha+\beta} \Gamma(\frac{2+\alpha+\beta+\gamma}{2}) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; & -1 \\ \frac{2-\alpha+\beta-\gamma}{2}; & \end{matrix} \right], \tag{21}
\end{aligned}$$

where  $\Re(\alpha) > -1, \Re(\beta) > -1, \Re(\alpha+\beta) > -2, \frac{2-\alpha-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ , when  $\alpha, \beta$

and  $\gamma$  are purely real numbers then we can equate real and imaginary parts

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta \{ \cos(\gamma t) + i \sin(\gamma t) \} dt \\ &= \frac{\pi \{ \cos(\frac{\pi}{2}\alpha) - i \sin(\frac{\pi}{2}\alpha) \}}{2^{\alpha+\beta} \Gamma(\frac{2+\alpha+\beta+\gamma}{2}) \Gamma(\frac{2-\alpha+\beta-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\beta-\gamma}{2}; \\ \frac{2-\alpha+\beta-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (22)$$

where  $\alpha > -1, \beta > -1, (\alpha + \beta) > -2$ .

Equating real and imaginary parts of Eq. (22), we get the required results (11) and (12).  $\blacksquare$

*Proof of integral (13)–(18).* The proof of integrals (13),(14),(15),(16),(17)and (18) can be obtained by using same technique and Gauss' classical summation theorem.  $\blacksquare$

#### 4. Other Associated Integrals as Special Cases

In Eq. (10), put  $\beta = 0$ , we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha e^{i\gamma t} dt = \frac{\pi e^{-i(\frac{\pi}{2}\alpha)}}{2^\alpha \Gamma(\frac{2+\alpha+\gamma}{2}) \Gamma(\frac{2-\alpha-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (23)$$

where  $\Re(\alpha) > -1$ .

In equations (11) and (12), put  $\beta = 0$ , we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha \cos(\gamma t) dt = \frac{\pi \cos(\frac{\pi}{2}\alpha)}{2^\alpha \Gamma(\frac{2+\alpha+\gamma}{2}) \Gamma(\frac{2-\alpha-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (24)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^\alpha \sin(\gamma t) dt = -\frac{\pi \sin(\frac{\pi}{2}\alpha)}{2^\alpha \Gamma(\frac{2+\alpha+\gamma}{2}) \Gamma(\frac{2-\alpha-\gamma}{2})} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (25)$$

where  $\alpha$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equations (24) and (25) is well defined and  $\alpha > -1$ .

In equation (10), put  $\alpha = 0$ , we get (Gröbner-Hofreiter Integral)[6, p. 138, Eq. (19)]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^\beta e^{i\gamma t} dt = \frac{\pi \Gamma(1+\beta)}{2^\beta \Gamma(\frac{2+\beta+\gamma}{2}) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (26)$$

where  $\Re(\beta) > -1, \frac{2-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

In equations (11) and (12), put  $\alpha = 0$ , we get (Nielsen) [9, p. 159, Eq.(6)]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^\beta \cos(\gamma t) dt = \frac{\pi \Gamma(1+\beta)}{2^\beta \Gamma(\frac{2+\beta+\gamma}{2}) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (27)$$

Nielsen [9, p. 159, Eq.(7)]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^\beta \sin(\gamma t) dt = 0, \quad (28)$$

where  $\beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equation (27) is well defined and  $\beta > -1$ .

In equation (13), put  $\beta = 0$ , we get

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^\alpha e^{i\gamma t} dt &= \frac{e^{i(\alpha+2\gamma-1)\frac{\pi}{2}} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1+\alpha)}{2^\alpha (\alpha+\gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} \\ &\quad - \frac{e^{-i(2\alpha+\gamma+1)\frac{\pi}{2}}}{2^\alpha (\alpha+\gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (29)$$

where  $\Re(\alpha) > -1, \frac{2-\alpha-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

In equations (14) and (15), put  $\beta = 0$ , we get

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^\alpha \cos(\gamma t) dt &= \frac{\cos\{(\alpha+2\gamma-1)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1+\alpha)}{2^\alpha (\alpha+\gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} \\ &\quad - \frac{\cos\{(2\alpha+\gamma+1)\frac{\pi}{2}\}}{2^\alpha (\alpha+\gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (30)$$

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\pi} (\sin t)^\alpha \sin(\gamma t) dt &= \frac{\sin\{(\alpha+2\gamma-1)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1+\alpha)}{2^\alpha (\alpha+\gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} \\ &\quad + \frac{\sin\{(2\alpha+\gamma+1)\frac{\pi}{2}\}}{2^\alpha (\alpha+\gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \end{aligned} \quad (31)$$

where  $\alpha$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equations (30) and (31) is well defined and  $\alpha > -1, (\alpha+\gamma) \neq 2, 4, 6, \dots$

In equation (13), put  $\alpha = 0$ , we get

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\pi} (\cos t)^\beta e^{i\gamma t} dt &= \frac{e^{i(2\beta+2\gamma-1)\frac{\pi}{2}}}{2^\beta (\beta+\gamma)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ &\quad - \frac{e^{-i(\beta+\gamma+1)\frac{\pi}{2}} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^\beta (\beta+\gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \end{aligned} \quad (32)$$

where  $\Re(\beta) > -1, \frac{2-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

In equations (14) and (15) put  $\alpha = 0$ , we get

$$\int_{-\frac{\pi}{2}}^{\pi} (\cos t)^{\beta} \cos(\gamma t) dt = \frac{\cos \{(2\beta + 2\gamma - 1)\frac{\pi}{2}\}}{2^{\beta}(\beta + \gamma)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{matrix} -1 \right] - \frac{\cos \{(\beta + \gamma + 1)\frac{\pi}{2}\} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1 + \beta)}{2^{\beta}(\beta + \gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (33)$$

$$\int_{-\frac{\pi}{2}}^{\pi} (\cos t)^{\beta} \sin(\gamma t) dt = \frac{\sin \{(2\beta + 2\gamma - 1)\frac{\pi}{2}\}}{2^{\beta}(\beta + \gamma)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{matrix} -1 \right] + \frac{\sin \{(\beta + \gamma + 1)\frac{\pi}{2}\} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1 + \beta)}{2^{\beta}(\beta + \gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (34)$$

where  $\beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equations (33) and (34) is well defined and  $\beta > -1, (\beta + \gamma) \neq 2, 4, 6, \dots$

In equation (16), put  $\beta = 0$ , we get

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\alpha} e^{i\gamma t} dt = \frac{e^{-i(\alpha-1)\frac{\pi}{2}} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1 + \alpha)}{2^{\alpha}(\alpha + \gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} - \frac{e^{i(\gamma+1)\frac{\pi}{2}}}{2^{\alpha}(\alpha + \gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (35)$$

where  $\Re(\alpha) > -1, \frac{2-\alpha-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

In equations (17) and (18) put  $\beta = 0$ , we get

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\alpha} \cos(\gamma t) dt = \frac{\cos \{(1 - \alpha)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1 + \alpha)}{2^{\alpha}(\alpha + \gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} - \frac{\cos \{(\gamma + 1)\frac{\pi}{2}\}}{2^{\alpha}(\alpha + \gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (36)$$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\alpha} \sin(\gamma t) dt = \frac{\sin \{(1 - \alpha)\frac{\pi}{2}\} \Gamma(\frac{2-\alpha-\gamma}{2}) \Gamma(1 + \alpha)}{2^{\alpha}(\alpha + \gamma) \Gamma(\frac{2+\alpha-\gamma}{2})} - \frac{\sin \{(\gamma + 1)\frac{\pi}{2}\}}{2^{\alpha}(\alpha + \gamma)} {}_2F_1 \left[ \begin{matrix} -\alpha, \frac{-\alpha-\gamma}{2}; \\ \frac{2-\alpha-\gamma}{2}; \end{matrix} -1 \right], \quad (37)$$

where  $\alpha$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equations (36) and (37) is well defined and  $\alpha > -1, (\alpha + \gamma) \neq 2, 4, 6, \dots$

In equation (16), put  $\alpha = 0$ , we get

$$\int_0^{\frac{\pi}{2}} (\cos t)^\beta e^{i\gamma t} dt = \frac{e^{\frac{i\pi}{2}}}{2^\beta(\beta + \gamma)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{matrix} -1 \right] - \frac{e^{i(1+\beta+\gamma)\frac{\pi}{2}} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^\beta(\beta + \gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (38)$$

where  $\Re(\beta) > -1$ ,  $\frac{2-\beta-\gamma}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ .

In equations (17) and (18) put  $\alpha = 0$ , we get Gröbner-Hofreiter Integral [6, p. 108, Eq. (9c)] (see also [9, p. 158, Eq.(5)])

$$\int_0^{\frac{\pi}{2}} (\cos t)^\beta \cos(\gamma t) dt = -\frac{\cos \{(1+\beta+\gamma)\frac{\pi}{2}\} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^\beta(\beta + \gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \quad (39)$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} (\cos t)^\beta \sin \gamma t dt &= \frac{1}{2^\beta(\beta + \gamma)} {}_2F_1 \left[ \begin{matrix} -\beta, \frac{-\beta-\gamma}{2}; \\ \frac{2-\beta-\gamma}{2}; \end{matrix} -1 \right] \\ &\quad + \frac{\sin \{(1+\beta+\gamma)\frac{\pi}{2}\} \Gamma(\frac{2-\beta-\gamma}{2}) \Gamma(1+\beta)}{2^\beta(\beta + \gamma) \Gamma(\frac{2+\beta-\gamma}{2})}, \end{aligned} \quad (40)$$

where  $\beta$  and  $\gamma$  are real numbers such that each Gamma function in the right hand side of equations (39) and (40) is well defined and  $\beta > -1$ ,  $(\beta + \gamma) \neq 2, 4, 6, \dots$

## 5. Applications in Weber-Anger-Type Functions

The Weber  $\mathbf{E}_\nu(z)$  and Anger  $\mathbf{J}_\nu(z)$  functions are defined by the formulas [3]

$$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu t - z \sin t) dt, \quad (41)$$

$$\mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu t - z \sin t) dt. \quad (42)$$

### Integrals Parallel to Weber function

$$\mathbf{E}_v^{(a)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(v\theta - z \sin \theta) d\theta = 0, \quad (43)$$

$$\begin{aligned} \mathbf{E}_v^{(b)}(z) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \sin(v\theta - z \sin \theta) d\theta \\ &= -\frac{\cos(\pi v)}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2-v}{2}, \frac{2+v}{2}; \end{matrix} \frac{-z^2}{4} \right] - \frac{z \cos(\pi v)}{\pi(1-v^2)} \end{aligned}$$

$$\begin{aligned} & \times {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3+v}{2}, \frac{3-v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{\cos(\frac{\pi v}{2})}{\pi v} A + \frac{z^2 \cos(\frac{\pi v}{2})}{8\pi(2-v)} B \\ & + \frac{z \cos\{(3+v)\frac{\pi}{2}\}}{2\pi(v+1)} D + \frac{z \cos\{(3+v)\frac{\pi}{2}\} \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G, \end{aligned} \quad (44)$$

$$\mathbf{E}_v^{(c)}(z) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \sin(v\theta - z \sin \theta) d\theta \quad (45)$$

$$\begin{aligned} &= \frac{1}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2-v}{2}, \frac{2+v}{2}; \end{matrix} \frac{-z^2}{4} \right] - \frac{z}{\pi(1-v^2)} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3+v}{2}, \frac{3-v}{2}; \end{matrix} \frac{-z^2}{4} \right] \\ & - \frac{\cos(\frac{\pi v}{2})}{\pi v} A - \frac{z^2 \cos(\frac{\pi v}{2})}{8\pi(2-v)} B - \frac{z \sin(\frac{\pi v}{2})}{2\pi(v+1)} D - \frac{z \sin(\frac{\pi v}{2}) \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G, \end{aligned}$$

$$\mathbf{E}_v^{(d)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(v\theta - z \cos \theta) d\theta \quad (46)$$

$$= \frac{-z}{2\Gamma(\frac{3+v}{2})\Gamma(\frac{3-v}{2})} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3+v}{2}, \frac{3-v}{2}; \end{matrix} \frac{-z^2}{4} \right],$$

$$\begin{aligned} & \mathbf{E}_v^{(e)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \sin(v\theta - z \cos \theta) d\theta \\ &= \frac{\cos(\frac{\pi v}{2})}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2-v}{2}, \frac{2+v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{z \cos\{(2+v)\frac{\pi}{2}\}}{\pi(1-v^2)} \\ & \times {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3+v}{2}, \frac{3-v}{2}; \end{matrix} \frac{-z^2}{4} \right] - \frac{\cos(\pi v)}{\pi v} A - \frac{z^2 \cos(\pi v)}{8\pi(2-v)} B \\ & + \frac{z \sin(\pi v)}{2\pi(v+1)} D + \frac{z \sin(\pi v) \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G, \end{aligned} \quad (47)$$

$$\begin{aligned} & \mathbf{E}_v^{(f)}(z) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \sin(v\theta - z \cos \theta) d\theta \\ &= \frac{z \cos\{(2+v)\frac{\pi}{2}\}}{\pi(1-v^2)} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3-v}{2}, \frac{3+v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{\cos(\frac{\pi v}{2})}{\pi v} \\ & \times {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2+v}{2}, \frac{2-v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{1}{v\pi} A + \frac{z^2}{8\pi(2-v)} B. \end{aligned} \quad (48)$$

### Integrals Parallel to Anger function

$$\mathbf{J}_v^{(a)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(v\theta - z \sin \theta) d\theta$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\frac{2+v}{2})\Gamma(\frac{2-v}{2})}A + \frac{z^2v}{8(2-v)\Gamma(\frac{2+v}{2})\Gamma(\frac{2-v}{2})}B \\
&\quad - \frac{z}{2\Gamma(\frac{3+v}{2})\Gamma(\frac{1-v}{2})}D - \frac{z\Gamma(\frac{v-1}{2})}{2\Gamma(\frac{1+v}{2})\Gamma(\frac{1-v}{2})\Gamma(\frac{1+v}{2})}G,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\mathbf{J}_v^{(b)}(z) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \cos(v\theta - z \sin \theta) d\theta \\
&= \frac{\sin(\pi v)}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2+v}{2}, \frac{2-v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{z \sin(\pi v)}{\pi(1-v^2)} \\
&\quad \times {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3-v}{2}, \frac{3+v}{2}; \end{matrix} \frac{-z^2}{4} \right] + \frac{\sin(\frac{\pi v}{2})}{\pi v} A + \frac{z^2 \sin(\frac{\pi v}{2})}{8\pi(2-v)} B \\
&\quad + \frac{z \sin\{(3+v)\frac{\pi}{2}\}}{2\pi(v+1)} D + \frac{z \sin\{(3+v)\frac{\pi}{2}\} \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G,
\end{aligned} \tag{50}$$

$$\begin{aligned}
\mathbf{J}_v^{(c)}(z) &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(v\theta - z \sin \theta) d\theta \\
&= \frac{\sin(\frac{\pi v}{2})}{\pi v} A + \frac{z^2 \sin(\frac{\pi v}{2})}{8\pi(2-v)} B - \frac{z \cos(\frac{\pi v}{2})}{2\pi(v+1)} D - \frac{z \cos(\frac{\pi v}{2}) \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G,
\end{aligned} \tag{51}$$

$$\mathbf{J}_v^{(d)}(z) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(v\theta - z \cos \theta) d\theta \tag{52}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\frac{2+v}{2})\Gamma(\frac{2-v}{2})} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2+v}{2}, \frac{2-v}{2}; \end{matrix} \frac{-z^2}{4} \right], \\
\mathbf{J}_v^{(e)}(z) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \cos(v\theta - z \cos \theta) d\theta \\
&= \frac{z \sin\{(2+v)\frac{\pi}{2}\}}{\pi(1-v^2)} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{3-v}{2}, \frac{3+v}{2}; \end{matrix} \frac{-z^2}{4} \right] \\
&\quad + \frac{\sin(\frac{\pi v}{2})}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2+v}{2}, \frac{2-v}{2}; \end{matrix} \frac{-z^2}{4} \right] \\
&\quad + \frac{\sin(\pi v)}{\pi v} A + \frac{z^2 \sin(\pi v)}{8\pi(2-v)} B + \frac{z \cos(\pi v)}{2\pi(v+1)} D + \frac{z \cos(\pi v) \Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G,
\end{aligned} \tag{53}$$

$$\begin{aligned}
\mathbf{J}_v^{(f)}(z) &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(v\theta - z \cos \theta) d\theta \\
&= \frac{\sin(\frac{\pi v}{2})}{\pi v} {}_1F_2 \left[ \begin{matrix} 1; \\ \frac{2+v}{2}, \frac{2-v}{2}; \end{matrix} \frac{-z^2}{4} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{z \sin \left\{ (2+v) \frac{\pi}{2} \right\}}{\pi(1-v^2)} {}_1F_2 \left[ \begin{array}{c} 1; \\ \frac{3-v}{2}, \frac{3+v}{2}; \end{array} \frac{-z^2}{4} \right] \\
& + \frac{z}{2\pi(v+1)} D + \frac{z\Gamma(\frac{v-1}{2})}{4\pi\Gamma(\frac{1+v}{2})} G. \tag{54}
\end{aligned}$$

Where  $A, B, D$  and  $G$  in the section 5 is given by:

$$\begin{aligned}
A &= F_{3: 1; 0}^{1: 2; 0} \left( \begin{array}{ccc} [1: 2, 2] & : [\frac{v}{2}: 1], [1: 1]; -; & -\frac{z^2}{16}, -\frac{z^2}{16} \\ & [1: 2, 1], [\frac{1}{2}: 1, 1], [1: 1, 1] : & [\frac{2+v}{2}: 1]; -; \end{array} \right) \\
B &= F_{3: 0; 1}^{1: 0; 2} \left( \begin{array}{ccc} [3: 2, 2] & : -; [\frac{2-v}{2}: 1], [1: 1]; & -\frac{z^2}{16}, -\frac{z^2}{16} \\ & [3: 1, 2], [\frac{3}{2}: 1, 1], [2: 1, 1] : -; & [\frac{4-v}{2}: 1]; \end{array} \right) \\
D &= F_{3: 1; 0}^{1: 2; 0} \left( \begin{array}{ccc} [2: 2, 2] & : [\frac{1+v}{2}: 1], [1: 1]; -; & -\frac{z^2}{16}, -\frac{z^2}{16} \\ & [2: 2, 1], [\frac{3}{2}: 1, 1], [1: 1, 1] : & [\frac{3+v}{2}: 1]; -; \end{array} \right) \\
G &= F_{3: 0; 1}^{1: 0; 2} \left( \begin{array}{ccc} [2: 2, 2] & : -; [\frac{1-v}{2}: 1], [1: 1]; & -\frac{z^2}{16}, -\frac{z^2}{16} \\ & [2: 1, 2], [\frac{3}{2}: 1, 1], [1: 1, 1] : -; & [\frac{3-v}{2}: 1]; \end{array} \right)
\end{aligned}$$

*Proof of Integral (43).*

$$\begin{aligned}
\mathbf{E}_v^{(a)}(z) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(v\theta - z \sin \theta) d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(v\theta) \cos(z \sin \theta) d\theta - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(v\theta) \sin(z \sin \theta) d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(v\theta) {}_0F_1 \left[ \begin{array}{c} -; \\ \frac{1}{2}; \end{array} \frac{-z^2 \sin^2 \theta}{4} \right] d\theta \\
&\quad - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(v\theta) (z \sin \theta) {}_0F_1 \left[ \begin{array}{c} -; \\ \frac{3}{2}; \end{array} \frac{-z^2 \sin^2}{4} \right] d\theta \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \theta)^{2n} \sin(v\theta) d\theta \\
&\quad - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \theta)^{2n+1} \cos(v\theta) d\theta. \tag{55}
\end{aligned}$$

Using equations (25) and (24) in equation (55), we get the required result (43). ■

*Proof of Integral (44).*

$$\begin{aligned}
\mathbf{E}_v^{(b)}(z) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \sin(v\theta - z \sin \theta) d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \sin(v\theta) \cos(z \sin \theta) d\theta - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \cos(v\theta) \sin(z \sin \theta) d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \sin(v\theta) \sum_{n=0}^{\infty} \frac{(-1)^n (z \sin \theta)^{2n}}{(2n)!} d\theta \\
&\quad - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \cos(v\theta) \sum_{n=0}^{\infty} \frac{(-1)^n (z \sin \theta)^{2n+1}}{(2n+1)!} d\theta \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} (\sin \theta)^{2n} \sin(v\theta) d\theta \\
&\quad - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} (\sin \theta)^{2n+1} \cos(v\theta) d\theta. \tag{56}
\end{aligned}$$

Using equations (31) and (30) in equation (56), we get

$$\begin{aligned}
\mathbf{E}_v^{(b)}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \frac{1}{\pi} \times \left\{ \frac{\sin \{(2n+2v-1)\frac{\pi}{2}\} \Gamma(\frac{2-2n-v}{2}) \Gamma(1+2n)}{2^{2n}(2n+v) \Gamma(\frac{2+2n-v}{2})} \right. \\
&\quad + \frac{\sin \{(4n+v+1)\frac{\pi}{2}\}}{2^{2n}(2n+v)} \times {}_2F_1 \left[ \begin{matrix} -2n, \frac{-2n-v}{2}; \\ \frac{2-2n-v}{2}; \end{matrix} -1 \right] \Big\} \\
&\quad - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \frac{1}{\pi} \left\{ \frac{\cos \{(2n+1+2v-1)\frac{\pi}{2}\}}{2^{2n+1}(2n+1+v)} \right. \\
&\quad \times \frac{\Gamma(\frac{2-2n-1-v}{2}) \Gamma(1+2n+1)}{\Gamma(\frac{2+2n+1-v}{2})} - \frac{\cos \{(4n+2+v+1)\frac{\pi}{2}\}}{2^{2n+1}(2n+1+v)} \\
&\quad \times {}_2F_1 \left[ \begin{matrix} -2n-1, \frac{-2n-1-v}{2}; \\ \frac{1-2n-v}{2}; \end{matrix} -1 \right] \Big\}.
\end{aligned}$$

Further applying Cauchy's double series identities and using the definition of double hypergeometric function of Srivastava-Daoust (1), we get the required result (44). ■

*Proof of integrals (45)–(54).* The proof of integrals (45), (46), (47), (48), (49), (50), (51), (52), (53) and (54), can be obtained by using the same approach, so we omit the details here. ■

## 6. Concluding Remarks

In this article, we evaluated Gröbner-Hofreiter-type integrals and also deduced some applications in the form of Weber-Anger-type functions by using hypergeometric approach, Euler's beta function and Gauss' classical summation theorem. We conclude this paper with these words that certain Gröbner-Hofreiter type integrals and other definite integrals which may be different from those presented here can also be evaluated in a similar way other than Cauchy's residue theorem. Moreover the mentioned integral is supposed to find various applications in Applied mathematics and theory of probability. We may consider the definite integrals whose limit varies from 0 to  $\pi$ , 0 to  $\frac{3\pi}{2}$  and 0 to  $2\pi$  (see [12]).

**Acknowledgement.** We are very thankful to the Editor and Reviewer for their kind suggestions to improve the paper in its present form.

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