Southeast Asian Bulletin of Mathematics (2023) 47: 563–571

Southeast Asian
Bulletin of
Mathematics

Probability of Cyclicity of Chains in Finite Groups

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Received 16 May 2020
Accepted 30 December 2020

Communicated by Wenbin Guo

AMS Mathematics Subject Classification(2020): 20D60, 20P05, 20F05, 20E15

Abstract. We introduce the probability of cyclicity of chains in finite groups and among other results we give explicit formulas for such probability of dihedral groups D_{2n}, quasi-dihedral groups QD_{2^n}, generalized quaternion groups Q_{2^n}, and modular p-groups M_{p^n}.

Keywords: Probability of cyclicity; Quasi-dihedral group; Generalized quaternion group; Modular p-group.

1. Introduction

The concept of different probabilities in finite groups have been studied by many authors, in recent years. One of the usual ones, is the probability of commutativity of two randomly chosen elements of a finite group (see [4, 3, 5]). Also the probablistic notion on the subgroup lattice, $L(G)$, of a finite group G has been investigated in [8]. For instance, assume $N(G)$ and $C(G)$ denote the normal subgroup lattice and the poset of cyclic subgroups of G. Then one can study the probrbilities of choosing an arbiterary subgroup of G, whether it is normal or cyclic and these probabilities are defined as the ratios of $\frac{|N(G)|}{|L(G)|}$ or $\frac{|C(G)|}{|L(G)|}$, respectively (for more information see [8] and the references in).

Now, in the present article we study the probability of a random chosen chain of subgroups of G, whether it will be cyclic chain. More precisely, let

$$H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_n = G$$

be a chain of subgroups of a given finite group G, which starts from any subgroup H_1 (including the identity) and ending in G. The set of all such chains of the group G is denoted by $Ch(G)$.

A chain of subgroups of G that ends in the group G is called *cyclic*, whenever all of its components are cyclic except possibly for $H_n = G$ (see also [2] for counting the number of cyclic subgroups of a group G). We denote the set of all cyclic chains of G by $C^*(G)$, and introduce the *probability of cyclicity (or cyclicity degree) of chains of* G, denoted by $pcc(G)$, as follows:

$$pcc(G) = \frac{|C^*(G)|}{|Ch(G)|}.$$

Throughout this article, we consider all groups to be finite and by a *chain* we mean a chain of subgroups of a group G, which starts from any subgroup and ends in G. The properties and counting the number of chains in a given group has been investigated by many authors in latter years. For example, the number of chains of finite cyclic groups and finite elementary abelian p-groups are obtained in [8]. Also, the number of chains of nonabelian groups D_{2n}, Q_{4n}, QD_{2^n}, and M_{p^n} are computed in [1]. Our aim is to compute the probability of cyclicity of chains of the groups D_{2n}, Q_{2^n}, QD_{2^n}, and M_{p^n}.

2. Preliminaries

Clearly the probability of cyclicity of chains of any finite group G satisfies $0 < pcc(G) \leq 1$, and $pcc(G) = 1$ if and only if G is cyclic.

Clearly the group G itself is a chain. So as in [1] using the subgroup lattice of a group G, we define the number of chains start from any non-trivial subgroup H of G and end in G in the following way

$$n(H) = \sum_{i=1}^{r} n(H_i),$$

in which H_i's are the subgroups of G containing H, properly. Clearly $n(G)$=1 and if $H = \langle e \rangle$ is the trivial subgroup, then the number of all chains of subgroups of G is $|Ch(G)| = 2n(\langle e \rangle)$.

The following example shows the subgroup lattice and the diagram of cyclic subgroups of dihedral group D_{18}.

Example 2.1. The lattice of subgroups and the poset of cyclic subgroups of dihedral group D_{18} are given below. One can also obtain the probability of cyclicity of chains of D_{18}.

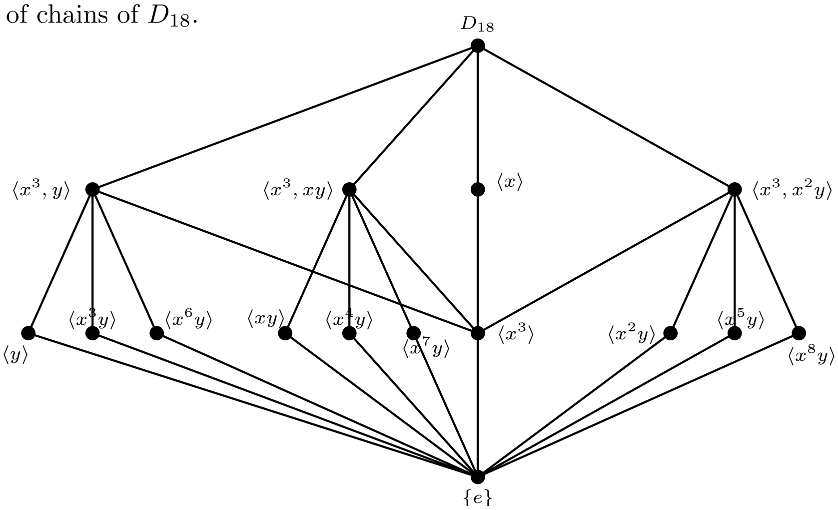

Fig. 1. Subgroup lattice of D_{18}.

Using the above method, the number of chains of subgroups of the group D_{18} can be obtained as follows:

$$\begin{aligned} n(D_{18}) &= 1, \\ n(\langle x\rangle) &= n(\langle x^3, y\rangle) = n(\langle x^3, xy\rangle) = n(\langle x^3, x^2y\rangle) = 1, \\ n(\langle x^3\rangle) &= 5, \\ n(\langle x^iy\rangle) &= 2 \quad (0 \le i \le 8), \\ n(\langle e\rangle) &= 1 + (1+1+1+1) + 5 + 9(2) = 28. \end{aligned}$$

Therefore $|Ch(D_{18})| = 2 \times 28 = 56$.

As above, the number of cyclic subgroup chains of D_{18} is as follows:

$$\begin{aligned} n(D_{18}) &= 1, \\ n(\langle x\rangle) &= 1, \\ n(\langle x^iy\rangle) &= 1 \quad (0 \le i \le 8), \\ n(\langle x^3\rangle) &= 2. \end{aligned}$$

Therefore $|C^*(D_{18})| = 2(1+1+9(1)+2) - 1 = 25$, and so $pcc(D_{18}) = \frac{25}{56}$.

The number of all subgroup chains of a group G with maximum length 2 that end in G is the same as $|L(G)|$. Similarly, the number of all cyclic chains of a group G with maximum length 2 is the same as the number of all cyclic subgroups of the group G. Hence the probability of cyclic chains with maximum

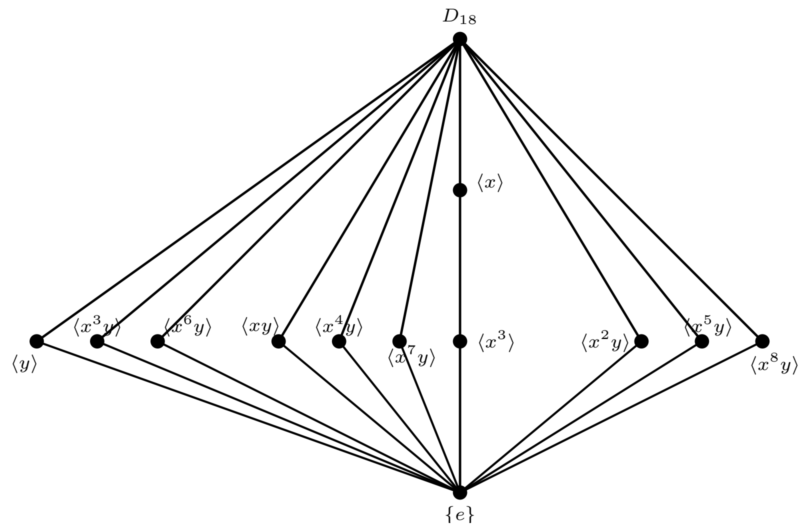

Fig. 2. Diagram of cyclic subgroups of D_{18}.

length 2 is equal to the probability cyclicity of chain of subgroups of G defined in [9] and

$$pcc(G) = |C(G)|/|L(G)|,$$

where $C(G)$ denotes the poset of all cyclic subgroups of G and $L(G)$ the subgroup lattice of the group G. We investigate finite p-groups having a cycle maximal subgroup.

Clearly, the following finite non-abelian groups are all have a cyclic maximal subgroup. Hence, we calculate the probability of cyclicity of their chains in final section.

$$\begin{aligned} D_{2n} &= \langle x, y : x^n = y^2 = 1, x^y = x^{-1}\rangle, \\ Q_{2^n} &= \langle x, y : x^{2^{n-1}} = y^4 = 1, x^y = x^{2^{n-1}-1}\rangle, \\ QD_{2^n} &= \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-2}-1}\rangle, \\ M_{p^n} &= \langle x, y : x^{p^{n-1}} = y^p = 1, x^y = x^{p^{n-2}+1}\rangle. \end{aligned}$$

The following theorem of [1] is needed in proving our main results.

Theorem 2.2. [1] *For any natural number n.*

(i) *The number of all subgroup chains of the dihedral group D_{2n} is*

$$|Ch(D_{2n})| = \sum_{k|n} \frac{n}{k} |Ch(\mathbb{Z}_{\frac{n}{k}})|(k + |Ch(\mathbb{Z}_k)|) - (2n-1)|Ch(\mathbb{Z}_n)| + n.$$

(ii) *The number of all subgroup chains of the generalized quaternion group Q_{4n} is*
$$|Ch(Q_{4n})| = |Ch(D_{2n})| + \sum_{d|m} |Ch(\mathbb{Z}_d)||Ch(D_{\frac{2n}{d}})|,$$
where m is an odd integer such that $n = 2^k m$, for some k.

(iii) *The number of all subgroup chains of the quasidihedral group $QD_{2^n}(n \geq 4)$ is*
$$|Ch(QD_{2^n})| = 3 \cdot 2^{2n-3}.$$

(iv) *The number of all subgroup chains of modular p-group M_{p^n} $(n \geq 3,\ p^n \neq 8)$ is*
$$2^{n-1}(n-1)p + 2^n.$$

Clearly, $pcc(G_1) = pcc(G_2)$ for any two isomorphic groups G_1 and G_2, but the converse is not true in general. For example, $pcc(\mathbb{Z}_6) = pcc(\mathbb{Z}_{15}) = 1$, while $\mathbb{Z}_6 \ncong \mathbb{Z}_{15}$.

Also, it is obvious that $pcc(G_1 \times G_2) = pcc(G_1)pcc(G_2)$ for any two groups G_1 and G_2 of coprime orders. For
$$pcc(G_1 \times G_2) = \frac{|C^*(G_1 \times G_2)|}{|Ch(G_1 \times G_2)|} = \frac{|C^*(G_1)||C^*(G_2)|}{|Ch(G_1)||Ch(G_2)|} = pcc(G_1)pcc(G_2).$$
However, the above equality does not hold in general, for instance
$$pcc(S_3 \times \mathbb{Z}_2) = \frac{23}{68} \neq \frac{9}{10} = pcc(S_3)pcc(\mathbb{Z}_2).$$

On the other hand, if the groups in the direct factors are pairwize coprime orders, then one can extend the above equality to arbitrary direct products of finite groups.

Proposition 2.3. *Assume $G_1, G_2 \ldots,$ and G_r are finite groups with pairwise coprime orders. Then $pcc(\Pi_{i=1}^k G_i) = \Pi_{i=1}^k pcc(G_i)$.*

The following corollary explains that the probability of cyclicity of chains of a finite nilpotent group can be obtained from its Sylow p-subgroups.

Corollary 2.4. *If G is a finite nilpotent group and $P_1, P_2, \ldots, P_r$ are its Sylow p_i-subgroups, then*
$$pcc(G) = \Pi_{i=1}^r pcc(P_i).$$

In general, if G and H are two lattices of isomorphic groups, then $pcc(G) = pcc(H)$, as lattice isomorphisms preserve cyclic subgroups (for more details see [6, Theorem 1.2.10]).

Clearly, for non-cyclic groups in which all their proper subgroups are cyclic, all chains are cyclic except the chain G. For such groups the probability of cyclicity of chain can be computed simply as follows:

Theorem 2.5. *Let G be a finite group. Then*

$$pcc(G) = \frac{|Ch(G)| - 1}{|Ch(G)|}$$

if and only if G is either a semidirect product of a normal subgroup of prime order p by a cyclic subgroup of prime power order q^n, an elementary abelian p-group of rank two, or the quaternion group Q_8.

Proof. According to [9, Theorem 2.1], such groups form the class of all noncyclic groups in which all proper subgroups are cyclic. Hence, the result follows. ■

3. Main Results

In this section, we obtain the probability of cyclicity of chains of the groups D_{2n}, Q_{2^n}, QD_{2^n}, and M_{p^n}.

Theorem 3.1. *The probability of cyclicity of chains of the dihedral group D_{2n} is*

$$pcc(D_{2n}) = \frac{\sum_{k|n} |Ch(\mathbb{Z}_k)| + 2n}{\sum_{k|n} \frac{n}{k}|Ch(\mathbb{Z}_{\frac{n}{k}})|(k + |Ch(\mathbb{Z}_k)|) - (2n-1)|Ch(\mathbb{Z}_n)| + n}.$$

Proof. One notes that the cyclic subgroups of D_{2n} are $\langle x^i y\rangle$ and $\langle x^k\rangle$, where $0 \le i \le n-1$ and $k|n$. Now, we count the number of chains of subgroups of D_{2n} in which all components are cyclic except D_{2n}. These chains are of the following form:

(1) $\cdots \subseteq \langle x^{\frac{n}{k}}\rangle \subseteq D_{2n}$,

(2) $\langle x^i y\rangle \subseteq D_{2n} \quad (0 \le i \le n-1)$,

(3) $1 \subseteq \langle x^i y\rangle \subseteq D_{2n} \quad (0 \le i \le n-1)$.

The above cases shows that, the number of cyclic chains of the group D_{2n} is equal to the sum of $\sum_{k|n} |Ch(\mathbb{Z}_k)|$, n and n, respectively. Therefore

$$|C^*(D_{2n})| = \sum_{k|n} |Ch(\mathbb{Z}_k)| + 2n.$$

Now, the proof is completed by using Theorem 2.2(i). ■

Corollary 3.2. *For a prime p, the probability of cyclicity of chains of dihedral group D_{2p^m} is as follows:*

$$pcc(D_{2p^m}) = \frac{2^{m+1} + 2p^m - 1}{\frac{2^m}{p-1}(p^{m+1} + p - 2)}.$$

In particular,

$$pcc(D_{2^m}) = \frac{2^{m+1}-1}{2^{2m-1}}.$$

Proof. By Theorem 3.1, the number of chains of subgroups of the group D_{2p^m} ending in G is equal to $|Ch(D_{2p^m})| = \frac{2^m}{p-1}(p^{m+1}+p-2)$. On the other hand, we have

$$\begin{aligned}|C^*(D_{2p^m})| &= \sum_{k|p^m} |Ch(\mathbb{Z}_k)| + 2p^m = \sum_{0\le i\le m} |Ch(\mathbb{Z}_{p^i})| + 2p^m \\ &= 1+2+\cdots+2^m+2p^m = 2^{m+1}-1+2p^m.\end{aligned}$$

Hence, the result holds. ∎

Now we calculate the probability of cyclicity of chains of the generalized quaternion group of order 2^n, as follows

$$Q_{2^n} = \langle x, y : x^{2^{n-1}} = y^4 = 1, x^y = x^{2^{n-1}-1}\rangle, n \ge 3.$$

Clearly, $Z(Q_{2^n}) = \langle x^{2^{n-2}}\rangle$ and $Q_{2^n}/Z(Q_{2^n}) \cong D_{2^{n-1}}$. Furthermore, if H is a subgroup of Q_{2^n} such that $H \cap Z(Q_{2^n}) = 1$, then either $H = 1$ or $H = \langle x^{2i}y\rangle$ for some $0 \le i \le 2^{n-2}$. Hence the cyclic subgroups of Q_{2^n} are $\langle x^{2^i}\rangle$ and $\langle x^j y\rangle$, where $0 \le i \le n-1$ and $0 \le j \le 2^{n-2}-1$. Thus, by applying a recursive argument, it follows that the number of cyclic subgroups of Q_{2^n} is equal to $n + 2^{n-2}$.

Theorem 3.3. *The probability of cyclicity of chains of the group Q_{2^n} is equal to*

$$pcc(Q_{2^n}) = \frac{3\cdot 2^{n-1}-1}{2^{2n-2}}.$$

Proof. Using the property $|Ch(Q_{2^n})| = 2n(\langle e\rangle)$ and the above discussion, the number of cyclic subgroups of Q_{2^n} is calculated as follows:

$$\begin{aligned}|C^*(Q_{2^n})| &= 2(1+2+2^2+\cdots+2^{n-2}+1+\underbrace{1+1+\cdots+1}_{2^{n-2}}) - 1 \\ &= 3\cdot 2^{n-1}-1.\end{aligned}$$

On the other hand, Theorem 2.2(ii) implies that

$$\begin{aligned}|Ch(Q_{2^n})| &= |Ch(Q_{4.2^{n-2}})| = |Ch(D_{2^{n-1}})| + 1\times|Ch(D_{2^{n-1}})| \\ &= 2|Ch(D_{2^{n-1}})| = 2\cdot 2^{2n-3} = 2^{2n-2},\end{aligned}$$

from which the result follows. ∎

Now, we obtain the probability of cyclicity of chains of quasi-dihedral group

$$QD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, x^y = x^{2^{n-2}-1}\rangle,$$

of order 2^n.

Clearly, $Z(QD_{2^n}) = \langle x^{2^{n-2}} \rangle$ and $QD_{2^n}/Z(QD_{2^n}) \cong D_{2^{n-1}}$. Furthermore, if H is a subgroup of QD_{2^n} such that $H \cap Z(QD_{2^n}) = 1$, then either $H = \langle 1 \rangle$ or $\langle x^{2i}y \rangle$, for some $0 \le i \le 2^{n-2}$. Hence the cyclic subgroups of QD_{2^n} are $\langle x^{2^i} \rangle$ with $0 \le i \le n-1$, $\langle x^i y \rangle$ with $0 \le i \le 2^{n-2}$, and $\langle x^{2^{n-2}+2j} y \rangle$ with $1 \le j \le 2^{n-2}-1$. Hence we obtain the following theorem.

Theorem 3.4. *The probability of cyclicity of chains of the group QD_{2^n} is*

$$pcc(QD_{2^n}) = \frac{7 \cdot 2^{n-2} - 1}{3 \cdot 2^{2n-3}}.$$

Proof. An argument similar to that of the proof of Theorem 3.3, it implies

$$\begin{aligned} |C^*(QD_{2^n})| &= 2(1 + 3.2^{n-3} + 1 + 2 + 2^2 + \cdots + 2^{n-2}) - 1 \\ &= 2(1 + 3.2^{n-3} + 2^{n-1} - 1) - 1 \\ &= 7 \cdot 2^{n-2} - 1. \end{aligned}$$

By Theorem 2.2(iii), we have $|Ch(QD_{2^n})| = 3.2^{2n-3}$, from which the result follows. ∎

Finally, we calculate the probability of cyclicity of chains of the modular p-groups of order p^n, with the following presentation

$$M_{p^n} = \langle x, y : x^{p^{n-1}} = y^p = 1, x^y = x^{p^{n-2}+1} \rangle, \quad n \ge 3.$$

One observes that $Z(M_{p^n}) = \langle x^{p^{n-2}} \rangle$ and $M_{p^n}/Z(M_{p^n}) \cong C_{p^{n-2}} \times C_p$. Now, if H is a subgroup of M_{p^n} such that $H \cap Z(M_{p^n}) = 1$, then either $H = \langle 1 \rangle$ or $\langle x^{ip^{n-2}} y \rangle$, for $0 \le i \le p$. Hence the group M_{p^n} has the following cyclic subgroups: $\langle x^{p^i} \rangle$ with $0 \le i \le n-1$ and $\langle x^{jp^i} y \rangle$ for all $0 \le j \le p-1$ and $0 \le i \le n-2$, in which p is a prime number that $p^n \ne 8$.

Theorem 3.5. *The probability of cyclicity of chains of the modular p-group M_{p^n} ($n \ge 3$ and $p^n \ne 8$) is*

$$pcc(M_{p^n}) = \frac{2^n + 2(p-1)(n-1) + 1}{2^{n-1}(n-1)p + 2^n}.$$

In particular

$$pcc(M_{2^n}) = \frac{2^n + 2n - 1}{n2^n}.$$

Proof. By the above discussion, one can easily calculate that

$$\begin{aligned} |C^*(M_{p^n})| &= 2(1 + 1 + 2 + 2^2 + \cdots + 2^{n-2} + (p-1)(n-1) + 1) - 1, \\ &= 2^n + 2(p-1)(n-1) + 1. \end{aligned}$$

Also, by Theorem 2.2(iv), the number of chains of the modular p-group M_{p^n} $(n \geq 3$ and $p^n \neq 8)$ is equal to $2^{n-1}(n-1)p + 2^n$. Hence, the result is obtained. ■

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