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# Briot-Bouquet Differential Subordination and Bernardi's Integral Operator

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**Abstract.** The conditions on $A$, $B$, $\beta$ and $\gamma$ are obtained for an analytic function $p$ defined on the open unit disc $\mathbb{D}$ and normalized by $p(0) = 1$ to be subordinate to $(1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$ when $p(z) + zp'(z)/(\beta p(z) + \gamma)$ is subordinate to $e^z$. The conditions on these parameters are derived for the function $p$ to be subordinate to $\sqrt{1+z}$ or $e^z$ when $p(z) + zp'(z)/(\beta p(z) + \gamma)$ is subordinate to $(1 + Az)/(1 + Bz)$. The conditions on $\beta$ and $\gamma$ are determined for the function $p$ to be subordinate to $e^z$ when $p(z)+zp'(z)/(\beta p(z)+\gamma)$ is subordinate to $\sqrt{1+z}$. Related result for the function $p(z) + zp'(z)/(\beta p(z) + \gamma)$ to be in the parabolic region bounded by the $\operatorname{Re} w = |w - 1|$ is investigated. Sufficient conditions for the Bernardi's integral operator to belong to the various subclasses of starlike functions are obtained as applications.

**Keywords:** Starlike functions; Briot–Bouquet differential subordination; Bernardi's integral operator; Lemniscate of Bernoulli; Parabolic starlike.

## 1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D}$. For a natural number $n$, let $\mathcal{H}[a,n]$ be the subset of $\mathcal{H}$ consisting of functions $p$ of the form $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots$. Suppose that $h$ is a univalent function defined on $\mathbb{D}$ with $h(0) = a$ and the function $p \in \mathcal{H}[a,n]$. The Briot–Bouquet differential subordination is the first order differential subordination of the form

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{1}$$

where $\beta \neq 0, \gamma \in \mathbb{C}$. This particular differential subordination has many interesting applications in the theory of univalent functions. Ruschewyh and Singh [28] proved that if the function $p \in \mathcal{H}[1,1]$, $\beta > 0, \operatorname{Re}\gamma \geq 0$ and $h(z) = (1+z)/(1-z)$ in (1) and the function $q \in \mathcal{H}$ satisfy the differential equation

$$q(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = \frac{1+z}{1-z},$$

then $\min_{|z|=r} \operatorname{Re} p(z) \geq \min_{|z|=r} \operatorname{Re} q(z)$. More related results are proved in [17, 19, 8]. For $c > -1$ and $f \in \mathcal{H}[0,1]$, the function $F \in \mathcal{H}[0,1]$ given by Bernardi's integral operator is defined as

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{2}$$

There is an important connection between Briot–Bouquet differential equations and the Bernardi's integral operator. If we set $p(z) = zF'(z)/F(z)$, where $F$ is given by (2), then the functions $f$ and $p$ are related through the following Briot–Bouquet differential equation

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z) + c}.$$

Several authors have investigated results on Briot–Bouquet differential subordination. For example, Ali et al. [3] determined the conditions on $A, B, D$ and $E$ for $p(z) \prec (1 + Az)/(1 + Bz)$ when $p(z) + zp'(z)/(\beta p(z) + \gamma)$ is subordinate to $(1 + Dz)/(1 + Ez)$, $(A, B, D, E \in [-1, 1])$. For related results, see [4, 8, 17, 19, 21, 22, 28]. Recently, Kumar and Ravichandran [14] obtained the conditions on $\beta$ so that $p(z)$ is subordinate to $e^z$ or $(1 + Az)/(1 + Bz)$ whenever $1 + \beta p(z)/p'(z)$ is subordinate to $\sqrt{1 + z}$ or $(1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$. We investigate generalised problems for regions that were considered recently by many authors. In Section 2, we find conditions on $\gamma$ and $\beta$ so that

$p(z) + zp'(z)/(\beta p(z) + \gamma)$ is subordinate to $\sqrt{1+z}$ implies $p(z) \prec e^z$. Conditions on $A, B, \beta$ and $\gamma$ are also determined so that $p(z) + zp'(z)/(\beta p(z) + \gamma) \prec (1 + Az)/(1 + Bz)$ implies $p(z) \prec \sqrt{1+z}$ or $e^z$. We determine conditions on $A, B, \beta$ and $\gamma$ so that $p(z) \prec (1 + Az)/(1 + Bz), (-1 \le B < A \le 1)$ when $p(z)+zp'(z)/(\beta p(z)+\gamma) \prec e^z$ or $\varphi_{PAR}(z)$. The function $\varphi_{PAR} : \mathbb{D} \to \mathbb{C}$ is given by

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad \operatorname{Im}\sqrt{z} \ge 0 \tag{3}$$

and $\varphi_{PAR}(\mathbb{D}) = \{w = u + iv : v^2 < 2u - 1\} = \{w : \operatorname{Re} w > |w - 1|\} =: \Omega_P$. As an application of our results, we give sufficient conditions for the Bernardi's integral operator to belong to the various subclasses of starlike functions which we define below.

Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. For an analytic function $\varphi$ with $\varphi(0) = 1$, let

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}.$$

This class unifies various classes of starlike functions when $\operatorname{Re}\varphi > 0$. Shanmugam [30] studied the convolution properties of this class when $\varphi$ is convex while Ma and Minda [15] investigated the growth, distortion and coefficient estimates under less restrictive assumption that $\varphi$ is starlike and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Notice that, for $-1 \le B < A \le 1$, the class $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ is the class of Janowski starlike functions [10, 23]. For $0 \le \alpha < 1$, the class $\mathcal{S}^*[1 - 2\alpha, -1] =: \mathcal{S}^*(\alpha)$ is the familiar class of starlike functions of order $\alpha$, introduced by Robertson [26]. The class $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike function. The class $\mathcal{S}_P := \mathcal{S}^*(\varphi_{PAR})$ is the class of parabolic starlike functions, introduced by Rønning [29], consists of function $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}.$$

Sokól and Stankiewicz [38] introduced and studied the class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$; the class $\mathcal{S}_L^*$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\Omega_L := \{w \in \mathbb{C} : |w^2 - 1| < 1\}$. Another class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$, introduced recently by Mendiratta *et al.* [16], consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$. There has been several works [9, 2, 13, 24, 31, 36, 37, 25, 35, 34, 32, 1, 39, 33] related to these classes.

The following results are required in our investigation.

**Lemma 1.1.** [20, Theorem 2.1, p. 2] *Let $\Omega \subset \mathbb{C}$ and suppose that $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z) \notin \Omega$, where $z \in \mathbb{D}$, $t \in [0, 2\pi]$ and $k \ge 1$. If $p \in \mathcal{H}[1, 1]$ and $\psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $p(z) \prec e^z$ in $\mathbb{D}$.*

**Lemma 1.2.** [27, Lemma 1.3, p. 28] *Let $w$ be a meromorphic function in $\mathbb{D}$, $w(0)=0$. If for some $z_0 \in \mathbb{D}$, $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|$, then it follows that $z_0w'(z_0)/w(z_0)\geq 1$.*

## 2. Briot–Bouquet Differential Subordination

In the first result, we find conditions on the real numbers $\beta$ and $\gamma$ so that $p(z)\prec e^z$, whenever $p(z)+(zp'(z))/(\beta p(z)+\gamma)\prec\sqrt{1+z}$, where $p\in\mathcal{H}$ with $p(0)=1$. This result gives the sufficient condition for $f\in\mathcal{A}$ to belong to the class $\mathcal{S}_e^*$ by substituting $p(z)=zf'(z)/f(z)$.

**Theorem 2.1.** *Let $\beta,\gamma\in\mathbb{R}$ satisfying $\max\{-\gamma/e,-\gamma e+e/(1-\sqrt{2}e)\}\leq\beta\leq-e\gamma$. Let $p\in\mathcal{H}$ with $p(0)=1$. If the function $p$ satisfies*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\sqrt{1+z},$$

*then $p(z)\prec e^z$.*

*Proof.* Define the functions $\psi:\mathbb{C}^2\times\mathbb{D}\to\mathbb{C}$ and $q:\mathbb{D}\to\mathbb{C}$ as follows:

$$\psi(r,s;z)=r+\frac{s}{\beta r+\gamma}\quad\text{and}\quad q(z)=\sqrt{1+z}\tag{4}$$

so that $\Omega:=q(\mathbb{D})=\left\{w\in\mathbb{C}:|w^2-1|<1\right\}$ and $\psi(p(z),zp'(z);z)\in\Omega$ for $z\in\mathbb{D}$. To prove $p(z)\prec e^z$, we use Lemma 1.1 so we need to show that $\psi(e^{e^{it}},ke^{it}e^{e^{it}};z)\notin\Omega$ which is equivalent to show that $|(\psi(e^{e^{it}},ke^{it}e^{e^{it}};z))^2-1|\geq 1$, where $z\in\mathbb{D}$, $t\in[-\pi,\pi]$ and $k\geq 1$. A simple computation and (4) yield that

$$\begin{aligned}\psi(e^{e^{it}},ke^{it}e^{e^{it}};z) &= e^{e^{it}}+\frac{ke^{it}e^{e^{it}}}{\beta e^{e^{it}}+\gamma},\\ |(\psi(e^{e^{it}},ke^{it}e^{e^{it}};z))^2-1|^2 &=: \frac{f(t)}{g(t)},\end{aligned}\tag{5}$$

for $-\pi\leq t\leq\pi$, where

$$\begin{aligned}f(t)=&\big(e^{2\cos t}\cos(2\sin t)((\gamma+k\cos t+\beta e^{\cos t}\cos(\sin t))^2\\ &-(k\sin t+\beta\sin(\sin t)e^{\cos t})^2)-2\sin(2\sin t)e^{2\cos t}(k\sin t\\ &+\beta\sin(\sin t)e^{\cos t})(\gamma+k\cos t+\beta e^{\cos t}\cos(\sin t))+\beta^2\sin^2(\sin t)e^{2\cos t}\\ &-(\gamma+\beta e^{\cos t}\cos(\sin t))^2\big)^2+\big(2e^{2\cos t}\cos(2\sin t)(k\sin t\\ &+\beta\sin(\sin t)e^{\cos t})(\gamma+k\cos t+\beta e^{\cos t}\cos(\sin t))\\ &+\sin(2\sin t)e^{2\cos t}((\gamma+k\cos t+\beta e^{\cos t}\cos(\sin t))^2\\ &-(k\sin t+\beta\sin(\sin t)e^{\cos t})^2)-2\beta\sin(\sin t)e^{\cos t}(\gamma+\beta e^{\cos t}\cos(\sin t))\big)^2\end{aligned}$$

and

$$g(t) = (\beta^2 \sin^2(\sin t) e^{2\cos t} + (\gamma + \beta e^{\cos t} \cos(\sin t))^2)^2.$$

Define the function $h : [-\pi, \pi] \to \mathbb{R}$ by $h(t) = f(t) - g(t)$. Since $h(-t) = h(t)$, we restrict to $0 \le t \le \pi$. It can be easily verified that the function $h$ attains its minimum value either at $t = 0$ or $t = \pi$. For $k \ge 1$, we have

$$h(0) = (e^2(e\beta + \gamma + k)^2 - (e\beta + \gamma)^2)^2 - (e\beta + \gamma)^4, \tag{6}$$

$$h(\pi) = \left(\left(\frac{\beta/e + \gamma - k}{e}\right)^2 - \left(\frac{\beta}{e} + \gamma\right)^2\right)^2 - \left(\frac{\beta}{e} + \gamma\right)^4. \tag{7}$$

The given relation $\beta \ge -\gamma/e$ gives $e\beta + \gamma \ge 0$ so that $e(k + e\beta + \gamma) > \sqrt{2}(e\beta + \gamma)$ which implies $e^2(k + e\beta + \gamma)^2 - (e\beta + \gamma)^2 > (e\beta + \gamma)^2$. Thus, the use of (6) yields $h(0) > 0$.

The given condition $1/(1 - \sqrt{2}e) \le \gamma + \beta/e \le 0$ leads to $(\gamma + \beta/e)(1 - \sqrt{2}e) \le 1$ which gives that $-k + \gamma + \beta/e \le -1 + \gamma + \beta/e \le \sqrt{2}e(\gamma + \beta/e)$ which implies $((-k + \gamma + \beta/e)/e)^2 \ge 2(\gamma + \beta/e)^2$ which further implies $((-k + \gamma + \beta/e)/e)^2 - (\gamma + \beta/e)^2 \ge (\gamma + \beta/e)^2$. Hence, by using (7), we get that $h(\pi) \ge 0$. So, $h(t) \ge 0, (0 \le t \le \pi)$ and thus, (5) implies $|(\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z))^2 - 1| \ge 1$ and therefore $p(z) \prec e^z$. ∎

We will illustrate Theorem 2.1 by the following example:

*Example 2.2.* By taking $\beta = 1$ and $\gamma = c$ $(c > -1)$ in Theorem 2.1, we get $-1/e + 1/(1 - \sqrt{2}e) \le c \le -1/e$. By taking $\beta = 1$, $-1/e + 1/(1 - \sqrt{2}e) \le \gamma \le -1/e$, $n = 1$, $h(z) = \sqrt{1+z}$, $a = 1$ in [18, Theorem 3.2d, p. 86], we get $\mathrm{Re}(a\beta + \gamma) > 0$ and $\beta h(z) + \gamma \prec R_{a\beta+\gamma,n}(z)$, where $R_{d,f}(z)$ is the open door mapping given by $R_{d,f}(z) := d(1+z)/(1-z) + (2fz)/(1-z^2)$. Thus by the use of [18, Theorem 3.2d, p. 86], we get

$$p(z) = -\gamma + \int_0^1 \frac{t^{-\gamma} e^{2\sqrt{z+1} - 2\sqrt{tz+1}} \left(\sqrt{tz+1} + 1\right)^2}{\left(\sqrt{z+1} + 1\right)^2} \, dt$$

which satisy Eq. $p(z) + zp'(z)/(\beta p(z) + \gamma) = h(z)$. Then $p(z) \prec e^z$.

Suppose that the function $F$ be given by Bernardi's integral (2). Now we discuss the sufficient conditions for the function $F$ to belong to various subclasses of starlike functions. We will illustrate Theorem 2.1 by the following corollary.

**Corollary 2.3.**

(i) *If the function $f \in \mathcal{S}_L^*$ and the conditions of Theorem 2.1 hold with $\beta = 1$ and $\gamma = c$, then $F \in \mathcal{S}_e^*$.*

(ii) *If the function $f'(z) \prec \sqrt{1+z}$ and the conditions of Theorem 2.1 hold with $\beta = 0$ and $\gamma = c + 1$, then $F'(z) \prec e^z$.*

*Proof.* (i) Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zF'(z)/F(z)$. Then $p$ is analytic in $\mathbb{D}$ with $p(0) = 1$. Upon differentiating Bernardi's integral given by (2), we obtain

$$(c+1)f(z) = zF'(z) + cF(z). \tag{8}$$

A computation now yields

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}.$$

By taking $\beta = 1$ and $\gamma = c$, the first part of the corollary follows from Theorem 2.1.

(ii) By defining a function $p$ by $p(z) = F'(z)$ and using (8), we get

$$f'(z) = \frac{zF''(z)}{c+1} + F'(z).$$

By taking $\beta = 0$ and $\gamma = c+1$, the result follows from Theorem 2.1. ∎

In the following result, we derive conditions on the real numbers $A, B$, $\beta$ and $\gamma$ so that $p(z)+(zp'(z))/(\beta p(z)+\gamma) \prec e^z$ implies $p(z) \prec (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)$, where $p \in \mathcal{H}$ with $p(0) = 1$. This result gives the sufficient condition for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}^*[A, B]$ by substituting $p(z) = zf'(z)/f(z)$.

**Theorem 2.4.** *Let $-1 < B < A \leq 1$ and $\beta, \gamma \in \mathbb{R}$. Suppose that*
(i) $\big(A-B\big)/\big((1 \mp B)((1 \mp A)\beta + (1 \mp B)\gamma)\big) \geq \pm(1 \mp A)/(1 \mp B) + e.$
(ii) $\beta(1 \pm A) + \gamma(1 \pm B) > 0.$

*Let $p \in \mathcal{H}$ with $p(0) = 1$. If the function $p$ satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec e^z,$$

*then $p(z) \prec (1+Az)/(1+Bz)$.*

*Proof.* Define the functions $P$ and $w$ as follows:

$$P(z) = p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \quad \text{and} \quad w(z) = \frac{p(z)-1}{A-Bp(z)} \tag{9}$$

so that $p(z) = (1+Aw(z))/(1+Bw(z))$. Clearly, $w(z)$ is analytic in $\mathbb{D}$ with $w(0) = 0$. In order to prove $p(z) \prec (1+Az)/(1+Bz)$, we need to show that $|w(z)| < 1$ in $\mathbb{D}$. If possible, suppose that there exists $z_0 \in \mathbb{D}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 1.2, it follows that there exists $k \geq 1$ so that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{it}$, $(-\pi \leq t \leq \pi)$ and $G := A\beta + B\gamma$. A simple calculation and by using (9), we get

$$P(z_0) = \frac{ke^{it}(A-B) + \left(1+Ae^{it}\right)\left(\beta+\gamma+Ge^{it}\right)}{\left(1+Be^{it}\right)\left(\beta+\gamma+Ge^{it}\right)} =: u + iv, \tag{10}$$

for $-\pi \le t \le \pi$. We derive a contradiction by showing $|\log P(z_0)|^2 \ge 1$. This later inequality is equivalent to

$$f(t) := 4(\arg(u+iv))^2 + (\log\left(u^2+v^2\right))^2 - 4 \ge 0 \quad (-\pi \le t \le \pi). \tag{11}$$

From (10), we get

$$\begin{aligned}u = {} & \frac{1}{(B^2+2B\cos t+1)\left((\beta+\gamma)^2+G^2+2G(\beta+\gamma)\cos t\right)}\big(G(A+B)(\beta+\gamma) \\ & \cos 2t + \cos t\big(A(BG(2(\beta+\gamma)+k)+G^2+(\beta+\gamma)(\beta+\gamma+k)) - B^2Gk \\ & + 2G(\beta+\gamma) + B\left(G^2-(\beta+\gamma)(-\beta-\gamma+k)\right)\big) + (\beta+\gamma)(AB(\beta+\gamma+k) \\ & + \beta - B^2k+\gamma) + G^2(AB+1) + G(A(\beta+\gamma+k)+B(\beta+\gamma-k))\big)\end{aligned}$$

and

$$v = \frac{(A-B)\sin t\left(-BGk+G^2+2G(\beta+\gamma)\cos t+(\beta+\gamma)(\beta+\gamma+k)\right)}{(B^2+2B\cos t+1)\left((\beta+\gamma)^2+G^2+2G(\beta+\gamma)\cos t\right)}.$$

Substituting these values of $u$ and $v$ in (11), we observe that $f(t)$ is an even function of $t$ and so, it is enough to show that $f(t) \ge 0$ for $t \in [0,\pi]$. It can be easily verified that the function $f(t)$ attains its minimum value either at $t=0$ or $t=\pi$. We show that both $f(0)$ and $f(\pi)$ are non negative. Note that, for $k \ge 1$,

$$f(0) = -4 + 4(\arg\psi(k))^2 + (\log(\psi^2(k)))^2 \tag{12}$$

and

$$f(\pi) = -4 + 4(\arg(-\phi(k)))^2 + (\log(\phi^2(k)))^2, \tag{13}$$

where $\psi(k) := \big(A^2\beta + A(2\beta+B\gamma+\gamma+k)+\beta+B(\gamma-k)+\gamma\big)/\big((1+B)(\beta(1+A)+\gamma(1+B))\big)$ and $\phi(k) := \big(A^2\beta - 2A\beta + (A-1)(B-1)\gamma - Ak + \beta + Bk\big)/\big((B-1)(-A\beta+\beta-B\gamma+\gamma)\big)$. The function $\psi$ is increasing as $\psi'(k) = (A-B)/\big((1+B)(\beta(1+A)+\gamma(1+B))\big) > 0$ using the given condition (ii) and therefore, the given hypothesis (i) yields that $\psi(k) \ge \psi(1) = (1+A)/(1+B) + (A-B)/\big((1+B)(\beta(1+A)+\gamma(1+B))\big) \ge e$ which gives that $\arg\psi(k)=0$ and $(\log(\psi^2(k)))^2 \ge (2\log e)^2 = 4$. Thus, the use of (12) yields $f(0) \ge 0$.

The function $\phi$ is increasing as $\phi'(k) = (A-B)/\big((1-B)(\beta(1-A)+\gamma(1-B))\big) > 0$ using the given condition (ii) and therefore, the given hypothesis (i) yields that $\phi(k) \ge \phi(1) = -(1-A)/(1-B) + (A-B)/\big((1-B)(\beta(1-A)+\gamma(1-B))\big) \ge e$ which further implies $\arg(-\phi(k)) = \pi$ and $(\log(\phi^2(k)))^2 \ge (2\log e)^2 = 4$. Hence, by using (13), we get $f(\pi) \ge 4\pi^2 > 0$. This completes the proof. ■

We will illustrate Theorem 2.4 by the following example:

*Example 2.5.* By taking $A=1/2$, $B=-1/2$, $\beta=1$ and $\gamma=c$ $(c>-1)$ in Theorem 2.4, we get $-1/3 \le c \le (1-e)/(1+3e)$. By taking $\beta=1$, $-1/3 \le \gamma \le (1-e)/(1+3e)$, $n=1$, $h(z)=e^z$, $a=1$ in [18, Theorem 3.2d, p. 86], we get $\mathrm{Re}(a\beta+\gamma) > 0$ and $\beta h(z)+\gamma \prec R_{a\beta+\gamma,n}(z)$, where $R_{d,f}(z)$ is the open door mapping given by $R_{d,f}(z) := d(1+z)/(1-z) + (2fz)/(1-z^2)$. Thus by using [18, Theorem 3.2d, p. 86], we get

$$p(z) = \int_0^1 t^{1-\gamma} e^{-\mathrm{Chi}(tz)+\mathrm{Chi}(z)-\mathrm{Shi}(tz)+\mathrm{Shi}(z)}\, dt - \gamma$$

which satisy Eq. $p(z) + zp'(z)/(\beta p(z)+\gamma) = h(z)$. Then $p(z) \prec (2+z)/(2-z)$. Here, $Chi(z)$ and $Shi(z)$ are the hyperbolic cosine integral function and the hyperbolic sine integral function respectively defined as follows:

$$Chi(z) = \eta + \log(z) + \int_0^z \frac{\cosh(t)-1}{t} dt \quad \text{and} \quad Shi(z) = \int_0^z \frac{\sinh(t)}{t} dt,$$

where $\eta$ is the Euler's constant.

The next corollary is obtained by substituting $p(z) = zf'(z)/f(z)$ with $\gamma = 0$, $B = 0$ and $A = 1-\alpha$, $(0 \le \alpha < 1)$ in Theorem 2.4.

**Corollary 2.6.** *Let $0 \le \alpha < 1$ and $\beta > 0$ satisfy the conditions $\alpha + e + \beta^{-1} \le (\alpha\beta)^{-1}$ and $1-\alpha \ge \beta(2-\alpha)(e-2+\alpha)$. If the function $f \in \mathcal{A}$ satisfies the subordination*

$$\frac{zf'(z)}{f(z)} + \frac{1}{\beta}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec e^z,$$

*then $f \in \mathcal{S}^*_\alpha$.*

Our next corollary deals with the class $\mathcal{R}[A,B]$ defined by

$$\mathcal{R}[A,B] = \left\{ f \in \mathcal{A} : f'(z) \prec \frac{1+Az}{1+Bz} \right\}.$$

The two parts of the following corollary are obtained by taking $p(z)$ to be $zF'(z)/F(z)$ with $\beta = 1$, $\gamma = c$ and $p(z) = F'(z)$ with $\beta = 0$, $\gamma = c+1$ respectively in Theorem 2.4.

**Corollary 2.7.**

(i) *If the function $f \in \mathcal{S}^*_e$ and the conditions of Theorem 2.4 hold with $\beta = 1$ and $\gamma = c$, then $F \in \mathcal{S}^*[A,B]$.*

(ii) *The function $f'(z) \prec e^z$ and the conditions of Theorem 2.4 hold with $\beta = 0$ and $\gamma = c+1$, then $F \in \mathcal{R}[A,B]$.*

In the next result, we find the conditions on the real numbers $A, B$, $\beta$ and $\gamma$ so that $p(z) \prec \sqrt{1+z}$, whenever $p(z)+(zp'(z))/(\beta p(z)+\gamma) \prec (1+Az)/(1+Bz)$, $-1 \le B < A \le 1$, where $p \in \mathcal{H}$ with $p(0) = 1$. As an application of the next result, it provides sufficient conditions for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}^*_L$.

**Theorem 2.8.** *Let $-1 \le B < A \le 1$ and $\beta, \gamma \in \mathbb{R}$ satisfy the following conditions:*

(i) $1+4(\sqrt{2}-1)\beta-2(\sqrt{2}-2)\gamma \geq B(-2A(2\beta+\sqrt{2}\gamma)+B(1+4(\sqrt{2}\beta+\gamma)))$.
(ii) $(1+4(\sqrt{2}-1)\beta-2(\sqrt{2}-2)\gamma)^2 \geq (-2A(2\beta+\sqrt{2}\gamma)+B(1+4(\sqrt{2}\beta+\gamma)))^2$.

*Let $p \in \mathcal{H}$ with $p(0)=1$. If the function $p$ satisfies*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Az}{1+Bz},$$

*then $p(z) \prec \sqrt{1+z}$.*

*Proof.* Define the functions $P$ and $w$ as follows:

$$P(z)=p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \quad \text{and} \quad w(z)=p^2(z)-1 \tag{14}$$

which implies $p(z)=\sqrt{1+w(z)}$. Clearly, $w(z)$ is analytic in $\mathbb{D}$ with $w(0)=0$. In order to complete our proof, we need to show that $|w(z)|<1$ in $\mathbb{D}$. Assume that there exists $z_0 \in \mathbb{D}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

then by Lemma 1.2, it follows that there exists $k \geq 1$ so that $z_0w'(z_0)=kw(z_0)$. Let $w(z_0)=e^{it}$, $(-\pi \leq t \leq \pi)$. By using (14), we get

$$P(z)=\sqrt{1+w(z)}+\frac{zw'(z)}{2\sqrt{1+w(z)}(\beta\sqrt{1+w(z)}+\gamma)}.$$

A simple computation shows that

$$P(z_0)=\frac{ke^{it}+2\left(1+e^{it}\right)\left(\gamma+\beta\sqrt{1+e^{it}}\right)}{2\sqrt{1+e^{it}}\left(\gamma+\beta\sqrt{1+e^{it}}\right)} \quad (-\pi \leq t \leq \pi)$$

and

$$\left|\frac{P(z_0)-1}{A-BP(z_0)}\right|^2 =: \frac{f(t)}{g(t)} \quad (-\pi \leq t \leq \pi), \tag{15}$$

where

$$\begin{aligned} f(t) =&\big((2\beta\cos t+2(\beta-\gamma))\sin(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)} \\ &+\sin t(k+2(\gamma+\beta(-1+\cos(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)}))))\big)^2 \\ &+\big(-\cos t(k+2(\gamma+\beta(-1+\cos(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)}))) \\ &+2\beta\sin t\sin(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)}+2(\beta-\gamma) \\ &(1-\cos(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)})\big)^2 \end{aligned}$$

and

$$\begin{aligned} g(t) =& \big( -2A(\beta \sin t + \gamma \sin(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)}) \\ &+ 4B\beta \cos^2(t/2)\sqrt{2\cos(t/2)}\sin(\arg(1+e^{it})/2) + B\sin t(k+2\gamma \\ &+ 2\beta\cos(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)})\big)^2 + \big( -4A\beta\cos^2(t/2) \\ &+ B(k+2\gamma)\cos t + 2\gamma - 2B\beta \sin t \sin(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)} \\ &+ 2(-A\gamma + B\beta\cos t + \beta)\cos(\arg(1+e^{it})/2)\sqrt{2\cos(t/2)}\big)^2. \end{aligned}$$

Define $h(t) = f(t) - g(t)$. Since $h(t)$ is an even function of $t$, we restrict to $0 \leq t \leq \pi$. It can be easily verified that for both the cases (i) and (ii), the function $h(t)$ attains its minimum value either at $t = 0$ or $t = \pi$. Note that for $k \geq 1$, $h(\pi) = (1 - B^2)k^2 > 0$ and

$$\begin{aligned} S(k) := h(0) = (4(\sqrt{2}-1)\beta - 2(\sqrt{2}-2)\gamma + k)^2 - (B(4(\sqrt{2}\beta + \gamma) + k) \\ - 2A(2\beta + \sqrt{2}\gamma))^2. \end{aligned} \tag{16}$$

The function $S'$ is increasing as $S''(k) = 2(1 - B^2) > 0$ and therefore, the given hypothesis (i) yields that $S'(k) \geq S'(1) = 2(1 + 4(\sqrt{2}-1)\beta - 2(\sqrt{2}-2)\gamma) - 2B(-2A(2\beta + \sqrt{2}\gamma) + B(1 + 4(\sqrt{2}\beta + \gamma))) \geq 0$ which gives that $S(k) \geq S(1) = (1 + 4(\sqrt{2}-1)\beta - 2(\sqrt{2}-2)\gamma)^2 - (-2A(2\beta + \sqrt{2}\gamma) + B(1 + 4(\sqrt{2}\beta + \gamma)))^2$. Thus, the use of given condition (ii) and (16) yields $h(0) \geq 0$. So, $h(t) \geq 0$ for all $t \in [0, \pi]$ and therefore, (15) implies $|(P(z_0) - 1)/(A - BP(z_0))| \geq 1$. This contradicts the fact that $P(z) \prec (1 + Az)/(1 + Bz)$ and completes the proof. ■

The next corollary is obtained by substituting $p(z) = zf'(z)/f(z)$ with $\gamma = 0$, $A = 1 - 2\alpha$, $(0 \leq \alpha < 1)$ and $B = -1$ in Theorem 2.8.

**Corollary 2.9.** *Let $0 \leq \alpha < 1$ and $f \in \mathcal{A}$. If the function $f$ satisfies the subordination*

$$\frac{zf'(z)}{f(z)} + \frac{1}{\beta}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1 + (1-2\alpha)z}{1 - z} \qquad \left(\frac{1}{4(\alpha - \sqrt{2})} \leq \beta < 0\right),$$

*then $f \in \mathcal{S}^*_L$.*

By taking $p(z) = zF'(z)/F(z)$ with $\beta = 1$ and $\gamma = c$ in Theorem 2.8 gives the following corollary:

**Corollary 2.10.** *Let $-1 \leq B < A \leq 1$ satisfy the following conditions:*

(i) $1 + 4(\sqrt{2}-1) - 2(\sqrt{2}-2)c \geq B(-2A(2 + \sqrt{2}c) + B(1 + 4(\sqrt{2} + c)))$.
(ii) $(1 + 4(\sqrt{2}-1) - 2(\sqrt{2}-2)c)^2 \geq (-2A(2 + \sqrt{2}c) + B(1 + 4(\sqrt{2} + c)))^2$.

*If $f \in \mathcal{S}^*[A, B]$ then $F \in \mathcal{S}^*_L$.*

By taking $p(z) = F'(z)$ with $\beta = 0$ and $\gamma = c + 1$ in Theorem 2.8 gives the following corollary:

**Corollary 2.11.** *Suppose that* $-1 \leq B < A \leq 1$ *satisfy the following conditions:*
(i) $5 - 2\sqrt{2} - 2(\sqrt{2} - 2)c \geq B(-2\sqrt{2}(c + 1)A + (5 + 4c)B)$.
(ii) $(5 - 2\sqrt{2} - 2(\sqrt{2} - 2)c)^2 \geq (-2\sqrt{2}(c + 1)A + (5 + 4c)B)^2$.
*If* $f \in \mathcal{R}[A, B]$ *then* $F'(z) \prec \sqrt{1 + z}$.

In the next result, we compute the conditions on the real numbers $A, B$, $\beta$ and $\gamma$ so that $p(z)+(zp'(z))/(\beta p(z)+\gamma) \prec (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)$ implies $p(z) \prec e^z$, where $p \in \mathcal{H}$ with $p(0) = 1$. As an application of the next result, it provides sufficient conditions for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}^*_e$.

**Theorem 2.12.** *Let* $-1 \leq B < A \leq 1$ *and* $\beta, \gamma \in \mathbb{R}$ *satisfy the following conditions:*
(i) $e^2\beta(1 - B^2) + e(-B(-A\beta + B\gamma + B) - \beta + \gamma + 1) + \gamma(AB - 1) \geq 0$.
(ii) $(e((A+e-1)\beta-(e\beta+1)B+1)+\gamma(A+e(1-B)-1))(e(-(A-e+1)\beta+B(e\beta+1)+1)+\gamma(-A+e(B+1)-1)) \geq 0$.
(iii) $e(\beta(1 - AB) + B^2(\gamma - 1) - \gamma + 1) + e^2\gamma(1 - AB) + \beta(B^2 - 1) \geq 0$.
(iv) $(e((A - 1)\beta + (1 - B)(\gamma - 1)) + e^2(A - 1)\gamma + \beta(1 - B))(-e((A + 1)\beta + (B + 1)(1 - \gamma)) - e^2(A + 1)\gamma + \beta(B + 1)) \geq 0$.
*Let* $p \in \mathcal{H}$ *with* $p(0) = 1$. *If the function* $p$ *satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Az}{1 + Bz},$$

*then* $p(z) \prec e^z$.

*Proof.* Define the functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ and $q : \mathbb{D} \to \mathbb{C}$ as follows:

$$\psi(r, s; z) = r + \frac{s}{\beta r + \gamma} \quad \text{and} \quad q(z) = \frac{1 + Az}{1 + Bz} \tag{17}$$

so that $\Omega := q(\mathbb{D}) = \{w \in \mathbb{C} : |(w - 1)/(A - Bw)| < 1\}$ and $\psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$. To prove $p(z) \prec e^z$, we use Lemma 1.1 so we need to show that $\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z) \notin \Omega$ which is equivalent to show that $|(\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z) - 1)/(A - B\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z))| \geq 1$, where $z \in \mathbb{D}$, $t \in [-\pi, \pi]$ and $k \geq 1$. A simple computation and (17) yield that

$$\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z) = e^{e^{it}} + \frac{ke^{it}e^{e^{it}}}{\beta e^{e^{it}} + \gamma} \quad (-\pi \leq t \leq \pi)$$

and

$$\left|\frac{\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z) - 1}{A - B\psi(e^{e^{it}}, ke^{it}e^{e^{it}}; z)}\right|^2 =: \frac{f(t)}{g(t)} \quad (-\pi \leq t \leq \pi), \tag{18}$$

where

$$\begin{aligned}f(t)=&e^{3\cos t}(2\beta k\sin t\sin(\sin t)+2\beta k\cos t\cos(\sin t)-2\beta^2\cos(\sin t)\\&+2\beta\gamma\cos(\sin t))+e^{2\cos t}((\beta-\gamma)^2+k^2-2\beta k\cos t+2\gamma k\cos t\\&+2\beta\gamma\sin^2(\sin t)-2\beta\gamma\cos^2(\sin t))+e^{\cos t}(2\gamma k\sin t\sin(\sin t)\\&-2\gamma k\cos t\cos(\sin t)+2\beta\gamma\cos(\sin t)-2\gamma^2\cos(\sin t))+\beta^2e^{4\cos t}+\gamma^2\end{aligned}$$

and

$$\begin{aligned}g(t)=&A^2\gamma^2+\beta^2B^2e^{4\cos t}+2\beta Be^{3\cos t}((B\gamma-A\beta)\cos(\sin t)+Bk\cos(t-\sin t))\\&+e^{2\cos t}(B(B(\gamma^2+k^2)-2A\beta\gamma)+2B(B\gamma-A\beta)k\cos t\\&-2AB\beta\gamma\cos(2\sin t)+A^2\beta^2)+2A\gamma e^{\cos t}((A\beta-B\gamma)\cos(\sin t)\\&-Bk\cos(t+\sin t)).\end{aligned}$$

Define $h(t)=f(t)-g(t)$. Since $h(-t)=h(t)$, we restrict to $0\le t\le\pi$. It can be easily verified that the function $h(t)$ attains its minimum value either at $t=0$ or $t=\pi$. For $k\ge1$, we have

$$\begin{aligned}\phi(k):=h(0)=&\,e^2((1-A^2)\beta^2+2k(\beta(AB-1)+(1-B^2)\gamma)+4\beta\gamma(AB\\&-1)+(1-B^2)(\gamma^2+k^2))+2e\gamma(-A^2\beta+(AB-1)(\gamma+k)+\beta)\\&+2e^3\beta(\beta(AB-1)+(1-B^2)(\gamma+k))+e^4\beta^2(1-B^2)\\&+(1-A^2)\gamma^2\end{aligned}\tag{19}$$

and

$$\begin{aligned}h(\pi)=&\frac{-1}{e^4}(e((A-1)\beta+(1-B)(\gamma-k))+e^2(A-1)\gamma+\beta(1-B))\\&(e((1+A)\beta+(B+1)(k-\gamma))+e^2(A+1)\gamma-\beta(B+1))=:\psi(k).\end{aligned}\tag{20}$$

The function $\phi'$ is increasing as $\phi''(k)=2(1-B^2)e^2>0$ and therefore, the given hypothesis (i) yields that $\phi'(k)\ge\phi'(1)=2e(e(-B(-A\beta+B\gamma+B)-\beta+\gamma+1)+\gamma(AB-1)+e^2\beta(1-B^2))\ge0$ which gives that $\phi(k)\ge\phi(1)=(e((A+e-1)\beta-(e\beta+1)B+1)+\gamma(A+e(1-B)-1))(e(-(A-e+1)\beta+B(e\beta+1)+1)+\gamma(-A+e(B+1)-1))$. Thus, the use of given condition (ii) and (19) yields $h(0)\ge0$.

In view of (iii), observe that $\psi''(k)=2(1-B^2)/e^2>0$ and therefore, $\min\psi'(k)=\psi'(1)=2(e(\beta(1-AB)+B^2(\gamma-1)-\gamma+1)+e^2\gamma(1-AB)+\beta(B^2-1))/e^3\ge0$ which implies $\min\psi(k)=\psi(1)=((e((A-1)\beta+(1-B)(\gamma-1))+e^2(A-1)\gamma+\beta(1-B))(-e((A+1)\beta+(B+1)(1-\gamma))-e^2(A+1)\gamma+\beta(B+1)))/e^4$. Hence, the use of given condition (iv) and (20) yields that $h(\pi)\ge0$. So, $h(t)\ge0,(0\le t\le\pi)$ and thus, (18) implies $|(\psi(e^{e^{it}},ke^{it}e^{e^{it}};z)-1)/(A-B\psi(e^{e^{it}},ke^{it}e^{e^{it}};z))|\ge1$ and therefore, $p(z)\prec e^z$. ∎

The next corollary is obtained by substituting $p(z)=zf'(z)/f(z)$ with $\gamma=0$, $B=0$ and $A=1-\alpha$, $(0\le\alpha<1)$ in Theorem 2.12.

**Corollary 2.13.** *Suppose* $0 \leq \alpha < 1$ *and* $\beta \geq 1/(1-e)$ *satisfy the conditions* $(-\alpha\beta+\beta e+1)(\beta(\alpha+e-2)+1) \geq 0$ *and* $(\beta - e((2-\alpha)\beta+1))(\beta+e(-\alpha\beta-1)) \geq 0$. *If the function* $f \in \mathcal{A}$ *satisfies the condition*

$$\left|\frac{zf'(z)}{f(z)} + \frac{1}{\beta}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) - 1\right| < 1 - \alpha,$$

*then* $f \in \mathcal{S}^*_e$.

The two parts of the following corollary are obtained by taking $p(z) = zF'(z)/F(z)$ with $\beta = 1$, $\gamma = c$ and $p(z) = F'(z)$ with $\beta = 0$, $\gamma = c+1$ respectively in Theorem 2.12.

**Corollary 2.14.**

(i) *If the function* $f \in \mathcal{S}^*[A,B]$ *and the conditions of the Theorem 2.12 hold with* $\beta = 1$ *and* $\gamma = c$, *then* $F \in \mathcal{S}^*_e$.

(ii) *The function* $f \in \mathcal{R}[A,B]$ *and the conditions of Theorem 2.12 hold with* $\beta = 0$ *and* $\gamma = c+1$, *then* $F'(z) \prec e^z$.

In the next result, we find the conditions on the real numbers $A, B$, $\beta$ and $\gamma$ so that $p(z) \prec (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)$, whenever $p(z) + (zp'(z))/(\beta p(z) + \gamma) \in \Omega_P$, where $p \in \mathcal{H}$ with $p(0) = 1$. As an application of the next result, it provides sufficient conditions for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}^*[A,B]$.

**Theorem 2.15.** *Let* $-1 \leq B < A \leq 1$ *and* $\beta, \gamma \in \mathbb{R}$. *For* $k \geq 1$ *and* $0 \leq m \leq 1$, *assume that* $G := A\beta + B\gamma, L := k + \beta + \gamma$. *Further assume that*

(i) $BG(\beta+\gamma) > 0$.

(ii) $\big(G(A^2L + 4(\beta+\gamma)) - 2B(AGL + 2(\beta+\gamma)^2 + 2G^2) + B^2G(4(\beta+\gamma) + L)\big)\big(G\left(A^2L - 4(\beta+\gamma)\right) + B\left(-2AGL + 4(\beta+\gamma)^2 + 4G^2\right) + B^2G(L - 4(\beta+\gamma))\big) \geq 2G(A-B)^2\big(GL(A^2L - 4(\beta+\gamma)) - 2B(AGL^2 + 2G^2(L - 2(\beta+\gamma)) - 2L(\beta+\gamma)(-\beta-\gamma+2L)) + B^2GL(L - 4(\beta+\gamma))\big)$.

(iii) $8G(A-B)^2(\beta+\gamma+k) \leq 2(B-1)^2G(\beta+\gamma) + 2B(\beta+\gamma-G)^2$.

(iv) $1 + \beta + \gamma \geq 0$, $G \geq 0$.

(v) $4m^4(A-B)^2(\beta+\gamma+G+1)^2 \geq (B+1)^2(\beta+\gamma+G)^2$.

*Let* $p \in \mathcal{H}$ *with* $p(0) = 1$. *If the function* $p$ *satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \varphi_{PAR}(z),$$

*then* $p(z) \prec (1+Az)/(1+Bz)$.

*Proof.* Define the functions $P$ and $w$ as given by Eq. (9) which implies $p(z) = (1 + Aw(z))/(1 + Bw(z))$. Proceeding as in Theorem 2.4, we need to show that

$|w(z)| < 1$ in $\mathbb{D}$. If possible suppose that there exists $z_0 \in \mathbb{D}$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 1.2, it follows that there exists $k \ge 1$ so that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{it}$, $(-\pi \le t \le \pi)$. A simple calculation and by using (9), we get

$$P(z_0) = \frac{ke^{it}(A-B) + \left(1+Ae^{it}\right)\left(\beta+\gamma+Ge^{it}\right)}{\left(1+Be^{it}\right)\left(\beta+\gamma+Ge^{it}\right)} \quad (-\pi \le t \le \pi). \tag{21}$$

Define the function $h$ by

$$h(z) = u + iv = \sqrt{(P(z)-1)\pi^2/2}. \tag{22}$$

We show that $|(e^{h(z_0)}-1)/(e^{h(z_0)}+1)|^2 \ge 1$; this condition is same as the inequality $\operatorname{Re} e^{h(z_0)} \le 0$. This last inequality is indeed equivalent to $\cos v \le 0$ or $1/2 \le |v/\pi| \le 1$. By using the definition of $h$ given in (22) together with (21), we get

$$\frac{|v|}{\pi} = \frac{\sqrt{A-B}|m(t)||Ge^{it}+L|^{1/2}}{\sqrt{2}|1+Be^{it}|^{1/2}|Ge^{it}+\beta+\gamma|^{1/2}} \quad (-\pi \le t \le \pi), \tag{23}$$

where $m(t) = \sin\left(\arg\left((e^{it}(A-B)(Ge^{it}+L))/((1+Be^{it})(Ge^{it}+\beta+\gamma))\right)/2\right)$.

(a) We will first show that $|v/\pi| \le 1$ which by using the fact that $|m(t)| \le 1$ and (23) is same as to show that $f(t) \ge 0$ $(-\pi \le t \le \pi)$, where

$$\begin{aligned} f(t) = {}& 4(1+B^2+2B\cos t)((\beta+\gamma)^2+G^2+2(\beta+\gamma)G\cos t) \\ & - (A-B)^2(L^2+G^2+2LG\cos t). \end{aligned}$$

After substituting $x = \cos t$ $(-\pi \le t \le \pi)$, the above inequality reduces to $F(x) \ge 0$ for all $x$ with $-1 \le x \le 1$, where

$$F(x) = 4(1+B^2+2Bx)((\beta+\gamma)^2+G^2+2(\beta+\gamma)Gx) - (A-B)^2(L^2+G^2+2LGx).$$

A simple computation shows that for

$$\begin{aligned} x_0 = {}& \frac{1}{16BG(\beta+\gamma)}\Big(G\left(A^2L-4(\beta+\gamma)\right) - 2B\left(AGL+2(\beta+\gamma)^2+2G^2\right) \\ & + B^2G(L-4(\beta+\gamma))\Big), \end{aligned}$$

$F'(x_0) = 0$ and $F''(x_0) = 32BG(\beta+\gamma) > 0$ by the given condition (i). Therefore, $F(x) \ge F(x_0)$. Observe that

$$\begin{aligned} F(x_0) = {}& \frac{1}{16BG(\beta+\gamma)}\Big(\big(G(A^2L+4(\beta+\gamma)) - 2B(AGL+2(\beta+\gamma)^2+2G^2) \\ & + B^2G(4(\beta+\gamma)+L)\big)\big(G(A^2L-4(\beta+\gamma)) \\ & + B^2G(L-4(\beta+\gamma)) + B(-2AGL+4(\beta+\gamma)^2 \\ & + 4G^2)\big) - 2G(A-B)^2\big(GL(A^2L-4(\beta+\gamma)) \\ & + B^2GL(L-4(\beta+\gamma)) - 2B(AGL^2 \\ & + 2G^2(L-2(\beta+\gamma)) - 2L(\beta+\gamma)(-\beta-\gamma+2L))\big)\Big) \end{aligned}$$

and $F(x_0) \geq 0$ by the given condition (ii).

(b) We will next show that $|v/\pi| \geq 1/2$ which by using (23) is same as to show that $g(t) \geq 0$ $(-\pi \leq t \leq \pi)$, where

$$g(t) = 4(A-B)^2 m^4(t)(L^2 + G^2 + 2LG\cos t) - (1 + B^2 + 2B\cos t)((\beta+\gamma)^2 + G^2 + 2(\beta+\gamma)G\cos t)$$

After substituting $x = \cos t$ $(-\pi \leq t \leq \pi)$ and $m = m(t)$, the above inequality reduces to $H(x) \geq 0$ for all $x$ with $-1 \leq x \leq 1$, where

$$H(x) = 4(A-B)^2 m^4(L^2+G^2+2LGx)-(1+B^2+2Bx)((\beta+\gamma)^2+G^2+2(\beta+\gamma)Gx).$$

In view of (i), (iii), (iv) and the fact that $-1 \leq m \leq 1$, we see that $H''(x) = -8BG(\beta+\gamma) < 0$ and hence $H'(x) \leq H'(-1) = 8m^4 G(A-B)^2(\beta+\gamma+k) - 2(B-1)^2 G(\beta+\gamma) - 2B(-G+\beta+\gamma)^2 \leq 0$. Thus, $H(x) \geq H(1) = 4m^4(A-B)^2(\beta+\gamma+G+k)^2 - (B+1)^2(\beta+\gamma+G)^2 =: \psi(k)$. Using (iv), we observe that $\psi''(k) = 8m^4(A-B)^2 \geq 0$ and hence for $k \geq 1$, we have $\psi'(k) \geq \psi'(1) = 8m^4(A-B)^2(\beta+\gamma+G+1) \geq 0$. Thus by using (v), we get $H(x) \geq \psi(k) \geq \psi(1) = 4m^4(A-B)^2(\beta+\gamma+G+1)^2 - (B+1)^2(\beta+\gamma+G)^2 \geq 0$. This completes the proof. ∎

The next corollary is obtained by substituting $p(z) = zf'(z)/f(z)$ with $\gamma = 0$, $B = -1$ and $A = 1 - 2\alpha$, $(0 \leq \alpha < 1)$ in Theorem 2.15.

**Corollary 2.16.** *Let $1/2 < \alpha < 1$, $-1 \leq \beta < 0$ and $k \geq 1$ satisfy the conditions $(2\alpha^2+\alpha-3)^2\beta^2+(4\alpha^4-12\alpha^3+13\alpha^2+2\alpha-3)k^2+2(4\alpha^4-20\alpha^3+17\alpha^2+2\alpha-3)\beta k \leq 0$ and $(\alpha^2+2\alpha-1)\beta^2 \leq 4(\alpha-1)^2(2\alpha-1)\beta(\beta+k)$. If the function $f \in \mathcal{A}$ satisfies the subordination*

$$\frac{zf'(z)}{f(z)} + \frac{1}{\beta}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \varphi_{PAR}(z),$$

*then $f \in \mathcal{S}^*(\alpha)$.*

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