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Total Edge Irregularity Strength of Modified Book Graphs*

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Abstract. Let $G(V, E)$ be a graph with vertex set V and edge set E. A map of f from the union of a vertex set and edge sets to $\{1, 2, \ldots, k\}$ such that for each different edge uv and $u'v'$ have different weights is called an irregular total edge k-graph labeling $G(V, E)$. The weight of the edge uv is the sum of the edge label uv, the vertex label u, and the vertex label v. The smallest k so that the graph $G(V, E)$ can be labeled with the edge irregular total k labeling is called the total edge irregularity strength of $G(V, E)$ and is denoted by $tes(G)$. Book graphs $B_d(G)$ have d copies of graph G with a common edge, and the common edge is the same fixed one in all copies of G. We modify the book graphs by replacing G with a wheel graph or a complete graph to obtain wheel book graphs or complete book graphs, respectively. In this paper, we determine the total edge irregularity strength of modified book graphs: wheel book graphs and complete book graphs.

Keywords: Total edge irregularity strength; Edge irregular total k-labeling; Book graphs; Wheel book graphs; Complete book graphs.

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1. Introduction

We consider a finite, simple and undirected graph. A graph labeling is defined as a function that assigns graph elements to numbers (usually positive integers). When the domain of the labelling is the vertex set (edge set, vertex and edge set) then the labelling is called vertex labeling (edge labeling, total labeling, respectively) [18].

Chartrand et al. [6] introduced irregular edge k-labeling on graph $G(V, E)$ as a mapping f from E to set $\{1, 2, ..., k\}$ such that the weights of all vertices in G are different, in which the weight of vertex v, respect to f, is denoted by $\omega_f(v)$ and is defined as the sum of all labels of edges that are incident to vertex v. The smallest number k such that graph G admits an irregular edge k-labeling is called irregularity strength of G and denoted by $s(G)$.

Bača et al. in [2] defined an edge irregular total labeling of graphs. Suppose $G(V, E)$ is a finite, simple and undirected graph with vertex set V and edge set E. An edge irregular total k-labeling of graph $G(V, E)$ is a total-k labelling $f : V \cup E \to \{1, 2, \ldots, k\}$ there is for any two different edges uv and $u'v'$, irregular in the sense of the $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are not the same. The smallest k for which the graph G admits an edge irregular total k-labeling is called the total edge irregularity strength of G and is denoted by $tes(G)$.

In [2] the authors have determined a lower bound of the total edge irregularity strength by the form $tes(G) \geq max\{\left\lceil \frac{|E|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\}$ where $\Delta(G)$ is the maximum vertex degree of G. The exact value of total edge irregularity strength is also provided for some graphs including path graphs, cycle graphs, wheel graphs, and fan graphs.

Several results of the total edge irregularity strength for certain classes of graphs can be found in [1, 4, 3, 17, 7, 8, 9, 10, 11, 12, 14, 13, 15, 16].

For arbitrary natural number d, book graph $B_d(G)$ has d copies of graph G with a common edge [5]. Particularly, we have the book graphs $B_d(C_e)$ if G is the cycle graph C_e. The total edge irregularity strength of book graphs $B_d(C_e)$ and double book graphs $2B_d(C_e)$ have been determined in [13]. Futhermore, in this paper we investigate the total edge irregularity strength of wheel book graphs $B_d(W_e)$ and complete book graphs $B_d(K_e)$.

2. Total Edge Irregularity Strength of Wheel Book Graphs

In this section, we construct an edge irregular total k labeling of wheel book graphs. First, we will define the wheel book graphs as below:

Definition 2.1. *Given wheel graphs W_e^s, $s = 1, 2, ..., d$ with*

$$V(W_e^s) = \{u, v, w_s\} \cup \{v_{s,t} : \ t = 1, 2, ..., e-2\}$$

with w_s as the center vertex and

$$E(W_e^s) = \{uv, uw_s, w_sv_{s,t}, w_sv, uv_{s,1}, v_{s,t}v_{s,t+1}, v_{s,e-2}v : \ t = 1, ..., e-3\}.$$

By a wheel book graph $B_d(W_e)$ we mean a graph obtained from wheel graphs W_e^s, $s = 1, 2, ..., d$ by merging the edge uv from the wheel graphs. Thus we have the vertex set and the edge set of $B_d(W_e)$, respectively, as below:

$$V(B_d(W_e)) = \bigcup_{s=1}^{d} V(W_e^s) = \{u, v, w_s\} \cup \{v_{s,t} : \ \}$$

$$E(B_d(W_e)) = \bigcup_{s=1}^{d} E(W_e^s) = \{uv\} \cup \{uw_s, w_sv_{s,t}, w_sv, uv_{s,1}, v_{s,t}v_{s,t+1}, v_{s,e-2}v\}$$

for $s = 1, 2, ..., d; t = 1, 2, ..., e-2$.

From Definition 2.1, we obtain $|E(B_d(W_e))| = (2e-1)d + 1$.

Example 2.2. The following is a wheel book graph $B_2(W_6)$ of 6 sides and two sheets, as shown in Figure 1. The following three lemmas show the total edge

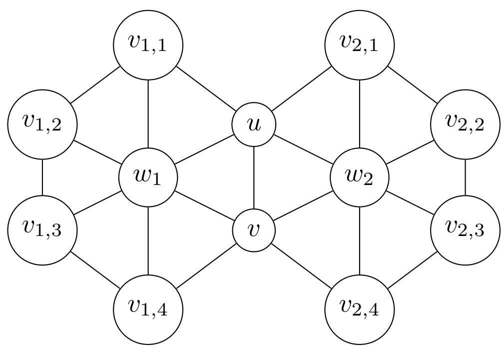

Figure 1: Wheel Book Graph $B_2(W_6)$

irregularity strength of some wheel book graphs.

Lemma 2.3. *Let $B_d(W_e)$ be a wheel book graph of d sheets. For $e = 0(mod\ 3)$, we have $tes(B_d(W_e)) = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$.*

Proof. For any wheel book graph $B_d(W_e)$ has maximum degree $\Delta(B_d(W_e)) = max\{2d+1, e\}$. By $tes(G) \geq max\left\{\left\lceil \frac{|E|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}$, we have $tes(B_d(W_e)) \geq \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$. Next we show that there is an edge irregular total k-labeling of $B_d(W_e)$ with $k = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$, for $e = 0(mod\ 3)$.

We define a total labeling f as below:
For $e = 3$, let

$$
\begin{array}{ll}
f(u) = 1, \quad f(v) = k & \\
f(v_{s,1}) = s, & 1 \le s \le d \\
f(w_s) = s + d, & 1 \le s \le k - d \\
f(w_s) = k, & k - d + 1 \le s \le d \\
f(uv) = d + \lceil \frac{d+1}{3} \rceil & \\
f(uv_{s,1}) = 1, & 1 \le s \le d \\
f(uw_s) = 1, & 1 \le s \le k - d \\
f(uw_s) = 1 + s - k + d, & k - d + 1 \le s \le d \\
f(w_s v_{s,1}) = d + 2 - s, & 1 \le s \le k - d, \\
f(w_s v_{s,1}) = 2d + 2 - k, & k - d + 1 \le s \le d \\
f(v_{s,1} v) = 3d + 3 - k, & 1 \le s \le d \\
f(w_s v) = 3d + 3 - k, & 1 \le s \le k - d \\
f(w_s v) = 4d + 3 - 2k + s, & k - d + 1 \le s \le d.
\end{array}
$$

From this labeling we obtain the weight of the edges for $e = 3$ as the following:

$$
\begin{array}{ll}
\omega_f(uv_{s,1}) = 2 + s, & 1 \le s \le d \\
\omega_f(uw_s) = d + 2 + s, & 1 \le s \le d \\
\omega_f(w_s v_{s,1}) = 2d + 2 + s, & 1 \le s \le d \\
\omega_f(uv) = k + d + \lceil \frac{d+1}{3} \rceil + 1, & 1 \le s \le d \\
\omega_f(v_{s,1} v) = 3d + 3 + s, & 1 \le s \le d \\
\omega_f(w_s v) = 4d + 3 + s, & 1 \le s \le d.
\end{array}
$$

Futhermore, we define the vertex and edge labeling for $\boldsymbol{e} \ge \mathbf{6}$. We define the vertex labeling as below:

$$
\begin{array}{ll}
f(u) = 1, \quad f(v) = k & \\
f(v_{s,1}) = s, & 1 \le s \le d \\
f(w_s) = s + d, & 1 \le s \le d \\
f(v_{s,t}) = s + td, & 1 \le s \le d, \quad 2 \le t \le \frac{2e-6}{3} \\
f(v_{s,t}) = s + td, & 1 \le s \le k - \frac{2e-3}{3} d, \quad t = \frac{2e-3}{3} \\
f(v_{s,t}) = k, & k + 1 - \frac{2e-3}{3} d \le s \le d, \quad t = \frac{2e-3}{3} \\
f(v_{s,t}) = k, & 1 \le s \le d, \quad \frac{2e}{3} \le t \le e - 2.
\end{array}
$$

And we define the edge labeling as the following:

$$
\begin{array}{ll}
f(uv) = 2d + \lceil \frac{d+1}{3} \rceil & \\
f(uv_{s,1}) = 1, & 1 \le s \le d \\
f(uw_s) = 1, & 1 \le s \le d.
\end{array}
$$

Based on the definition of labeling of vertex $v_{s,t}$ for $t = \frac{2e-3}{3}$, the labeling for edge $w_s v_{s,t}$ is defined as below:

$f(w_sv_{s,t}) = d+2-s,$ $\quad 1 \le s \le d,\ 1 \le t \le 2,$ (for $e \ge 6$)
$f(w_sv_{s,t}) = (t-2)d+2-s,$ $\quad 1 \le s \le d,\ 3 \le t \le \frac{e+3}{3},$ (for $e \ge 9$)
$f(w_sv_{s,t}) = \frac{e-3}{3}d+3-s,$ $\quad 1 \le s \le d,\ \frac{e+6}{3} \le t \le \frac{2e-6}{3},$ (for $e \ge 12$)
$f(w_sv_{s,t}) = \frac{e-3}{3}d+2-s,$ $\quad 1 \le s \le k-\frac{2e-3}{3}d,\ t = \frac{2e-3}{3}$ (for $e = 6$)
$f(w_sv_{s,t}) = (t+\frac{e-3}{3})d+2-k,$ $\quad k+1-\frac{2e-3}{3}d \le s \le d,\ t = \frac{2e-3}{3},$ (for $e = 6$)
$f(w_sv_{s,t}) = \frac{e-3}{3}d+3-s,$ $\quad 1 \le s \le k-\frac{2e-3}{3}d,\ t = \frac{2e-3}{3}$ (for $e \ge 9$)
$f(w_sv_{s,t}) = (t+\frac{e-3}{3})d+3-k,$ $\quad k+1-\frac{2e-3}{3}d \le s \le d,\ t = \frac{2e-3}{3}$ (for $e \ge 9$)
$f(w_sv_{s,t}) = (t+\frac{e-3}{3})d+3-k,$ $\quad 1 \le s \le d,\ \frac{2e}{3} \le t \le e-2,$ (for $e \ge 6$).

Furthermore, the labeling for edges w_sv and $v_{s,t}v_{s,t+1}$ is defined as below:

$f(w_sv) = 2(\frac{2e-3}{3})d+3-k,$
$f(v_{s,t}v_{s,t+1}) = d+2-s,$ $\quad 1 \le s \le d,\ 1 \le t \le \frac{e-3}{3}$
$f(v_{s,t}v_{s,t+1}) = (e-t-2)d+3-s,$ $\quad 1 \le s \le d,\ \frac{e}{3} \le t \le \frac{2e-9}{3}$ (for $e \ge 9$)
$f(v_{s,t}v_{s,t+1}) = (e-1-t)d+3-s,$ $\quad 1 \le s \le k-\frac{2e-3}{3}d,\ t = \frac{2e-6}{3}$
$f(v_{s,t}v_{s,t+1}) = ed+3-k,$ $\quad k+1-\frac{2e-3}{3}d \le s \le d,\ t = \frac{2e-6}{3}$
$f(v_{s,t}v_{s,t+1}) = ed+3-k,$ $\quad 1 \le s \le k-\frac{2e-3}{3}d,\ t = \frac{2e-3}{3}$
$f(v_{s,t}v_{s,t+1}) = ed+3-k+p,$ $\quad k+1-\frac{2e-3}{3}d \le s \le d,\ t = \frac{2e-3}{3}$
$\quad 1 \le p \le \frac{2e}{3}d-k$
$f(v_{s,t}v_{s,t+1}) = (t+e)d+3+s-2k,$ $\quad 1 \le s \le d,\ \frac{2e}{3} \le t \le e-3$ (for $e \ge 9$)
$f(v_{s,e-2}v) = (2e-2)d+3+s-2k,$ $\quad 1 \le s \le d.$

From this labeling we get the weight of the edges for $\boldsymbol{e \ge 6}$ as below:

$\omega_f(uv_{s,1}) = 2+s,$ $\quad 1 \le s \le d$
$\omega_f(uw_s) = d+2+s,$ $\quad 1 \le s \le d$
$\omega_f(w_sv_{s,1}) = 2d+2+s,$ $\quad 1 \le s \le d$
$\omega_f(w_sv_{s,t}) = 4d+2+s,$ $\quad 1 \le s \le d,\ t = 2$
$\omega_f(w_sv_{s,t}) = (2t-1)d+2+s,$ $\quad 1 \le s \le d,\ 3 \le t \le \frac{e}{3}$(for $e \ge 9$)
$\omega_f(uv) = 2d+\lceil\frac{d+1}{3}\rceil+1+k,$ $\quad 1 \le s \le d$
$\omega_f(w_sv_{s,t}) = (\frac{e}{3}+t)d+2+s,$ $\quad 1 \le s \le d,\ t = \frac{2e-3}{3}$ (for $e = 6$)
$\omega_f(w_sv_{s,t}) = (\frac{e}{3}+t)d+3+s,$ $\quad 1 \le s \le d,\ \frac{e+3}{3} \le t \le e-2$
$\omega_f(w_sv) = (\frac{4e-3}{3})d+3+s,$ $\quad 1 \le s \le d$
$\omega_f(v_{s,t}v_{s,t+1}) = (t+2)d+2+s,$ $\quad 1 \le s \le d,\ 1 \le t \le \frac{e-3}{3}$ (for $e \ge 6$)
$\omega_f(v_{s,t}v_{s,t+1}) = (t+e)d+3+s,$ $\quad 1 \le s \le d,\ \frac{e}{3} \le t \le \frac{2e-9}{3}$ (for $e \ge 9$)
$\omega_f(v_{s,t}v_{s,t+1}) = (t+e)d+3+s,$ $\quad 1 \le s \le d,\ t = \frac{2e-6}{3}$
$\omega_f(v_{s,t}v_{s,t+1}) = ed+3+k,$ $\quad 1 \le s \le k-\frac{2e-3}{3}d,\ t = \frac{2e-3}{3}$
$\omega_f(v_{s,t}v_{s,t+1}) = ed+3+k+p,$ $\quad k-\frac{2e-3}{3}d+1 \le s \le d,\ t = \frac{2e-3}{3}$
$\omega_f(v_{s,t}v_{s,t+1}) = (t+e)d+3+s,$ $\quad 1 \le s \le d,\ \frac{2e}{3} \le t \le e-3$ (for $e \ge 9$)
$\omega_f(v_{s,e-2}v) = (2e-2)d+3+s,$ $\quad 1 \le s \le d.$

We obtain that the weight of the edges of $B_d(W_e)$ for $e = 0(mod\ 3)$ under the labeling f constitutes the set $\{3, 4, ..., (2e-1)d+3\}$. ■

Lemma 2.4. *Let $B_d(W_e)$ be a wheel book graph of d sheets. For $e = 1(mod\ 3)$,* *we have* $tes(B_d(W_e)) = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$.

Proof. For wheel book graphs $B_d(W_e)$ with $e = 1(mod\ 3)$ also has maximum degree $\Delta(B_d(W_e)) = max\{2d+1, e\}$, then we have $tes(B_d(W_e)) \geq \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$. To prove the upper bound, we define a mapping $g : V(B_d(W_e)) \cup E(B_d(W_e)) \rightarrow \{1, 2, ...k\}$, with $k = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$, as below:

For $e = 4$ we have,

$$\begin{array}{ll}
g(u) = 1, \quad g(v) = k & \\
g(v_{s,1}) = s, & 1 \leq s \leq d \\
g(w_s) = d + s, & 1 \leq s \leq k - d \\
g(v_{s,2}) = 2d + s, & 1 \leq s \leq k - 2d \\
g(v_{s,2}) = k, & k + 1 - 2d \leq s \leq d.
\end{array}$$

We define the edge labeling for $e = 4$ as below:

$$\begin{array}{ll}
g(uv) = 2d - \lfloor \frac{d-1}{3} \rfloor & \\
g(uv_{s,1}) = 1, & 1 \leq s \leq d \\
g(uw_s) = 1, & 1 \leq s \leq k - d \\
g(w_s v_{s,1}) = d + 2 - s, & 1 \leq s \leq d, \\
g(w_s v_{s,2}) = d + 3 - s, & 1 \leq s \leq k - 2d, \\
g(w_s v_{s,2}) = 3d + 3 - k, & k + 1 - 2d \leq s \leq d \\
g(w_s v) = 4d + 3 - k, & 1 \leq s \leq d \\
g(v_{s,1} v_{s,2}) = d + 2 - s, & 1 \leq s \leq k - 2d \\
g(v_{s,1} v_{s,2}) = 3d + 2 - k, & k - 2d + 1 \leq s \leq d \\
g(v_{s,2} v) = 4d + 3 - k, & 1 \leq s \leq k - 2d \\
g(v_{s,2} v) = 4d + 3 - k + p, & k - 2d + 1 \leq s \leq d, \quad 1 \leq p \leq 3d - k.
\end{array}$$

From this labeling for $e = 4$ we get the edge weights below:

$$\begin{array}{ll}
\omega_g(uv_{s,1}) = 2 + s, & 1 \leq s \leq d \\
\omega_g(uw_s) = d + 2 + s, & 1 \leq s \leq d \\
\omega_g(w_s v_{s,1}) = 2d + 2 + s, & 1 \leq s \leq d \\
\omega_g(v_{s,1} v_{s,2}) = 3d + 2 + s, & 1 \leq s \leq d \\
\omega_g(uv) = 1 + k + 2d - \lfloor \frac{d-1}{3} \rfloor, & 1 \leq s \leq d \\
\omega_g(w_s v_{s,2}) = 4d + 3 + s, & 1 \leq s \leq d \\
\omega_g(w_s v) = 5d + 3 + s, & 1 \leq s \leq d \\
\omega_g(v_{s,2} v) = 6d + 3 + s, & 1 \leq s \leq d.
\end{array}$$

We define the vertex and edge labeling for $e \geq 7$ as below:

$$\begin{array}{ll}
g(u) = 1, \quad g(v) = k & \\
g(v_{s,1}) = s, & 1 \leq s \leq d \\
g(w_s) = s + d, & 1 \leq s \leq d
\end{array}$$

$$
\begin{array}{ll}
g(v_{s,t}) = s + td, & 1 \le s \le d,\ 2 \le t \le \frac{2e-5}{3} \\
g(v_{s,t}) = s + td, & 1 \le s \le k - \left(\frac{2e-2}{3}\right) d,\ t = \frac{2e-2}{3} \\
g(v_{s,t}) = k, & k + 1 - \left(\frac{2e-2}{3}\right) d \le s \le d,\ t = \frac{2e-2}{3} \\
g(v_{s,t}) = k, & 1 \le s \le d,\ \frac{2e+1}{3} \le t \le e-2 \\
g(uv) = 2d + 1 & \\
g(uv_{1,t}) = 1, & 1 \le s \le d \\
g(uw_s) = 1, & 1 \le s \le d \\
g(w_s v_{s,t}) = d + 2 - s, & 1 \le s \le d,\ 1 \le t \le 3 \\
g(w_s v_{s,t}) = (t-2)d + 2 - s, & 1 \le s \le d,\ 4 \le t \le \frac{e+2}{3}, \\
 & (\text{for } e \ge 10) \\
g(w_s v_{s,t}) = \frac{e-4}{3}d + 3 - s, & 1 \le s \le d,\ \frac{e+5}{3} \le t \le \frac{2e-5}{3}, \\
 & (\text{for } e \ge 10) \\
g(w_s v_{s,t}) = \frac{e-4}{3}d + 3 - s, & 1 \le s \le k - \frac{2e-5}{3}d,\ t = \frac{2e-2}{3}, \\
 & (\text{for } e \ge 7) \\
g(w_s v_{s,t}) = (t + \frac{e-4}{3})d + 3 - k, & k - \frac{2e-5}{3}d + 1 \le s \le d,\ t = \frac{2e-2}{3} \\
g(w_s v_{s,t}) = (t + \frac{e-4}{3})d + 3 - k, & 1 \le s \le d,\ \frac{2e+1}{3} \le t \le e-2 \\
g(w_s v) = (\frac{4e-7}{3})d + 3 - k, & 1 \le s \le d \\
g(v_{s,t}v_{s,t+1}) = d + 2 - s, & 1 \le s \le d,\ 1 \le t \le \frac{e-4}{3} \\
g(v_{s,t}v_{s,t+1}) = (e-1-t)d + 3 - s, & 1 \le s \le d,\ \frac{e-1}{3} \le t \le \frac{2e-8}{3} \\
g(v_{s,t}v_{s,t+1}) = ed - 2d + sd + 3 - s, & 1 \le s \le k - \frac{2e-2}{3}d,\ t = \frac{2e-5}{3} \\
g(v_{s,t}v_{s,t+1}) = (e-1)d + 3 - k, & k + 1 - \frac{2e-2}{3}d \le s \le d,\ t = \frac{2e-5}{3} \\
g(v_{s,t}v_{s,t+1}) = (e-1)d + 3 - k, & 1 \le s \le k - \frac{2e-2}{3}d,\ t = \frac{2e-2}{3} \\
g(v_{s,t}v_{s,t+1}) = (e-1)d + 3 - k + p, & k - \frac{2e-2}{3}d + 1 \le s \le d,\ t = \frac{2e-2}{3} \\
 & 1 \le p \le \frac{2e+1}{3}d - k \\
g(v_{s,t}v_{s,t+1}) = (t + \frac{e-1}{3})d + 2 + s, & 1 \le s \le d,\ \frac{2e+1}{3} \le t \le e-3 \\
g(v_{s,e-2}v) = (\frac{2e-4}{3})d + 2 + s, & 1 \le s \le d.
\end{array}
$$

We have the weight of the edges for $e \ge 7$ as below:

$$
\begin{array}{ll}
\omega_g(uv_{s,t}) = 2 + s, & 1 \le s \le d,\ t = 1 \\
\omega_g(uw_s) = d + 2 + s, & 1 \le s \le d \\
\omega_g(w_s v_{s,t}) = (t+1)d + 2 + s, & 1 \le s \le d,\ 1 \le t \le 3 \\
\omega_g(uv) = 2d + 2 + k, & \\
\omega_g(w_s v_{s,t}) = (2t-1)d + 2 + s, & 1 \le s \le d,\ 4 \le t \le \frac{e-1}{3} \\
\omega_g(w_s v_{s,t}) = (\frac{e-1}{3} + t)d + 3 + s, & 1 \le s \le d,\ \frac{e+2}{3} \le t \le \frac{2e-5}{3} \\
\omega_g(w_s v_{s,t}) = (e + t - 1)d + 3 + s, & 1 \le s \le d,\ \frac{e+3}{3} \le t \le e-3 \\
\omega_g(w_s v) = (\frac{4e-7}{3})d + 3 + s, & 1 \le s \le d \\
\omega_g(v_{s,t}v_{s,t+1}) = 3d + 2 + s, & 1 \le s \le d,\ t = 1 \\
\omega_g(v_{s,t}v_{s,t+1}) = (2t+2)d + 2 + s, & 1 \le s \le d,\ 2 \le t \le \frac{e-4}{3} \\
\omega_g(v_{s,t}v_{s,t+1}) = (e + t - 1)d + 3 + s, & 1 \le s \le d,\ \frac{e-1}{3} \le t \le \frac{2e-11}{3} \\
\omega_g(v_{s,t}v_{s,t+1}) = (e + t - 1)d + 3 + s, & 1 \le s \le k - \frac{2e-5}{3}d,\ t = \frac{2e-8}{3} \\
\omega_g(v_{s,t}v_{s,t+1}) = (e-2)d + 3 + k, & k - \frac{2e-5}{3}d + 1 \le s \le d,\ t = \frac{2e-8}{3} \\
\omega_g(v_{s,t}v_{s,t+1}) = (e + t - 1)d + 3 + s & 1 \le s \le k - \frac{2e-5}{3}d,\ t = \frac{2e-5}{3} \\
\omega_g(v_{s,t}v_{s,t+1}) = (e-1)d + 3 + k + p, & k - \frac{2e-5}{3}d + 1 \le s \le d,\ t = \frac{2e-5}{3} \\
 & 1 \le p \le \frac{2e-2}{3}d - k
\end{array}
$$

$$\begin{array}{ll} \omega_g(v_{s,t}v_{s,t+1}) = (t-e-1)d+3+s, & 1 \leq s \leq d,\ \frac{2e-2}{3} \leq t \leq e-3 \\ \omega_g(v_{s,e-2}v) = (2e-4)d+3+s, & 1 \leq s \leq d. \end{array}$$

We found that the weight of the edges of $B_d(W_e)$ for $e = 1(mod\ 3)$ under the labeling g constitutes the set $\{3, 4, ..., (2e-1)d+3\}$. ■

Lemma 2.5. *Let $B_d(W_e)$ be a wheel book graph of d sheets. For $e = 2(mod\ 3)$ we have $tes(B_d(W_e)) = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$.*

Proof. For wheel book graphs $B_d(W_e)$ with $e = 2(mod\ 3)$ has maximum degree $\Delta(B_d(W_e)) = max\{2d+1, e\}$, similar to Lemmas 2.3 and 2.4, we have $tes(B_d(W_e)) \geq \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$. To prove the upper bound, we define a mapping $h : V(B_d(W_e)) \cup E(B_d(W_e)) \to \{1, 2, ...k\}$, with $k = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$, as below:

For $e = 2(mod\ 3)$, we define the vertex and edge labeling h as the following:

$$\begin{array}{ll} h(u) = 1, \quad h(v) = k, & \\ h(v_{s,1}) = s, & 1 \leq s \leq d \\ h(w_s) = d+s, & 1 \leq s \leq d \\ h(v_{s,t}) = s+td, & 1 \leq s \leq d,\ 2 \leq t \leq \frac{2e-4}{3} \\ h(v_{s,t}) = k, & 1 \leq s \leq d,\ \frac{2e-1}{3} \leq t \leq e-2. \\ h(uv) = 2d+1 & \\ h(uv_{s,t}) = 1, & 1 \leq s \leq d,\ t = 1 \\ h(uw_s) = 1, & 1 \leq s \leq d. \end{array}$$

Based on the definition of labeling of vertex $v_{s,t}$ for $t = \frac{2e-1}{3}$, the labeling for edge $w_s v_{s,t}$ is defined as below:

$$\begin{array}{ll} h(w_s v_{s,t}) = d+2-s, & 1 \leq s \leq d,\ 1 \leq t \leq 2, \text{(for } e = 5) \\ h(w_s v_{s,t}) = d+2-s, & 1 \leq s \leq d,\ 1 \leq t \leq 3, \text{(for } e \geq 8) \\ h(w_s v_{s,t}) = (t-2)d+2-s, & 1 \leq s \leq d,\ 4 \leq t \leq \frac{e+1}{3}, \text{(for } e \geq 11) \\ h(w_s v_{s,t}) = \frac{e-2}{3}d+3-s, & 1 \leq s \leq d,\ \frac{e+4}{3} \leq t \leq \frac{2e-4}{3}, \text{(for } e \geq 8) \\ h(w_s v_{s,t}) = (t-\frac{e+1}{3})d+2, & 1 \leq s \leq d,\ \frac{2e-1}{3} \leq t \leq e-2 \text{ (for } \ e \geq 5) \end{array}$$

Furthermore, the labeling for edges w_s, v and $v_{s,t}v_{s,t+1}$ is defined as below:

$$\begin{array}{ll} h(w_s v) = (\frac{2e-4}{3})d+2, & 1 \leq s \leq d \\ h(v_{s,t}v_{s,t+1}) = d+2-s, & 1 \leq s \leq d, \quad 1 \leq t \leq \frac{e-2}{3} \\ h(v_{s,t}v_{s,t+1}) = (e-1-t)d+3-s, & 1 \leq s \leq d, \quad \frac{e+1}{3} \leq t \leq \frac{2e-4}{3} \\ h(v_{s,t}v_{s,t+1}) = (t-\frac{e-2}{3})d+1+s, & 1 \leq s \leq d, \quad \frac{2e-1}{3} \leq t \leq e-3 \\ h(v_{s,e-2}v) = (\frac{2e-4}{3})d+1+s, & 1 \leq s \leq d. \end{array}$$

Under that total labeling h we have the weight of the edges $e \geq 5$ as below:

$$\omega_h(uv_{s,t}) = 2+s, \qquad 1 \leq s \leq d,\ t = 1$$

$\omega_h(uw_s) = d+2+s,$ $\quad 1 \leq s \leq d$
$\omega_h(w_sv_{s,t}) = 2d+2+s,$ $\quad 1 \leq s \leq d,\ t=1$
$\omega_h(w_sv_{s,t}) = (t+2)d+2+s,$ $\quad 1 \leq s \leq d,\ t=2,$ (for $e \geq 5$)
$\omega_h(w_sv_{s,t}) = (t+2)d+2+s,$ $\quad 1 \leq s \leq d,\ t=3,$ (for $e \geq 8$)
$\omega_h(w_sv_{s,t}) = (2t-1)d+2+s,$ $\quad 1 \leq s \leq d,\ 4 \leq t \leq \frac{e+1}{3},$ (for $e \geq 8$)
$\omega_h(uv) = k+2d+2,$
$\omega_h(w_sv_{s,t}) = ((\frac{e+1}{3})+t)d+3+s,$ $\quad 1 \leq s \leq d,\ \frac{e+4}{3} \leq t \leq \frac{2e-4}{3},$ (for $e \geq 8$)
$\omega_h(w_sv_{s,t}) = (t-(\frac{e-2}{3}))d+2+k+s,$ $\quad 1 \leq s \leq d,\ \frac{2e-1}{3} \leq t \leq e-2,$ (for $e \geq 5$)
$\omega_h(w_sv) = (\frac{2e-1}{3})d+2+k+s,$ $\quad 1 \leq s \leq d$
$\omega_h(v_{s,t}v_{s,t+1}) = 3d+2+s,$ $\quad 1 \leq s \leq d,\ t=1$
$\omega_h(v_{s,t}v_{s,t+1}) = 4d+2+k+s,$ $\quad 1 \leq s \leq d,\ t=2,$ (for $e=5$)
$\omega_h(v_{s,t}v_{s,t+1}) = (2t+2)d+2+s,$ $\quad 1 \leq s \leq d,\ 2 \leq t \leq \frac{e-2}{3},$ (for $e \geq 8$)
$\omega_h(v_{s,t}v_{s,t+1}) = (t+e)d+3+s,$ $\quad 1 \leq s \leq d,\ \frac{e+1}{3} \leq t \leq \frac{2e-4}{3}$
$\omega_h(v_{s,t}v_{s,t+1}) = (t-\frac{e-2}{3})d+1+2k+s,$ $\quad 1 \leq s \leq d,\ \frac{2e-1}{3} \leq t \leq e-3$
$\omega_h(v_{s,e-2}v) = (\frac{2e-4}{3})d+1+2k+s,$ $\quad 1 \leq s \leq d.$

We found that the weight of the edges of $B_d(W_e)$ for $e = 2(mod\ 3)$ under the labeling h constitutes the set $\{3, 4, ..., (2e-1)d+3\}$. ■

From Lemmas 2.3, 2.4 and 2.5, we obtain the following theorem.

Theorem 2.6. *Let $B_d(W_e)$ be a wheel book graph of d sheets. It below that* $tes(B_d(W_e)) = \left\lceil \frac{(2e-1)d+3}{3} \right\rceil$.

Example 2.7. The following is an edge irregular total k-labeling of wheel book graph $B_2(W_6)$ with $k=9$, as shown in Figure 2.

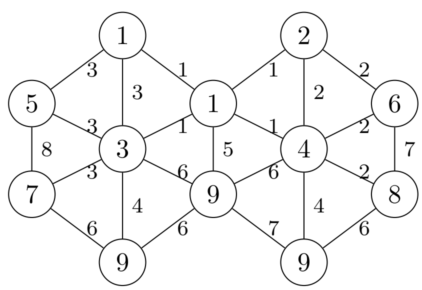

Figure 2: Edge irregular total 9-labeling of wheel book graph $B_2(W_6)$

3. Total Edge Irregularity Strength of Complete Book Graphs

In the next discussion, we investigate the total edge irregularity strength of complete book graphs. We define a complete book graph as below:

Definition 3.1. *Given complete graphs K_e^s, $s = 1, 2, ..., d$ with*

$$V(K_e^s) = \{u, v\} \cup \{a_{s,p}, b_{s,q}, c_{s,p} : \ p = 1, 2, ..., x-1, \ q = 1, 2, ..., y\}$$

with $x = \lfloor \frac{e+1}{3} \rfloor$ and $y = e - 2\lfloor \frac{e+1}{3} \rfloor$ and

$$E(K_e^s) = \{\alpha\beta | \alpha \neq \beta, \alpha, \beta \in V(K_e^s)\}.$$

The e-complete book graph with d sheets denoted by $B_d(K_e)$ is obtained from complete graphs K_e^s, $s = 1, 2, ..., d$ by merging the edge uv from the complete graphs. Therefore, the set of vertex and the set of edge set of $B_d(K_e)$, respectively, are:

$$V(B_d(K_e)) = \bigcup_{s=1}^{d} V(K_e^s), \text{ and } E(B_d(K_e)) = \bigcup_{s=1}^{d} E(K_e^s).$$

From the Definition 3.1, we have $|E(B_d(K_e))| = \frac{(e^2-e-2)d+2}{2}$.

Example 3.2. Figure 3 shows the complete book graph $B_2(K_8)$.

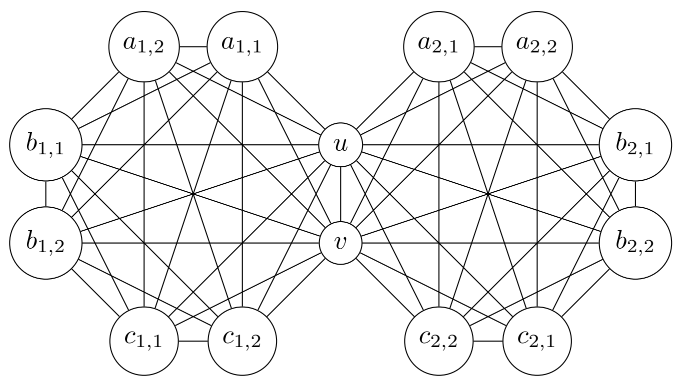

Figure 3: Complete Book Graph $B_2(K_8)$

The following lemmas are needed to prove the total edge irregularity strength of complete book graphs, which is technically motivated by that in [8].

Lemma 3.3. *Let $B_d(K_e)$ be the e-complete book graph with $e \equiv 0(mod\ 3)$ and d sheets. Then $tes(B_d(K_e)) = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.*

Proof. The e-complete book graph $B_d(K_e)$ with $e \equiv 0(mod\ 3)$ has maximum degree $\Delta(B_d(K_e)) = d(e-2)+1$. Since $|E(B_d(K_e))| = \frac{(e^2-e-2)d+2}{2}$ and

$tes(G) \geq max\left\{\left\lceil\frac{|E|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$, we have $tes(B_d(K_e)) \geq \left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$. For the upper bound, we prove by constructing an edge irregular total k-labeling for $(B_d(K_e))$ with $k = \left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$.

We define a function $f' : V(B_d(K_e)) \cup E(B_d(K_e)) \rightarrow \{1, 2, ...k\}$, with $k = \left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$. The 3-complete book graph $B_d(K_3)$ is indeed similar to $B_d(C_3)$, the tes of $B_d(C_3)$ can be found in [13].

Furthermore, we define a vertex and edge labeling for $B_d(K_e)$, $e \geq 6$, as below:

Let $e = 3l$. We have $l = \frac{e}{3}$. We define a vertex labeling f' in the following way:

$$\begin{aligned}
&f'(u) = 1, \quad f'(v) = k \\
&f'(a_{s,p}) = s, && 1 \leq s \leq d, \quad 1 \leq p \leq x-1 \\
&f'(b_{s,q}) = d\left(\binom{l}{2} + 1 + l.q - l\right) - d + s && 1 \leq s \leq d, \quad 1 \leq q \leq y \\
&f'(c_{s,p}) = k, && 1 \leq s \leq d, \quad 1 \leq p \leq x-1.
\end{aligned}$$

with $x = \lfloor\frac{e+1}{3}\rfloor$ and $y = e - 2\lfloor\frac{e+1}{3}\rfloor$.

We define the edge labeling f' in the following way:

$$\begin{aligned}
f(ua_{s,p}) &= (p-1)d+1, \quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right) \\
f'(a_{s,p}a_{s,q}) &= \left(p\left(\tfrac{e}{3}-1\right) + q - \left(\tfrac{p(p+1)}{2}+1\right)\right)d + 2 - s, \\
&\quad 1 \leq p \leq \left(\tfrac{e-6}{3}\right), \quad 1 \leq s \leq d, \quad e \geq 9, \quad p+1 \leq q \leq \left(\tfrac{e-3}{3}\right) \\
f'(u, b_{s,q}) &= 1, \quad 1 \leq s \leq d, \quad 1 \leq q \leq \tfrac{e}{3} \\
f'(a_{s,p}b_{s,q}) &= pd + 2 - s, \quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right), \quad 1 \leq q \leq \tfrac{e}{3} \\
f'(uc_{s,1}) &= \left(\tfrac{d-2}{3}\right) + s, \quad 1 \leq s \leq d \\
f'(uc_{s,q}) &= (q-1)d + s, \quad 1 \leq s \leq d, \quad 2 \leq q \leq \tfrac{e-3}{3} \\
f'(uv) &= \left(\tfrac{e}{3}-1\right)d + \left\lceil\tfrac{d+1}{3}\right\rceil, \\
f'(a_{s,p}c_{s,q}) &= \left(p\left(\tfrac{e}{3}\right) + q - 2\right)d + \left\lceil\tfrac{d+4}{3}\right\rceil \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right), \quad 1 \leq q \leq \left(\tfrac{e-3}{3}\right) \\
f'(a_{s,p}v) &= \left((p+1)\left(\tfrac{e}{3}\right) - 2\right)d + \left\lceil\tfrac{d+4}{3}\right\rceil, \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right) \\
f'(b_{s,p}b_{s,q}) &= \left(4 + \sum_{t=1}^{\frac{e}{3}-2}(7+3(t-1)) + (2-q)\left(\tfrac{e}{3}-1\right) - \left(\tfrac{p(p+1)}{2}-1\right)\right)d \\
&\quad +3 - s, \quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right), \quad p+1 \leq q \leq \left(\tfrac{e}{3}\right) \\
f'(b_{s,p}c_{s,q}) &= \left(\left(q + \tfrac{e-3}{3}\right)\left(\tfrac{e}{3}\right) - \left(\tfrac{q(q+1)}{2}\right)\right)d + \left\lceil\tfrac{d+4}{3}\right\rceil \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right), \quad 1 \leq q \leq \left(\tfrac{e-3}{3}\right) \\
f'(b_{s,p}v) &= \left(\left(p + \tfrac{e}{3}\right)\left(\tfrac{e}{3}\right) - \left(\tfrac{p(p+1)}{2}+2\right)\right)d + \left(\tfrac{e}{3}-p\right)\left\lceil\tfrac{d+4}{3}\right\rceil, \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right) \\
f'(c_{s,p}c_{s,q}) &= \left(\left(p + \tfrac{e-3}{3}\right)\left(\tfrac{e}{3}\right) + q - \left(\tfrac{p(p+1)}{2}+2\right)\right)n + 2\left\lceil\tfrac{d+1}{3}\right\rceil - 1 + s, \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-6}{3}\right), \quad e \geq 9, \quad p+1 \leq q \leq \left(\tfrac{e-3}{3}\right) \\
f'(c_{s,p}v) &= \left(\left(p + \tfrac{e}{3}\right)\left(\tfrac{e}{3}\right) - \left(\tfrac{p(p+1)}{2}+2\right)\right)d + 2\left\lceil\tfrac{d+1}{3}\right\rceil - 1 + i \\
&\quad 1 \leq s \leq d, \quad 1 \leq p \leq \left(\tfrac{e-3}{3}\right).
\end{aligned}$$

Considering the Definition 3.1. then we are grouping the vertices and the edges of graph $B_d(K_e)$ as below:

$A_s = \{u, a_{s,1}, a_{s,2}, ..., a_{s,x-1}\}$, $B_s = \{b_{s,1}, b_{s,2}, ..., b_{s,y}\}$, $C_s = \{c_{s,1}, c_{s,2}, ..., c_{s,x-1}, v\}$.

It follows that $|A_s| = |C_s| = x = \left\lfloor \frac{e+1}{3} \right\rfloor$ and $|B_s| = y = e - 2x$, with $s = 1, 2, ..., d$.

Let $(X_s, Y_s) = \{uv | u \in X_s, v \in Y_s\}$, $X_s \in \{A_s, B_s, C_s\}$, $Y_s \in \{A_s, B_s, C_s\}$, $s = 1, ..., d$. Under the labeling f', we obtain the weights of the edges of $(A_s, A_s), (A_s, B_s)$,
$(A_s, C_s), (B_s, B_s), (B_s, C_s)$, (C_s, C_s) as below:

(i) $e = 0(mod\ 3)$ **and** $d = 0(mod\ 3)$**,**

1. $\bigcup_{i=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, A_s)\} = \left[3, d\binom{x}{2} + 2\right]$
2. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, B_s)\} = \left[d\binom{x}{2} + 3, d\binom{x}{2} + 2 + xyd\right]$
3. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, C_s)\} = \left[d\binom{x}{2} + 3 + xyd, d\binom{x}{2} + 1 + xyd + x^2 d\right]$
4. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (B_s, B_s)\} =$
$\left[d\binom{x}{2} + 2 + xyd + x^2 d, 3k - d\binom{x}{2} - yxd - d\binom{y}{2}\right]$
5. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (B_s, C_s)\} = \left[3k + 1 - d\binom{x}{2} - yxd, 3k - d\binom{x}{2}\right]$
6. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (C_s, C_s)\} = \left[3k - d\binom{x}{2} + 1, 3k\right]$

(ii) $e = 0(mod\ 3)$ **and** $d = 1(mod\ 3)$**,**

1. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, A_s)\} = \left[3, d\binom{x}{2} + 1\right]$
2. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, B_s)\} = \left[d\binom{x}{2} + 2, d\binom{x}{2} + 1 + xyd\right]$
3. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (A_s, C_s)\} = \left[d\binom{x}{2} + 2 + xyd, d\binom{x}{2} + xyd + x^2 d\right]$
4. $\bigcup_{s=1}^{d} \{\omega_{f'}(uv) | uv \in (B_s, B_s)\} =$
$\left[d\binom{x}{2} + 1 + xyd + x^2 d, 3k - d\binom{x}{2} - yxd - d\binom{y}{2} - 1\right]$

5. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (B_s, C_s)\} = \left[3k - d\binom{x}{2} - yxd, 3k - d\binom{x}{2} - 1\right]$

6. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (C_s, C_s)\} = \left[3k - d\binom{x}{2}, 3k - 1\right]$

(iii) $e = 0(mod\ 3)$ **and** $d = 2(mod\ 3)$**,**

1. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (A_s, A_s)\} = \left[3, d\binom{x}{2}\right]$

2. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (A_s, B_s)\} = \left[d\binom{x}{2} + 1, d\binom{x}{2} + xyd\right]$

3. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (A_s, C_s)\} = \left[d\binom{x}{2} + 1 + xyd, d\binom{x}{2} + xyd + x^2d - 1\right]$

4. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (B_s, B_s)\} =$
$\left[d\binom{x}{2} + xyd + x^2d, 3k - d\binom{x}{2} - xyd - d\binom{y}{2} - 2\right]$

5. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (B_s, C_s)\} = \left[3k - d\binom{x}{2} - xyd - 1, 3k - d\binom{x}{2} - 2\right]$

6. $\bigcup_{s=1}^{d}\{\omega_{f'}(uv)|uv \in (C_s, C_s)\} = \left[3k - d\binom{x}{2} - 1, 3k - 2\right]$

where $[a, b]$ denotes a discrete interval from a to b.

The weight of the edges of $B_d(K_e)$ for $e = 0(mod\ 3)$ under the labeling f' are distinct and form consecutive integer from 3 to $3k$, for $d \equiv 0(mod\ 3)$, from 3 to $3k - 1$, for $d \equiv 1(mod\ 3)$, and from 3 to $3k - 2$, for $d \equiv 2(mod\ 3)$, where $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$. ∎

Lemma 3.4. *Let $B_d(K_e)$ be the e-complete book graph with $e \equiv 1(mod\ 3)$ and d sheets. Then $tes(B_d(K_e)) = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.*

Proof. The e-complete book graph $B_d(K_e)$ for $e \equiv 1(mod\ 3)$ has maximum degree $\Delta(B_d(K_e)) = d(e-2) + 1$, we have $tes(B_d(K_e)) \geq \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$. For the upper bound, we prove by constructing an edge irregular total k-labeling for $(B_d(C_b))$ with $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$. We define a function $g' : V(B_d(K_e)) \cup E(B_d(K_e)) \to \{1, 2, ...k\}$, with $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.

For $B_d(K_4)$, the vertex and edge labeling g' is defined as below:

$g'(u) = 1, \quad g'(v) = k$

$$\begin{array}{ll}
g'(b_{s,1}) = s, & 1 \le s \le d \\
g'(b_{s,2}) = d+s, & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
g'(b_{s,2}) = k, & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d \\
g'(ub_{s,1}) = 1 & 1 \le s \le d \\
g'(ub_{s,2}) = 1 & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
g'(ub_{s,2}) = 2 + s - (d - \left\lfloor \frac{d-6}{3} \right\rfloor) & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d \\
g'(b_{s,1}b_{s,2}) = d + 2 - s & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
g'(b_{s,1}b_{s,2}) = 1 + \left(\left\lceil \frac{d}{3} \right\rceil\right) & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d \\
g'(uv) = d + \left\lfloor \frac{d+3}{3} \right\rfloor & \\
g'(b_{s,1}v) = d + \left\lfloor \frac{d+6}{3} \right\rfloor & 1 \le s \le d \\
g'(b_{s,2}v) = d + \left\lfloor \frac{d+6}{3} \right\rfloor & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
g'(b_{s,2}v) = d + \left\lfloor \frac{d+6}{3} \right\rfloor + s - (d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1) & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d.
\end{array}$$

with $x = \lfloor \frac{e+1}{3} \rfloor$ and $y = e - 2\lfloor \frac{e+1}{3} \rfloor$.

From this labeling, we get the edge weights as below:

$$\begin{array}{ll}
\omega_{g'}(ub_{s,1}) = 2 + s, & 1 \le s \le d \\
\omega_{g'}(ub_{s,2}) = d + 2 + s, & 1 \le s \le d \\
\omega_{g'}(ub_{s,2}) = k + 2 + s - (d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1) & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d \\
\omega_{g'}(b_{s,1}b_{s,2}) = 2d + 2 + s, & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
\omega_{g'}(b_{s,1}b_{s,2}) = \left(\left\lceil \frac{d}{3} \right\rceil\right) + k + 1 + s, & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d \\
\omega_{g'}(uv) = \left\lfloor \frac{d+3}{3} \right\rfloor + d + k + 1, & \\
\omega_{g'}(b_{s,1}v) = \left\lfloor \frac{d+6}{3} \right\rfloor + d + k + s, & 1 \le s \le d \\
\omega_{g'}(b_{s,2}v) = \left\lfloor \frac{d+6}{3} \right\rfloor + 2d + k + s, & 1 \le s \le d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1 \\
\omega_{g'}(b_{s,2}v) = \left\lfloor \frac{d-6}{3} \right\rfloor + d + 2k + s - (d - \left\lfloor \frac{d-6}{3} \right\rfloor - 1) & d - \left\lfloor \frac{d-6}{3} \right\rfloor \le s \le d.
\end{array}$$

For $e \ge 7$, we define a vertex and edge labeling g' in the following way:

Let $e = 3l + 1$. We have $l = \frac{e-1}{3}$. Then we define a vertex labeling g' as below:

$$\begin{array}{ll}
g'(u) = 1, \quad g'(v) = k & \\
g'(a_{s,p}) = s, & 1 \le s \le d, \quad 1 \le p \le x - 1 \\
g'(b_{s,q}) = d\left(\binom{l}{2} + 1 + l.q - l\right) - d + s & 1 \le s \le d, \quad 1 \le q \le y \\
g'(c_{s,p}) = k, & 1 \le s \le d, \quad 1 \le p \le x - 1.
\end{array}$$

with $x = \lfloor \frac{e+1}{3} \rfloor$ and $y = e - 2\lfloor \frac{e+1}{3} \rfloor$.

We define the edge labeling g' in the following way:

$$\begin{array}{lcl}
g'(ua_{s,p}) & = & (p-1)d + 1, \quad 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-4}{3}\right) \\
g'(u, b_{s,q}) & = & 1, \quad 1 \le s \le d, \quad 1 \le q \le \frac{e-1}{3} \\
g'(a_{s,p}a_{s,q}) & = & \left(p\left(\frac{e-1}{3} - 1\right) + q - \left(\frac{p(p+1)}{2} + 1\right)\right) d + 2 - s \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-7}{3}\right), \quad e \ge 10, \quad p + 1 \le q \le \left(\frac{e-4}{3}\right) \\
g'(a_{s,p}b_{s,q}) & = & pd + 2 - s,
\end{array}$$

$$
\begin{array}{lcl}
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-4}{3}\right),\ \ 1 \le q \le \frac{e-1}{3} \\
g'(uc_{s,1}) & = & \left(\frac{d-2}{3}\right) + s,\ \ 1 \le s \le d \\
g'(uc_{s,q}) & = & (q-1)d + s,\ \ 1 \le s \le d,\ \ 2 \le q \le \frac{e-1}{3} \\
g'(uv) & = & \left(\frac{e}{3} - 1\right) d + \left\lceil \frac{d+1}{3} \right\rceil, \\
g'(a_{s,p}c_{s,q}) & = & \left(p\left(\frac{e-1}{3}\right) + q - 2\right) d + \left\lceil \frac{d+4}{3} \right\rceil \\
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-4}{3}\right),\ \ 1 \le q \le \left(\frac{e-4}{3}\right) \\
g'(a_{s,p}v) & = & \left((p+1)\left(\frac{e-1}{3}\right) - 2\right) d + \left\lceil \frac{d+4}{3} \right\rceil, 1 \le s \le d, 1 \le p \le \left(\frac{e-1}{3} - 1\right) \\
g'(b_{s,p}b_{s,q}) & = & \left(6 + \sum_{t=1}^{\frac{e-7}{3}} (8 + 3(t-1)) + (2-q)\left(\frac{e-4}{3}\right) - \left(\frac{p(p-1)}{2}\right)\right) d + 3 - s, \\
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-4}{3}\right),\ \ p+1 \le q \le \left(\frac{e-1}{3}\right) \\
g'(b_{s,p}c_{s,q}) & = & \left(\left(q + \frac{e-1}{3}\right)\left(\frac{e-1}{3}\right) - \left(\frac{q(q+1)}{2}\right)\right) d + \left\lceil \frac{d+4}{3} \right\rceil \\
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-4}{3}\right),\ \ 1 \le q \le \left(\frac{e-1}{3}\right) \\
g'(b_{s,p}v) & = & \left(\left(p + \frac{e-1}{3}\right)\left(\frac{e-1}{3}\right) - \left(\frac{p(p+1)}{2} + 4\right)\right) d + \left(\frac{e-1}{3} - p\right) \left\lceil \frac{d+4}{3} \right\rceil, \\
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-4}{3}\right) \\
g'(c_{s,p}c_{s,q}) & = & \left(\left(p + \frac{e-1}{3} - 1\right)\left(\frac{e-1}{3}\right) + q - \left(\frac{p(p+1)}{2} + 3\right)\right) d + 2\left\lceil \frac{d+1}{3} \right\rceil - 1 + s \\
 & & 1 \le s \le d,\ \ 1 \le p \le \left(\frac{e-7}{3}\right),\ \ e \ge 10,\ \ p+1 \le q \le \left(\frac{e-4}{3}\right) \\
g'(c_{s,p}v) & = & \left(\left(p + \frac{e-1}{3}\right)\left(\frac{e-1}{3}\right) - \left(\frac{p(p+1)}{2} + 3\right)\right) d + 2\left\lceil \frac{d+1}{3} \right\rceil - 1 + s, \\
 & & 1 \le p \le \left(\frac{e-4}{3}\right),\ \ 1 \le s \le d.
\end{array}
$$

Similar to Lemma 3.3, we obtain the weights of the edges of $(A_s, A_s), (A_s, B_s), (A_s, C_s), (B_s, B_s), (B_s, C_s), (C_s, C_s)$ under the labeling g' for cases:

(i) $e = 1(mod\ 3)$ and $d = 0(mod\ 3)$ similar to $e = 0(mod\ 3)$ and $d = 0(mod\ 3)$.
(ii) $e = 1(mod\ 3)$ and $d = 1(mod\ 3)$ similar to $e = 0(mod\ 3)$ and $d = 1(mod\ 3)$.
(iii) $e = 1(mod\ 3)$ and $d = 2(mod\ 3)$ similar to $e = 0(mod\ 3)$ and $d = 2(mod\ 3)$.

The weight of the edges of $B_d(K_e)$ for $e = 1(mod\ 3)$ under the labeling g' are distinct and form consecutive integer from 3 to $3k$, for $d \equiv 0(mod\ 3)$, from 3 to $3k - 1$, for $d \equiv 1(mod\ 3)$, and from 3 to $3k - 2$, for $d \equiv 2(mod\ 3)$, where $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$. ■

Lemma 3.5. *Let $B_d(K_e)$ be the e-complete book graph with $e \equiv 2(mod\ 3)$ and d sheets. Then $tes(B_d(K_e)) = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.*

Proof. The e-complete book graph $B_d(K_e)$ for $e \equiv 2(mod\ 3)$ also has maximum degree $\Delta(B_d(K_e)) = d(e-2) + 1$, we have $tes(B_d(K_e)) \ge \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$. For the upper bound, we prove by constructing an edge irregular total k-labeling for $B_d(K_e)$ with $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.

We define a function $h' : V(B_d(K_e)) \cup E(B_d(K_e)) \to \{1, 2, ...k\}$, with $k = \left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.

For $B_d(K_5)$, the vertex and edge labeling h' is defined as below:

$$\begin{array}{ll}
h'(u)=1, \quad h'(v)=k & \\
h'(a_{s,1})=s, & 1\le s\le d\\
h'(b_{1,1})=d+1, & \\
h'(b_{s,1})=d+s+1, & 2\le s\le d\\
h'(c_{s,1})=k, & 1\le s\le d\\
h'(ua_{s,1})=1 & 1\le s\le d\\
h'(ub_{s,1})=1 & 1\le s\le d\\
h'(a_{1,1}b_{1,1})=2 & \\
h'(a_{s,1}b_{s,1})=d-s+1 & 2\le s\le d\\
h'(uc_{s,1})=s & 1\le s\le d\\
h'(uv)=d+1 & \\
h'(a_{s,1}c_{s,1})=d+2 & 1\le s\le d\\
h'(a_{s,1}v)=2d+2 & 1\le s\le d\\
h'(b_{1,1}v)=2d+3 & \\
h'(b_{s,1}c_{s,1})=2d+2 & 1\le s\le d\\
h'(b_{s,1}v)=3d+1 & 2\le s\le d\\
h'(c_{s,1}v)=2d+1+s & 1\le s\le d.
\end{array}$$

From this labeling, we get the edge weights below:

$$\begin{array}{ll}
\omega_{h'}(ua_{s,1})=2+s, & 1\le s\le d\\
\omega_{h'}(ub_{1,1})=d+3, & \\
\omega_{h'}(a_{1,1}b_{1,1})=d+4, & \\
\omega_{h'}(ub_{s,1})=d+3+s, & 2\le s\le d\\
\omega_{h'}(a_{s,1}b_{s,1})=2d+2+s, & 2\le s\le d\\
\omega_{h'}(uc_{s,1})=3d+2+s, & 1\le s\le d\\
\omega_{h'}(uv)=4d+3, & \\
\omega_{h'}(a_{s,1}c_{s,1})=4d+3+s, & 1\le s\le d\\
\omega_{h'}(a_{s,1}v)=5d+3+s, & 1\le s\le d\\
\omega_{h'}(b_{1,1}c_{1,1})=6d+4, & \\
\omega_{h'}(b_{1,1}v)=6d+5, & \\
\omega_{h'}(b_{s,1}c_{s,1})=6d+4+s, & 2\le s\le d\\
\omega_{h'}(b_{s,1}v)=7d+3+s, & 2\le s\le d\\
\omega_{h'}(c_{s,1}v)=8d+3+s, & 1\le s\le d
\end{array}$$

For $e\ge 8$, we define a vertex and edge labeling as below:
Let $e=3l-1$. We have $l=\frac{e+1}{3}$. We define a vertex labeling h' as follow:

$$\begin{array}{ll}
h'(u)=1, & \\
h'(v)=k & \\
h'(a_{s,p})=s, & 1\le s\le d,\quad 1\le p\le x-1\\
h'(b_{s,q})=d\left(\binom{l}{2}+1+l.q-l\right)-d+s & 1\le s\le d,\quad 1\le q\le y
\end{array}$$

$h'(c_{s,p}) = k, \quad 1 \le s \le d, \quad 1 \le p \le x-1.$
with $x = \lfloor \frac{e+1}{3} \rfloor$ and $y = e - 2\lfloor \frac{e+1}{3} \rfloor$.

We define the edge labeling h' in the following way:

$$
\begin{array}{lcl}
h'(ua_{s,p}) & = & (p-1)d+1, \quad 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right) \\
h'(a_{s,p}a_{s,q}) & = & \left(p\left(\frac{e+1}{3}-1\right)+q-\left(\frac{p(p+1)}{2}+1\right)\right)d+2-s \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-5}{3}\right), \quad p+1 \le q \le \left(\frac{e-2}{3}\right) \\
h'(u, b_{s,q}) & = & 1, \quad 1 \le s \le d, \quad 1 \le q \le \frac{e-2}{3} \\
h'(a_{s,p}b_{s,q}) & = & pd+2-s, \quad 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right), \quad 1 \le q \le \frac{e-2}{3} \\
h'(uc_{s,1}) & = & \left(\frac{d-2}{3}\right)+s, \quad 1 \le s \le d \\
h'(uc_{s,q}) & = & (q-1)d+s, \quad 1 \le s \le d, \quad 2 \le q \le \frac{e-2}{3} \\
h'(uv) & = & \left(\frac{e-2}{3}\right)d+\left\lceil\frac{d+1}{3}\right\rceil, \\
h'(a_{s,p}c_{s,q}) & = & \left(p\left(\frac{e+1}{3}\right)+q-2\right)d+\left\lceil\frac{d+4}{3}\right\rceil \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right), \quad 1 \le q \le \left(\frac{e-2}{3}\right) \\
h'(a_{s,p}v) & = & \left((p+1)\left(\frac{e+1}{3}\right)-2\right)d+\left\lceil\frac{d+4}{3}\right\rceil, \quad 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right) \\
h'(b_{s,p}b_{s,q}) & = & \left(5+\sum_{t=1}^{\frac{e-5}{3}}(6+3(t-1))+(2-q)\left(\frac{e-2}{3}\right)-\left(\frac{p(p+3)}{2}+1\right)\right)d+3-s, \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-5}{3}\right), \quad p+1 \le q \le \left(\frac{e-2}{3}\right) \\
h'(b_{s,p}c_{s,q}) & = & \left(\left(q+\frac{e-2}{3}\right)\left(\frac{e+1}{3}\right)-\left(\frac{q(q+1)}{2}+2\right)\right)d+\left\lceil\frac{c+4}{3}\right\rceil \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right), \quad 1 \le q \le \left(\frac{e-2}{3}\right) \\
h'(b_{s,p}v) & = & \left(\left(p+\frac{e+1}{3}\right)\left(\frac{e+1}{3}\right)-\left(\frac{p(p+1)}{2}+4\right)\right)d+\left(\frac{e+1}{3}-p\right)\left\lceil\frac{d+4}{3}\right\rceil, \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right) \\
h'(c_{s,p}c_{s,q}) & = & \left(\left(p+\frac{e+1}{3}-1\right)\left(\frac{e+1}{3}\right)+q-\left(\frac{p(p+1)}{2}+4\right)\right)d+2\left\lceil\frac{d+1}{3}\right\rceil-1+s, \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-5}{3}\right), \quad p+1 \le q \le \left(\frac{e-2}{3}\right) \\
h'(c_{s,p}v) & = & \left(\left(p+\frac{e+1}{3}\right)\left(\frac{e+1}{3}\right)-\left(\frac{p(p+1)}{2}+4\right)\right)d+2\left\lceil\frac{d+1}{3}\right\rceil-1+s \\
 & & 1 \le s \le d, \quad 1 \le p \le \left(\frac{e-2}{3}\right).
\end{array}
$$

We obtain the weights of the edges of $(A_s, A_s), (A_s, B_s), (A_s, C_s)$, $(B_s, B_s), (B_s, C_s)$, (C_s, C_s) under the labeling h' as the following:

1. $\bigcup_{s=1}^{d} \{\omega_{h'}(uv) | uv \in (A_s, A_s)\} = \left[3, d\binom{x}{2}+2\right]$
2. $\bigcup_{s=1}^{d} \{\omega_{h'}(uv) | uv \in (A_s, B_s)\} = \left[d\binom{x}{2}+3, d\binom{x}{2}+2+xyd\right]$
3. $\bigcup_{s=1}^{d} \{\omega_{h'}(uv) | uv \in (A_s, C_s)\} = \left[d\binom{x}{2}+3+xyd, d\binom{x}{2}+1+xyd+x^2d\right]$
4. $\bigcup_{s=1}^{d} \{\omega_{h'}(uv) | uv \in (B_s, B_s)\} =$
   $\left[d\binom{x}{2}+2+xyd+x^2d, 3k-d\binom{x}{2}-yxd-d\binom{y}{2}\right]$

5. $\bigcup_{s=1}^{d}\{\omega_{h'}(uv)|uv\in(B_s,C_s)\}=\left[3k+1-d\binom{x}{2}-yxd,3k-d\binom{x}{2}\right]$

6. $\bigcup_{s=1}^{d}\{\omega_{h'}(uv)|uv\in(C_s,C_s)\}=\left[3k-d\binom{x}{2}+1,3k\right]$

where $[a,b]$ denotes a discrete interval from a to b.

The weight of the edges of $B_d(K_e)$ for $e=2(mod\ 3)$ and arbitrary d under the labeling h' are distinct and form consecutive integer from 3 to $3k$ where $k=\left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$. ∎

From Lemmas 3.3, 3.4 and 3.5 can be concluded that $tes(B_d(K_e))$ for arbitrary e and d is $\left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$. We obtain the following theorem.

Theorem 3.6. *Let $B_d(K_e)$ be a wheel book graph of d sheets. It follows that $tes(B_d(K_e))=\left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$.*

In the following example, we show that there is an edge irregular total k-labeling for $B_2(K_8)$ with $k=\left\lceil\frac{(e^2-e-2)d+6}{6}\right\rceil$ as constructed in Lemma 3.5.

Example 3.7. Let $A_1=\{u,a_{1,1},a_{1,2}\}$, $A_2=\{u,a_{2,1},a_{2,2}\}$, $B_1=\{b_{1,1},b_{1,2}\}$, $B_2=\{b_{2,1},b_{2,2}\}$, $C_1=\{c_{1,1},c_{1,2},v\}$, $C_2=\{c_{2,1},c_{2,2},v\}$. We have $x=3$ and $y=2$.

Based on the construction in Lemma 3.5 for graph $B_2(K_8)$ we have the vertex and edge labeling h' as below:

$h'(u)=1$, $h'(v)=k=19$, $h'(a_{1,1})=1$,
$h'(a_{1,2})=1$, $h'(a_{2,1})=2$, $h'(a_{2,2})=2$,
$h'(b_{1,1})=7$, $h'(b_{1,2})=13$, $h'(b_{2,1})=8$,
$h'(b_{2,2})=14$, $h'(c_{1,1})=19$, $h'(c_{1,2})=19$,
$h'(c_{2,1})=19$, $h'(c_{2,2})=19$,
$h'(ua_{s,1})=1$, $h'(ua_{s,2})=3$, $h'(a_{s,1}a_{s,2})=6-s$,
$h'(ub_{s,q})=1$, $h'(a_{s,1}b_{s,q})=4-s$, $h'(a_{s,2}b_{s,q})=6-s$,
$h'(uc_{s,1})=s$, $h'(uc_{s,2})=2+s$, $h'(uv)=5$,
$h'(a_{s,1}c_{s,1})=6$, $h'(a_{s,1}c_{s,2})=8$, $h'(a_{s,1}v)=10$,
$h'(a_{s,2}c_{s,1})=12$, $h'(a_{s,2}c_{s,2})=14$, $h'(a_{s,2}v)=16$,
$h'(b_{s,1}b_{s,2})=19-s$, $h'(b_{s,q}c_{s,1})=14$, $h'(b_{s,q}c_{s,2})=16$,
$h'(b_{s,q}v)=18$, $h'(c_{s,1}c_{s,2})=13+s$, $h'(c_{s,1}v)=15+s$,
$h'(c_{s,2}v)=17+s$,
with $s=1,2$, $q=1,2$.

The weights of edges in $B_2(K_8)$ under the labeling h' are as below:

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (A_s, A_s)\} = \{3, 4, ..., 8\}$$

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (A_s, B_s)\} = \{9, 10, ..., 20\}$$

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (A_s, C_s)\} = \{21, 22, ..., 37\}$$

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (B_s, B_s)\} = \{38, 39\}$$

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (B_s, C_s)\} = \{40, ..., 51\}$$

$$\bigcup_{s=1}^{2}\{\omega_{h'}(uv)|uv \in (C_s, C_s)\} = \{52, ..., 57\}.$$

Under the function h', the weights of the edges in graph $B_2(K_8)$ are distinct and form consecutive integers from 3 to 57, we obtain $tes(B_2(K_8)) = 19$.

4. Conclusion

In this paper we studied the total edge irregularity strength of any wheel book graph $B_d(W_e)$ and complete book graph $B_d(K_e)$. We found that the exact value of the total edge irregularity strength of wheel book graphs $tes(B_d(W_e))$ is equal to $\left\lceil \frac{(2e-1)d+3}{3} \right\rceil$. We found also that the edge irregularity strength of any complete book graph $tes(B_d(K_e))$ is equal to $\left\lceil \frac{(e^2-e-2)d+6}{6} \right\rceil$.

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