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# Weak GE-Morphisms and Qualified GE-Algebras

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**Abstract.** The notions of weak GE-morphism, weak GE-endomorphism are introduced and investigated its properties. Conditions for a self-map on a GE-algebra to be an idempotent and to be a weak GE-endomorphism are provided. The notion of qualified GE-algebra is introduced and its properties are investigated. Condition for a qualified GE-algebra to be an idempotent weak GE-endomorphism is given. The notion of qualified self-map is defined and a necessary and condition for a given self-map on a GE-algebra, the product of two GE-algebras under the qualified self-map to be a qualified GE-algebra is given. For a given qualified GE-algebra, using its kernel, an equivalence relation is induced, and then the quotient set is constructed. A binary operation is given on the quotient set to create a quotient qualified GE-algebra.

**Keywords:** GE-algebra; GE-filter; Weak GE-morphism; (Idempotent) Weak GE-endomorphism.

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## 1. Introduction

BCK-algebras (see [9, 10]) were introduced by Y. Imai and K. Iséki in 1966 as the algebraic semantics for a non-classical logic possessing only implication. Since then, the generalized concepts of BCK-algebras have been studied by various scholars. Hilbert algebras were introduced by L. Henkin and T. Skolem in the fifties for investigations in intuitionistic and other non-classical logics. A. Diego established that Hilbert algebras form a locally finite variety (see [7]). Later several researchers extended the theory on Hilbert algebras (see [5, 6, 8, 11, 13]). The notion of BE-algebra was introduced by H.S. Kim and Y.H. Kim as a generalization of a dual BCK-algebra (see [14]). A. Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see [16]). In the study of algebraic structures, the generalization process is also an important topic. As a generalization of Hilbert algebras, R.K. Bandaru et al. introduced the notion of GE-algebras, and investigated several properties (see [1]). A. Rezaei et al. introduced the concept of prominent GE-filters in GE-algebras and discussed its properties (see [17]). M.A. Ozturk et al. introduced the concept of Strong GE-filters, GE-ideals of bordered GE-algebras and investigated its properties (see [15]). S.Z. Song et al. introduced the concept of Imploring GE-filters of GE-algebras and discussed its properties (see [18]). The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches. Jun et al. derived isomorphism theorems by using Chinese Remainder Theorem in BCI-algebras (see [12]). Recently, R.K. Bandaru et al. introduced the notion of GE-morphism and established fundamental GE-morphism theorem. They investigated some isomorphism theorems in GE-algebras (see [3]).

In this paper, we introduce the notions of weak GE-morphism, weak GE-endomorphism and study its properties. We provide conditions for a given self-map on a GE-algebra to be an idempotent and to be a weak GE-endomorphism. We introduce the notion of qualified GE-algebra and investigate its properties. We give condition for a qualified GE-algebra to be an idempotent weak GE-endomorphism. We define the notion of qualified self-map and give a necessary and condition for a given self-map on a GE-algebra, the product of two GE-algebras under the qualified self-map to be a qualified GE-algebra. In qualified GE-algebras, we use kernel to induce an equivalence relation, and then construct the quotient set. We define a binary operation on the quotient set to construct a quotient qualified GE-algebra.

## 2. Preliminaries

**Definition 2.1.** [1] *By a GE-algebra we mean a nonempty set $X$ with a constant 1 and a binary operation "$*$" satisfying the following axioms:*
(GE1) $u * u = 1$,
(GE2) $1 * u = u$,
(GE3) $u * (v * w) = u * (v * (u * w))$

*for all* $u, v, w \in X$*.*

**Definition 2.2.** [1, 2] *A GE-algebra* $X$ *is said to be*

(i) *transitive if it satisfies:*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{1}$$

(ii) *antisymmetric if the binary relation "*$\leq$*" is antisymmetric.*

**Proposition 2.3.** [1] *Every GE-algebra* $X$ *satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2}$$
$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{3}$$
$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{4}$$
$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{5}$$
$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{6}$$
$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{7}$$
$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{8}$$
$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{9}$$

*If* $X$ *is transitive, then*

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{10}$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{11}$$
$$(\forall u, v, w \in X)\,(u \leq v,\, v \leq w \;\Rightarrow\; u \leq w)\,. \tag{12}$$

**Definition 2.4.** [1] *A subset* $F$ *of a GE-algebra* $X$ *is called a GE-filter of* $X$ *if it satisfies:*

$$1 \in F, \tag{13}$$
$$(\forall u, v \in X)(u \in F,\, u * v \in F \;\Rightarrow\; v \in F). \tag{14}$$

**Lemma 2.5.** [1] *In a GE-algebra* $X$*, every GE-filter* $F$ *of* $X$ *satisfies:*

$$(\forall x, y \in X)\,(x \leq y,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{15}$$

In [17], the concept of GE-morphisms in GE-algebras is defined as follows:

**Definition 2.6.** [17] *Let* $(X, *_X, 1_X)$ *and* $(Y, *_Y, 1_Y)$ *be GE-algebras. A mapping* $\xi : X \to Y$ *is called a GE-morphism if it satisfies:*

$$(\forall x_1, x_2 \in X)(\xi(x_1 *_X x_2) = \xi(x_1) *_Y \xi(x_2)). \tag{16}$$

If a GE-morphism $\xi : X \to Y$ is onto (resp., one-to-one), we say it is a *GE-epimorphism* (resp., *GE-monomorphism*). If a GE-morphism $\xi : X \to Y$ is both onto and one-to-one, we say it is a *GE-isomorphism.*

It is clear that the identity mapping $\xi : X \to X$ is a GE-isomorphism.

## 3. Weak GE-Morphisms

Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. Given a mapping $\xi : X \to Y$, consider the following condition:

$$(\forall x_1, x_2 \in X)(\xi(x_1 *_X x_2) \leq_Y \xi(x_1) *_Y \xi(x_2)). \tag{17}$$

If a mapping $\xi : X \to Y$ satisfies the condition (17), then $\xi(1_X) \leq_Y 1_Y$. In fact, $\xi(1_X) = \xi(x *_X x) \leq_Y \xi(x) *_Y \xi(x) = 1_Y$ for all $x \in X$. But, the inequality

$$1_Y \leq_Y \xi(1_X) \tag{18}$$

does not hold in general as seen in the following example.

*Example 3.1.* Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$*_X$" and "$*_Y$", respectively, which are given by Table 1.

Table 1: Cayley tables for the binary operations "$*_X$" and "$*_Y$"

| $*_X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

| $*_Y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 3 | 3 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 |

Then $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ are GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(x) = \begin{cases} 0 & \text{if } x \in \{0, 1, 2, 3\}, \\ 2 & \text{if } x = 4. \end{cases}$$

Then $\xi$ satisfies (17). But $\xi$ does not satisfy (18) since

$$1_Y *_Y \xi(1_X) = 1_Y *_Y 0 = 0 \neq 1_Y.$$

**Definition 3.2.** *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. A mapping $\xi : X \to Y$ is called a weak GE-morphism if it satisfies* (17) *and* (18).

If $X = Y$, the weak GE-morphism $\xi : X \to X$ is called a *weak GE-endomorphism.*

If a weak GE-morphism $\xi : X \to Y$ is onto (resp., one-to-one), we say it is a *weak GE-epimorphism* (resp., *weak GE-monomorphism*). If a weak GE-morphism $\xi : X \to Y$ is both onto and one-to-one, we say it is a *weak GE-isomorphism.*

It is clear that every GE-morphism is a weak GE-morphism. But the converse is not true in general as seen in the following example.

*Example 3.3.* Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$*_X$" and "$*_Y$", respectively, which are given by Table 2.

Table 2: Cayley tables for the binary operations "$*_X$" and "$*_Y$"

| $*_X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

| $*_Y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 0 | 4 |
| 3 | 1 | 1 | 2 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 |

Then $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ are GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(x) = \begin{cases} 0 & \text{if } x \in \{0, 2\}, \\ 1 & \text{if } x = 1, \\ 3 & \text{if } x = 3, \\ 4 & \text{if } x = 4. \end{cases}$$

Then $\xi$ is a weak GE-morphism. But $\xi$ is not a GE-morphism since

$$\xi(0 *_X 2) = \xi(2) = 0 \neq 1 = 0 *_Y 0 = \xi(0) *_Y \xi(2).$$

**Proposition 3.4.** *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. Given a weak GE-morphism $\xi : X \to Y$, we have*

(i) $\xi(1_X) = 1_Y$.

(ii) $(\forall x, y \in X)$ $(x \leq_X y \;\Rightarrow\; \xi(x) \leq_Y \xi(y))$.

(iii) $(\forall x, y \in X)$ $(\xi(x *_X y) \leq_Y \xi((x *_X y) *_X y) *_Y \xi(y))$.

(iv) *The set* $\operatorname{Ker}(\xi) := \{x \in X \mid \xi(x) = 1_Y\}$, *which is called the kernel of $\xi$, is a GE-filter of $X$.*

(v) *The inverse image $\xi^{-1}(F_Y)$ of a GE-filter $F_Y$ of $Y$ under $\xi$ is a GE-filter of $X$.*

*Proof.* (i) is obtained by the combination of (6) and (18).

(ii) Let $x, y \in X$ be such that $x \leq_X y$. Then $x *_X y = 1_X$, and so

$$1_Y = \xi(1_X) = \xi(x *_X y) \leq_Y \xi(x) *_Y \xi(y).$$

Hence $1_Y = \xi(x) *_Y \xi(y)$ by (6), and so $\xi(x) \leq_Y \xi(y)$.

(iii) Let $x, y \in X$. By (GE3), (GE1) and (2), we get

$$\begin{aligned} &(x *_X y) *_X (((x *_X y) *_X y) *_X y) \\ =&(x *_X y) *_X (((x *_X y) *_X y) *_X ((x *_X y) *_X y)) \\ =&(x *_X y) *_X *1_X = 1_X, \end{aligned}$$

and so $x *_X y \leq_X ((x *_X y) *_X y) *_X y$. It follows from (ii) and (17) that

$$\xi(x *_X y) \leq_Y \xi(((x *_X y) *_X y) *_X y) \leq_Y \xi((x *_X y) *_X y) *_Y \xi(y).$$

(iv) By (i), we get $1_X \in \mathrm{Ker}(\xi)$. Let $x, y \in X$ be such that $x \in \mathrm{Ker}(\xi)$ and $x *_X y \in \mathrm{Ker}(\xi)$. Then $\xi(x) = 1_Y$ and $\xi(x *_X y) = 1_Y$. Hence, which imply from (17) and (GE2) that

$$1_Y = \xi(x *_X y) \leq_Y \xi(x) *_Y \xi(y) = 1_Y *_Y \xi(y) = \xi(y).$$

Hence $\xi(y) = 1_Y$ by (6), that is, $y \in \mathrm{Ker}(\xi)$. Thus $\mathrm{Ker}(\xi)$ is a GE-filter of $X$.

(v) Let $F_Y$ be a GE-filter of $Y$. The result (i) induces $1_X \in \xi^{-1}(F_Y)$. Let $x, y \in X$ be such that $x \in \xi^{-1}(F_Y)$ and $x *_X y \in \xi^{-1}(F_Y)$. Then $\xi(x) \in F_Y$ and $\xi(x *_X y) \in F_Y$. It follows from Lemma 2.5 and (17) that $\xi(x) *_Y \xi(y) \in F_Y$. Thus $\xi(y) \in F_Y$ and so $y \in \xi^{-1}(F_Y)$. Therefore $\xi^{-1}(F_Y)$ is a GE-filter of $X$. ■

**Corollary 3.5.** *Let $\xi : X \to Y$ be a weak GE-morphism from a GE-algebra $(X, *_X, 1_X)$ to a GE-algebra $(Y, *_Y, 1_Y)$. Then*

$$(\forall x, y \in X)(x \in \mathrm{Ker}(\xi),\ x \leq_X y \ \Rightarrow\ y \in \mathrm{Ker}(\xi)). \tag{19}$$

**Theorem 3.6.** *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. If $\xi : X \to Y$ is a weak GE-monomorphism, then* $\mathrm{Ker}(\xi) = \{1_X\}$.

*Proof.* Assume that $\xi : X \to Y$ is a weak GE-monomorphism. If $x \in \mathrm{Ker}(\xi)$, then $\xi(x) = 1_Y = \xi(1_X)$ by Proposition 3.4(i), and so $x = 1_X$. Hence $\mathrm{Ker}(\xi) = \{1_X\}$. ■

The converse of Theorem 3.6 is not true in general as seen in the following example.

*Example 3.7.* Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$*_X$" and "$*_Y$", respectively, which are given by Table 3.

Table 3: Cayley tables for the binary operations "$*_X$" and "$*_Y$"

| $*_X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 2 | 1 | 1 |

| $*_Y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 1 | 4 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 2 | 3 | 1 |

Then $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ are GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(x) = \begin{cases} 0 & \text{if } x \in \{0, 2, 3, 4\}, \\ 1 & \text{if } x = 1. \end{cases}$$

Then $\xi$ is a weak GE-morphism and $\mathrm{Ker}(\xi) = \{1_X\}$. But $\xi$ is not a weak GE-monomorphism since $\xi(0) = 0 = \xi(2)$ but $0 \neq 2$.

We want to strengthen the conditions so that the converse of Theorem 3.6 can be established.

**Theorem 3.8.** *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras, and let $\xi : X \to Y$ be a weak GE-morphism. If $X$ is antisymmetric and $\mathrm{Ker}(\xi) = \{1_X\}$, then $\xi$ is a weak GE-monomorphism.*

*Proof.* Assume that $Y$ is antisymmetric and $\mathrm{Ker}(\xi) = \{1_X\}$. Let $x_1, x_2 \in X$ be such that $\xi(x_1) = \xi(x_2)$. Then $\xi(x_1 *_X x_2) = \xi(x_1) *_Y \xi(x_2) = 1_Y$, and thus $x_1 *_X x_2 \in \mathrm{Ker}(\xi) = \{1_X\}$, that is, $x_1 \leq_X x_2$. The similar way induces $x_2 \leq_X x_1$. Thus $x_1 = x_2$ by the antisymmetry of $X$, and therefore $\xi$ is a weak GE-monomorphism. ∎

**Definition 3.9.** *A weak GE-endomorphism $\xi$ on a GE-algebra $(X, *, 1)$ is said to be idempotent if $\xi^2(x) := (\xi \circ \xi)(x) = \xi(x)$ for all $x \in X$.*

*Example 3.10.* Consider a set $X := \{0, 1, 2, 3, 4\}$ with the binary operation "$*$", which is given by Table 4.
Then $(X, *, 1)$ is a GE-algebra. Let $\xi : X \to X$ be a mapping defined by

$$\xi(x) = \begin{cases} 4 & \text{if } x \in \{0, 3, 4\}, \\ 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2. \end{cases}$$

Then $\xi$ is an idempotent weak GE-endomorphism.

Table 4: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 4 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 3 | 3 |
| 3 | 1 | 1 | 2 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 |

**Proposition 3.11.** *Let $\xi$ be a weak GE-endomorphism on a GE-algebra $(X, *, 1)$. If $\xi$ is idempotent, then* $\mathrm{Ker}(\xi) \cap \xi(X) = \{1\}$.

*Proof.* Assume that $\xi$ is idempotent and let $x \in \mathrm{Ker}(\xi) \cap \xi(X)$. Then $\xi(x) = 1$ and there exists $y \in X$ such that $x = \xi(y)$. Hence

$$1 = \xi(x) = \xi(\xi(y)) = \xi(y) = x,$$

and therefore $\mathrm{Ker}(\xi) \cap \xi(X) = \{1\}$. ■

The following example shows that if $\xi$ is not idempotent, then Proposition 3.11 is not valid.

*Example 3.12.* Consider a set $X := \{0, 1, 2, 3, 4\}$ with the binary operation "$*$", which is given by Table 5.

Table 5: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 2 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra. Let $\xi : X \to X$ be a mapping defined by

$$\xi(x) = \begin{cases} 2 & \text{if } x = 0, \\ 1 & \text{if } x \in \{1, 2\}, \\ 4 & \text{if } x = 3, \\ 3 & \text{if } x = 4. \end{cases}$$

Then $\xi$ is a weak GE-endomorphism. We can observe that $\xi$ is not idempotent because of $\xi(\xi(0) = \xi(2) = 1 \neq 2 = \xi(0)$. Also, $\mathrm{Ker}(\xi) = \{1, 2\}$ and $\xi(X) = \{1, 2, 3, 4\}$. But $\mathrm{Ker}(\xi) \cap \xi(X) = \{1, 2\} \neq \{1\}$.

Given a self-map $\xi$ on a GE-algebra $(X, *, 1)$, consider the next assertions:

$$(\forall x, y \in X)(\xi(\xi(x) * \xi(y)) \leq \xi(x) * \xi(y)). \tag{20}$$

$$(\forall x, y \in X)(\xi(x * y) \leq \xi((x * y) * y) * \xi(y)). \tag{21}$$

By Proposition 3.4 (iii), every weak GE-endomorphism $\xi$ on a GE-algebra $(X, *, 1)$ satisfies the conditions (21).

**Question 3.13.** Does every weak GE-endomorphism $\xi$ on a GE-algebra $(X, *, 1)$ satisfy the condition (20)?

The answer to Question 3.13 is negative as seen in the following example.

*Example 3.14.* Consider a set $X := \{0, 1, 2, 3, 4\}$ with the binary operation "$*$", which is given by Table 6.

Table 6: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 | 1 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 4 | 1 | 1 | 3 | 4 |
| 3 | 0 | 1 | 2 | 1 | 0 |
| 4 | 1 | 1 | 2 | 3 | 1 |

Then $(X, *, 1)$ is a GE-algebra. Let $\xi : X \to X$ be a mapping defined by

$$\xi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in \{1, 2\}, \\ 2 & \text{if } x = 3, \\ 4 & \text{if } x = 4. \end{cases}$$

Then $\xi$ is a weak GE-endomorphism. But $\xi$ does not satisfy (20) because of

$$(\xi(\xi(2) * \xi(3))) * (\xi(2) * \xi(3)) = \xi(1 * 2) * (1 * 2) = \xi(2) * 2 = 1 * 2 = 2 \neq 1.$$

**Proposition 3.15.** *Every weak GE-endomorphism $\xi$ on a GE-algebra $(X, *, 1)$ satisfies the condition (20) when it is idempotent.*

*Proof.* Let $\xi : X \to X$ be an idempotent weak GE-endomorphism. Then

$$\xi(\xi(x) * \xi(y)) \leq \xi^2(x) * \xi^2(y) = \xi(x) * \xi(y)$$

for all $x, y \in X$. ■

Given a self-map $\xi$ on a GE-algebra $(X, *, 1)$, consider the next assertions:

$$(\forall x, y \in X)(\xi(\xi(x) * \xi(y)) = \xi(x) * \xi(y)). \tag{22}$$

$$(\forall x, y \in X)(\xi(x * y) = \xi((x * y) * y) * \xi(y)). \tag{23}$$

**Question 3.16.** Does every weak GE-endomorphism $\xi$ on a GE-algebra $(X, *, 1)$ satisfy the conditions (22) and (23)?

The answer to Question 3.16 is negative as seen in the following example.

*Example 3.17.* Consider the weak GE-endomorphism $\xi$ in Example 3.14. It does not satisfy (22) because of

$$(\xi(\xi(2) * \xi(3))) = \xi(1 * 2) = \xi(2) = 1 \neq 2 = 1 * 2 = \xi(2) * \xi(3).$$

Also, the weak GE-endomorphism $\xi$ in Example 3.12 does not satisfy (23) because of

$$\begin{aligned} \xi(2 * 3) = \xi(4) = 3 \neq 4 = \xi(3) &= 1 * \xi(3) = \xi(1) * \xi(3) \\ &= \xi(4 * 3) * \xi(3) = \xi((2 * 3) * 3) * \xi(3). \end{aligned}$$

**Proposition 3.18.** *Let $\xi$ be a self-map on a GE-algebra $(X, *, 1)$. If $\xi$ satisfies:*

$$(\forall x, y \in X)(\xi((x * y) * y) * \xi(y) \leq \xi(x * y)), \tag{24}$$

*then $\xi(1) = 1$. If $\xi$ satisfies (22) and (24), then $\xi^2(x) = \xi(x)$ for all $x \in X$.*

*Proof.* Assume that $\xi$ satisfies the condition (24). Using (GE1) and (24), we get

$$1 = \xi(1) * \xi(1) = \xi((1 * 1) * 1) * \xi(1) \leq \xi(1 * 1) = \xi(1),$$

and so $\xi(1) = 1$ by (6). If $\xi$ satisfies (22) and (24), then

$$\xi^2(x) = \xi(\xi(x)) = \xi(1 * \xi(x)) = \xi(\xi(1) * \xi(x)) = \xi(1) * \xi(x) = 1 * \xi(x) = \xi(x)$$

for all $x \in X$. ■

**Corollary 3.19.** *Let $\xi$ be a self-map on a GE-algebra $(X, *, 1)$. If $\xi$ satisfies (23), then $\xi(1) = 1$.*

We provide conditions for a self-map on a GE-algebra to be a weak GE-endomorphism.

**Proposition 3.20.** *Let $\xi$ be a self-map on a transitive GE-algebra $(X, *, 1)$. If $\xi$ satisfies* (23) *and*

$$(\forall x, y \in X)(x \leq y \ \Rightarrow \ \xi(x) \leq \xi(y)), \tag{25}$$

*then $\xi$ is a weak GE-endomorphism.*

*Proof.* Suppose $X$ is transitive and $\xi$ satisfies (23) and (25). Corollary 3.19 shows that $1 \leq \xi(1)$. By the combination of (8) and (25), we get $\xi(x) \leq \xi((x * y) * y)$ for all $x, y \in X$. It follows from (10) and (23) that

$$\xi(x * y) = \xi((x * y) * y) * \xi(y) \leq \xi(x) * \xi(y)$$

for all $x, y \in X$. Therefore $\xi$ is a weak GE-endomorphism. ∎

**Question 3.21.** If a self-map $\xi$ on a GE-algebra $(X, *, 1)$ satisfies (20), (21) and (25), then is $\xi$ a weak GE-endomorphism?

The next example verify that the answer to Question 3.21 is negative.

*Example 3.22.* Consider a set $X := \{0, 1, 2, 3, 4, 5\}$ with the binary operation "$*$", which is given by Table 7.

Table 7: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 4 | 4 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 5 | 5 | 5 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 0 | 1 | 1 |
| 5 | 0 | 1 | 1 | 0 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra. Let $\xi : X \to X$ be a mapping defined by

$$\xi(x) = \begin{cases} 2 & \text{if } \ x = 0, \\ 1 & \text{if } \ x = 1, \\ 3 & \text{if } \ x = 3, \\ 4 & \text{if } \ x \in \{2, 4, 5\} \end{cases}$$

Then $\xi$ satisfies satisfies (20), (21) and (25). But $\xi$ is not a weak GE-endomorphism because of

$$\xi(2 * 3) * (\xi(2) * \xi(3)) = \xi(5) * (4 * 3) = 4 * 0 = 0 \neq 1.$$

**Definition 3.23.** *A couple $(X, \xi)$ is called a qualified GE-algebra (briefly, qGE-algebra) if $(X, *, 1)$ is a GE-algebra and $\xi$ is a self-map on $X$ that satisfies conditions* (22), (23) *and* (25).

*Example 3.24.* Consider a set $X := \{0, 1, 2, 3, 4\}$ with the binary operation "$*$", which is given by Table 8.

Table 8: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 3 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 1 | 4 | 4 |
| 3 | 0 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra. Let $\xi : X \to X$ be a mapping defined by

$$\xi(x) = \begin{cases} 1 & \text{if } x \in \{0, 1\}, \\ 2 & \text{if } x = 2, \\ 4 & \text{if } x \in \{3, 4\}. \end{cases}$$

Then it is easy to verify that $(X, \xi)$ is a qGE-algebra.

**Theorem 3.25.** *Let $(X, \xi)$ be a qGE-algebra. If $X$ is transitive, then $\xi$ is an idempotent weak GE-endomorphism.*

*Proof.* This is induced by Propositions 3.18 and 3.20. ■

**Lemma 3.26.** [2] *Let $(X_1, *_1, 1_1)$ and $(X_2, *_2, 1_2)$ be GE-algebras with with binary relations $\leq_1$ and $\leq_2$, respectively, and consider $\tilde{X} := X_1 \times X_2$. Define a binary operation "$\tilde{*}$", the special element $\tilde{1}$ and a binary relation $\leq_{(1,2)}$ on $X$ as follows:*

$$(x_1, x_2) \,\tilde{*}\, (y_1, y_2) = (x_1 *_1 y_1,\ x_2 *_2 y_2), \tag{26}$$

$$\tilde{1} = (1_1, 1_2), \tag{27}$$

$$(x_1, x_2) \leq_{(1,2)} (y_1, y_2) \ \Leftrightarrow\ x_1 \leq_1 y_1,\ x_2 \leq_2 y_2 \tag{28}$$

*for all $(x_1, x_2), (y_1, y_2) \in \tilde{X}$. Then $(\tilde{X}, \tilde{*}, \tilde{1})$ is a GE-algebra.*

**Theorem 3.27.** *Let $(X_1, \xi_1)$ and $(X_2, \xi_2)$ be qGE-algebras. If we define a self-map $\tilde{\xi}$ on $\tilde{X}$ as follows:*

$$\tilde{\xi} : \tilde{X} \to \tilde{X},\ (x_1, x_2) \mapsto (\xi_1(x_1), \xi_2(x_2)), \tag{29}$$

*then* $(\tilde{X}, \tilde{\xi})$ *is a qGE-algebra.*

*Proof.* By Lemma 3.26, $(\tilde{X},\ \tilde{*},\ \tilde{1})$ is a GE-algebra. Let $(x_1, x_2), (y_1, y_2) \in \tilde{X}$. Then

$$\begin{aligned}
&\tilde{\xi}(\tilde{\xi}(x_1, x_2)\,\tilde{*}\,\tilde{\xi}(y_1, y_2))\\
=&\tilde{\xi}((\xi_1(x_1), \xi_2(x_2))\,\tilde{*}\,(\xi_1(y_1), \xi_2(y_2)))\\
=&\tilde{\xi}((\xi_1(x_1) * \xi_1(y_1)), (\xi_2(x_2) * \xi_2(y_2)))\\
=&(\xi_1(\xi_1(x_1) * \xi_1(y_1)), \xi_2(\xi_2(x_2) * \xi_2(y_2)))\\
=&(\xi_1(x_1) * \xi_1(y_1), \xi_2(x_2) * \xi_2(y_2))\\
=&(\xi_1(x_1), \xi_2(x_2))\,\tilde{*}\,(\xi_1(y_1), \xi_2(y_2))\\
=&\tilde{\xi}(x_1, x_2)\,\tilde{*}\,\tilde{\xi}(y_1, y_2)
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\xi}(((x_1, x_2)\,\tilde{*}\,(y_1, y_2))\,\tilde{*}\,(y_1, y_2))\,\tilde{*}\,\tilde{\xi}(y_1, y_2)\\
=&\tilde{\xi}((x_1 * y_1, x_2 * y_2)\,\tilde{*}\,(y_1, y_2))\,\tilde{*}\,\tilde{\xi}(y_1, y_2)\\
=&\tilde{\xi}((x_1 * y_1) * y_1, (x_2 * y_2) * y_2)\,\tilde{*}\,\tilde{\xi}(y_1, y_2)\\
=&(\xi_1((x_1 * y_1) * y_1), \xi_2((x_2 * y_2) * y_2))\,\tilde{*}\,(\xi_1(y_1), \xi_2(y_2))\\
=&(\xi_1((x_1 * y_1) * y_1) * \xi_1(y_1), \xi_2((x_2 * y_2) * y_2) * \xi_2(y_2))\\
=&(\xi_1(x_1 * y_1), \xi_2(x_2 * y_2))\\
=&\tilde{\xi}(x_1 * y_1, x_2 * y_2)\\
=&\tilde{\xi}((x_1, x_2)\,\tilde{*}\,(y_1, y_2)).
\end{aligned}$$

Assume that $(x_1, x_2) \leq_{(1,2)} (y_1, y_2)$. Then $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$, and so $\xi_1(x_1) \leq_1 \xi_1(y_1)$ and $\xi_2(x_2) \leq_2 \xi_2(y_2)$. It follows that

$$\begin{aligned}
\tilde{\xi}(x_1, x_2)\,\tilde{*}\,\tilde{\xi}(y_1, y_2) &= (\xi_1(x_1), \xi_2(x_2))\,\tilde{*}\,(\xi_1(y_1), \xi_2(y_2))\\
&= (\xi_1(x_1) * \xi_1(y_1),\ \xi_2(x_2) * \xi_2(y_2))\\
&= (1_1, 1_2) = \tilde{1},
\end{aligned}$$

that is, $\tilde{\xi}(x_1, x_2) \leq_{(1,2)} \tilde{\xi}(y_1, y_2)$. Therefore $(\tilde{X}, \tilde{\xi})$ is a qGE-algebra. ■

**Definition 3.28.** *Let $g$ be a self-map on a GE-algebra $(X, *, 1)$. A self-map $\xi_g$ on $(\tilde{X},\ \tilde{*},\ \tilde{1})$ given by*

$$(\forall (x, y) \in \tilde{X})(\xi_g(x, y) = (g(x), g(y))) \tag{30}$$

*is called a qualified self-map on $(\tilde{X},\ \tilde{*},\ \tilde{1})$ .*

*Example 3.29.* Consider a set $X = \{0, 1, 2\}$ with binary operations "$*$", which is given by Table 9.

Table 9: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 1 | 1 |

Then $(X, *, 1)$ is a GE-algebra. We can observe that

$$\tilde{X} := X \times X = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}$$

where $w_1 = (0, 0)$, $w_2 = (0, 1)$, $w_3 = (0, 2)$, $w_4 = (1, 0)$, $w_5 = (1, 1)$, $w_6 = (1, 2)$, $w_7 = (2, 0)$, $w_8 = (2, 1)$, and $w_9 = (2, 2)$. Define a binary operation "$\tilde{*}$" on $\tilde{X}$ by Table 10.

Table 10: Cayley table for the binary operation "$\tilde{*}$"

| $\tilde{*}$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $w_1$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ |
| $w_2$ | $w_4$ | $w_5$ | $w_6$ | $w_4$ | $w_5$ | $w_6$ | $w_4$ | $w_5$ | $w_6$ |
| $w_3$ | $w_4$ | $w_5$ | $w_5$ | $w_4$ | $w_5$ | $w_5$ | $w_4$ | $w_5$ | $w_5$ |
| $w_4$ | $w_2$ | $w_2$ | $w_2$ | $w_5$ | $w_5$ | $w_5$ | $w_8$ | $w_8$ | $w_8$ |
| $w_5$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ |
| $w_6$ | $w_1$ | $w_2$ | $w_2$ | $w_4$ | $w_5$ | $w_5$ | $w_7$ | $w_8$ | $w_8$ |
| $w_7$ | $w_2$ | $w_2$ | $w_2$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ | $w_5$ |
| $w_8$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_4$ | $w_5$ | $w_6$ |
| $w_9$ | $w_1$ | $w_2$ | $w_2$ | $w_4$ | $w_5$ | $w_5$ | $w_4$ | $w_5$ | $w_5$ |

Then $(\tilde{X}, \tilde{*}, \tilde{1})$, where $\tilde{1} = (1, 1) = w_5$, is a GE-algebra. Define a self-map $g$ on $(X, *, 1)$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in \{0, 2\}, \\ 1 & \text{if } x = 1. \end{cases}$$

Let $\xi_g : \tilde{X} \to \tilde{X}$ be a mapping defined by (30). Then

$$\xi_g(x) = \begin{cases} w_1 & \text{if } x \in \{w_1, w_3, w_7, w_9\}, \\ w_2 & \text{if } x \in \{w_2, w_8\}, \\ w_4 & \text{if } x \in \{w_4, w_6\}, \\ w_5 & \text{if } x = w_5, \end{cases}$$

and it is a qualified self-map on $(\tilde{X}, \tilde{*}, \tilde{1})$.

**Theorem 3.30.** *Given a self-map $g$ on a GE-algebra $(X, *, 1)$, let $\xi_g$ be a qualified self-map on $(\tilde{X}, \tilde{*}, \tilde{1})$ where $\tilde{X} = X \times X$. Then $(X, g)$ is a qGE-algebra if and only if $(\tilde{X}, \xi_g)$ is a qGE-algebra.*

*Proof.* Assume that $(X, g)$ is a qGE-algebra. Then $(\tilde{X}, \tilde{*}, \tilde{1})$ is a GE-algebra by Lemma 3.26. For every $(x_1, x_2), (y_1, y_2) \in \tilde{X}$, we have

$$\begin{aligned}
&\xi_g(\xi_g(x_1, x_2) \,\tilde{*}\, \xi_g(y_1, y_2)) \\
=&\xi_g((g(x_1), g(x_2)) \,\tilde{*}\, (g(y_1), g(y_2))) \\
=&\xi_g((g(x_1) * g(y_1)), (g(x_2) * g(y_2))) \\
=&(g(g(x_1) * g(y_1)), g(g(x_2) * g(y_2))) \\
=&(g(x_1) * g(y_1), g(x_2) * g(y_2)) \\
=&(g(x_1), g(x_2)) \,\tilde{*}\, (g(y_1), g(y_2)) \\
=&\xi_g(x_1, x_2) \,\tilde{*}\, \xi_g(y_1, y_2)
\end{aligned}$$

and

$$\begin{aligned}
&\xi_g(((x_1, x_2) \,\tilde{*}\, (y_1, y_2)) \,\tilde{*}\, (y_1, y_2)) \,\tilde{*}\, \xi_g(y_1, y_2) \\
=&\xi_g((x_1 * y_1, x_2 * y_2) \,\tilde{*}\, (y_1, y_2)) \,\tilde{*}\, \xi_g(y_1, y_2) \\
=&\xi_g((x_1 * y_1) * y_1, (x_2 * y_2) * y_2) \,\tilde{*}\, \xi_g(y_1, y_2) \\
=&(g((x_1 * y_1) * y_1), g((x_2 * y_2) * y_2)) \,\tilde{*}\, (g(y_1), g(y_2)) \\
=&(g((x_1 * y_1) * y_1) * g(y_1), g((x_2 * y_2) * y_2) * g(y_2)) \\
=&(g(x_1 * y_1), g(x_2 * y_2)) \\
=&\xi_g(x_1 * y_1, x_2 * y_2) \\
=&\xi_g((x_1, x_2) \,\tilde{*}\, (y_1, y_2)).
\end{aligned}$$

Suppose that $(x_1, x_2) \,\tilde{*}\, (y_1, y_2) = (1, 1)$. Then $(x_1 * y_1, x_2 * y_2) = (1, 1)$ and hence $x_1 * y_1 = 1$ and $x_2 * y_2 = 1$. Thus $g(x_1) * g(y_1) = 1$ and $g(x_2) * g(y_2) = 1$, which imply that

$$\begin{aligned}
\xi_g(x_1, x_2) \,\tilde{*}\, \xi_g(y_1, y_2) &= (g(x_1), g(x_2)) \,\tilde{*}\, (g(y_1), g(y_2)) \\
&= (g(x_1) * g(y_1), g(x_2) * g(y_2)) = (1, 1) = \tilde{1}.
\end{aligned}$$

Hence $(\tilde{X}, \xi_g)$ is a qGE-algebra.

Conversely, assume that $(\tilde{X}, \xi_g)$ is a qGE-algebra. Then

$$\begin{aligned}
(g(1), g(1)) &= \xi_g(1, 1) = \xi_g((1, 1) \,\tilde{*}\, (1, 1)) \\
&= \xi_g(((1, 1) \,\tilde{*}\, (1, 1)) \,\tilde{*}\, (1, 1)) \,\tilde{*}\, \xi_g(1, 1) \\
&= \xi_g((1 * 1, 1 * 1) \,\tilde{*}\, (1, 1)) \,\tilde{*}\, \xi_g(1, 1) \\
&= \xi_g((1, 1) \,\tilde{*}\, (1, 1)) \,\tilde{*}\, \xi_g(1, 1) \\
&= \xi_g((1 * 1, 1 * 1)) \,\tilde{*}\, \xi_g(1, 1) \\
&= \xi_g(1, 1) \,\tilde{*}\, \xi_g(1, 1) \\
&= (1, 1),
\end{aligned}$$

and so $g(1) = 1$. It follows from that

$$\begin{aligned}
(1, g(x) * g(y)) &= (g(x) * g(x), g(x) * g(y)) \\
&= (g(x), g(x)) \,\tilde{*}\, (g(x), g(y)) \\
&= \xi_g(x, x) \,\tilde{*}\, \xi_g(x, y) \\
&= \xi_g(\xi_g(x, x) \,\tilde{*}\, \xi_g(x, y)) \\
&= \xi_g((g(x), g(x)) \,\tilde{*}\, (g(x), g(y))) \\
&= \xi_g(g(x) * g(x), g(x) * g(y)) \\
&= \xi_g(1, g(x) * g(y)) \\
&= (g(1), g(g(x) * g(y))) \\
&= (1, g(g(x) * g(y)))
\end{aligned}$$

and

$$\begin{aligned}
(1, g(x * y)) &= (g(1), g(x * y)) \\
&= \xi_g(1, x * y) \\
&= \xi_g(1 * 1, x * y) \\
&= \xi_g((1, x) \,\tilde{*}\, (1, y)) \\
&= \xi_g(((1, x) \,\tilde{*}\, (1, y)) \,\tilde{*}\, (1, y)) \,\tilde{*}\, \xi_g(1, y) \\
&= \xi_g((1 * 1, x * y) \,\tilde{*}\, (1, y)) \,\tilde{*}\, \xi_g(1, y) \\
&= \xi_g((1 * 1) * 1, (x * y) * y) \,\tilde{*}\, \xi_g(1, y) \\
&= \xi_g(1, (x * y) * y) \,\tilde{*}\, \xi_g(1, y) \\
&= (g(1), g((x * y) * y)) \,\tilde{*}\, (g(1), g(y)) \\
&= (g(1) * g(1), g((x * y) * y) * g(y)) \\
&= (1, g((x * y) * y) * g(y))
\end{aligned}$$

for all $x, y \in X$. Hence $g(g(x)*g(y)) = g(x)*g(y)$ and $g((x*y)*y)*g(y)) = g(x*y)$ for all $x, y \in X$. Let $x, y \in X$ be such that $x \leq y$. Then

$$(1, x) \,\tilde{*}\, (1, y) = (1 * 1, x * y) = (1, 1) = \tilde{1}$$

and hence

$$\begin{aligned}
(1, g(x) * g(y)) &= (g(1) * g(1), g(x) * g(y)) \\
&= (g(1), g(x)) \,\tilde{*}\, (g(1), g(y)) \\
&= \xi_g(1, x) \,\tilde{*}\, \xi_g(1, y) \\
&= (1, 1)
\end{aligned}$$

which implies that $g(x) \leq g(y)$. Therefore $(X, g)$ is qGE-algebra. ■

For every qGE-algebra $(X, \xi)$, we define the image $\mathrm{Im}(\xi)$, kernel $\mathrm{Ker}(\xi)$ and diagonal set $\Delta(\xi)$ of $\xi$ as follows:

$$\mathrm{Im}(\xi) = \{\xi(x) \in X \mid x \in X\}, \tag{31}$$

$$\mathrm{Ker}(\xi) = \{x \in X \mid \xi(x) = 1\}, \tag{32}$$

$$\Delta(\xi) = \{x \in X \mid \xi(x) = x\}. \tag{33}$$

**Proposition 3.31.** *If* $(X, \xi)$ *is a qGE-algebra, then* $\mathrm{Im}(\xi)$ *is a sub-GE-algebra of* $X$*,* $\mathrm{Ker}(\xi) \cap \Delta(\xi) = \{1\}$ *and* $\mathrm{Ker}(\xi)$ *is a GE-filter of* $X$*.*

*Proof.* Let $x, y \in \mathrm{Im}(\xi)$. Then there exist $a, b \in X$ such that $x = \xi(a)$ and $y = \xi(b)$. Thus $x * y = \xi(a) * \xi(b) = \xi(\xi(a) * \xi(b)) \in \mathrm{Im}(\xi)$, and hence $\mathrm{Im}(\xi)$ is a sub-GE-algebra of $X$. Let $x \in \mathrm{Ker}(\xi) \cap \Delta(\xi)$. Then $x = \xi(x) = 1$ and so $\mathrm{Ker}(\xi) \cap \Delta(\xi) = \{1\}$. Since

$$1 = \xi(1) * \xi(1) = \xi(1 * 1) * \xi(1) = \xi((1 * 1) * 1) * \xi(1) = \xi(1 * 1) = \xi(1),$$

we have $1 \in \mathrm{Ker}(\xi)$. Let $x, y \in X$ be such that $x \in \mathrm{Ker}(\xi)$ and $x * y \in \mathrm{Ker}(\xi)$. Then $\xi(x) = 1$ and $\xi(x * y) = 1$. Since $x \le (x * y) * y$, we get

$$1 = \xi(x) \le \xi((x * y) * y)$$

by (25), and thus $\xi((x * y) * y) = 1$ by (6). It follows from (GE2) and (23) that

$$1 = \xi(x * y) = \xi((x * y) * y) * \xi(y) = 1 * \xi(y) = \xi(y).$$

Hence $y \in \mathrm{Ker}(\xi)$ and therefore $\mathrm{Ker}(\xi)$ is a GE-filter of $X$. ∎

Given a qGE-algebra $(X, \xi)$, let $\delta_{\mathrm{Ker}(\xi)}$ be a subset of $X \times X$ constructed to satisfy the following conditions:

$$(\forall x, y \in X)((x, y) \in \delta_{\mathrm{Ker}(\xi)} \;\Leftrightarrow\; x * y \in \mathrm{Ker}(\xi),\, y * x \in \mathrm{Ker}(\xi)). \tag{34}$$

It is routine to verify that $\delta_{\mathrm{Ker}(\xi)}$ is a congruence relation in $X$. Denote by $[x]_{\mathrm{Ker}(\xi)}$ the equivalence class of $x$ in $X$ under $\delta_{\mathrm{Ker}(\xi)}$, that is,

$$[x]_{\mathrm{Ker}(\xi)} := \{y \in X \mid (x, y) \in \delta_{\mathrm{Ker}(\xi)}\},$$

and the collection of all such equivalence classes is denoted by $X/\delta_{\mathrm{Ker}(\xi)}$, i.e.,

$$X/\delta_{\mathrm{Ker}(\xi)} = \{[x]_{\mathrm{Ker}(\xi)} \mid x \in X\}.$$

**Theorem 3.32.** *Let* $\delta_{\mathrm{Ker}(\xi)}$ *be a congruence relation in a qGE-algebra* $(X, \xi)$ *where* $X$ *is transitive and antisymmetric. Define a binary operation* $*_{\delta_{\mathrm{Ker}(\xi)}}$ *on* $X/\delta_{\mathrm{Ker}(\xi)}$ *and a self-map* $\tilde{\xi}$ *on* $X/\delta_{\mathrm{Ker}(\xi)}$ *as follows:*

$$[x]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [y]_{\mathrm{Ker}(\xi)} = [x * y]_{\mathrm{Ker}(\xi)} \tag{35}$$

*and*

$$\tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) = [\xi(x)]_{\mathrm{Ker}(\xi)} \tag{36}$$

*respectively, for all* $[x]_{\mathrm{Ker}(\xi)}, [y]_{\mathrm{Ker}(\xi)} \in X/\delta_{\mathrm{Ker}(\xi)}$. *Then* $(X/\delta_{\mathrm{Ker}(\xi)}, \tilde{\xi})$ *is a qGE-algebra with the constant* $[1]_{\mathrm{Ker}(\xi)}$.

*Proof.* Since $\mathrm{Ker}(\xi)$ is a GE-filter of $X$ by Proposition 3.31, it is routine to verify that $(X/\delta_{\mathrm{Ker}(\xi)}, *_{\delta_{\mathrm{Ker}(\xi)}}, [1]_{\mathrm{Ker}(\xi)})$ is a GE-algebra. Let $x, y \in X$ be such that $[x]_{\mathrm{Ker}(\xi)} = [y]_{\mathrm{Ker}(\xi)}$ in $X/\delta_{\mathrm{Ker}(\xi)}$. Then $(x, y) \in \delta_{\mathrm{Ker}(\xi)}$ and hence $x * y \in \mathrm{Ker}(\xi)$ and $y * x \in \mathrm{Ker}\xi$, that is, $\xi(x * y) = 1$ and $\xi(y * x) = 1$. Since $x \leq (x * y) * y$ and $X$ is transitive, we have $\xi(x) \leq \xi((x * y) * y)$ and so

$$1 = \xi(x * y) = \xi((x * y) * y) * \xi(y) \leq \xi(x) * \xi(y)$$

Hence $\xi(x) * \xi(y) = 1$. Similarly, we get $\xi(y) * \xi(x) = 1$. Thus $\xi(x) = \xi(y)$ since $X$ is antisymmetric. Therefore

$$\tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) = [\xi(x)]_{\mathrm{Ker}(\xi)} = [\xi(y)]_{\mathrm{Ker}(\xi)} = \tilde{\xi}([y]_{\mathrm{Ker}(\xi)})$$

which shows that $\tilde{\xi}$ is well-defined. Let $x, y \in X$ be such that $[x]_{\mathrm{Ker}(\xi)}, [y]_{\mathrm{Ker}(\xi)} \in X/\delta_{\mathrm{Ker}(\xi)}$. Then

$$\begin{aligned}
&\tilde{\xi}(\tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)})) = \tilde{\xi}\{[\xi(x)]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [\xi(y)]_{\mathrm{Ker}(\xi)}\} \\
=&\tilde{\xi}([\xi(x) * \xi(y)]_{\mathrm{Ker}(\xi)}) = [\xi(\xi(x) * \xi(y))]_{\mathrm{Ker}(\xi)} = [\xi(x) * \xi(y)]_{\mathrm{Ker}(\xi)} \\
=&[\xi(x)]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [\xi(y)]_{\mathrm{Ker}(\xi)} = \tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)})
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\xi}([x]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [y]_{\mathrm{Ker}(\xi)}) = \tilde{\xi}([x * y]_{\mathrm{Ker}(\xi)}) \\
=&[\xi(x * y)]_{\mathrm{Ker}(\xi)} = [\xi((x * y) * y) * \xi(y)]_{\mathrm{Ker}(\xi)} \\
=&[\xi((x * y) * y)]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [\xi(y)]_{\mathrm{Ker}(\xi)} \\
=&\tilde{\xi}([(x * y) * y]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)}) \\
=&\tilde{\xi}(([x]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [y]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} [y]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)}).
\end{aligned}$$

Let $x, y \in X$ be such that $[x]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [y]_{\mathrm{Ker}(\xi)} = [1]_{\mathrm{Ker}(\xi)}$. Then $[x*y]_{\mathrm{Ker}(\xi)} = [1]_{\mathrm{Ker}(\xi)}$, and so $\xi(x*y) = 1$. Since $\xi$ is a weak GE-endomorphism by Proposition 3.20, we have

$$\begin{aligned}
[1]_{\mathrm{Ker}(\xi)} &= [\xi(x * y)]_{\mathrm{Ker}(\xi)} \subseteq [\xi(x) * \xi(y)]_{\mathrm{Ker}(\xi)} \\
&= [\xi(x)]_{\mathrm{Ker}(\xi)} *_{\delta_{\mathrm{Ker}(\xi)}} [\xi(y)]_{\mathrm{Ker}(\xi)} \\
&= \tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)})
\end{aligned}$$

and so $\tilde{\xi}([x]_{\mathrm{Ker}(\xi)}) *_{\delta_{\mathrm{Ker}(\xi)}} \tilde{\xi}([y]_{\mathrm{Ker}(\xi)}) = [1]_{\mathrm{Ker}(\xi)}$. Therefore $(X/\delta_{\mathrm{Ker}(\xi)}, \tilde{\xi})$ is a qGE-algebra with the constant $[1]_{\mathrm{Ker}(\xi)}$. ■

## References


[1] R.K. Bandaru, A. Borumand Saeid, Y.B. Jun, On GE-algebras, *Bulletin of the Section of Logic* **50** (1) (2021) 81–96. https://doi.org/10.18778/0138-0680.2020.20

[2] R.K. Bandaru, A. Borumand Saeid, Y.B. Jun, Belligerent GE-filter in GE-algebras, *Journal of the Indonesian Mathematical Society* **28** (1) (2022) 31–43.

[3] R.K. Bandaru, M.K. Shaik, Y.B. Jun, GE-morphisms of GE-algebras, *Southeast Asian Bull. Math.* **45** (6) (2021) 931–944.

[4] A. Borumand Saeid, A. Rezaei, R.K. Bandaru, Y.B. Jun, Voluntary GE-filters and further results of GE-filters in GE-algebras, *Journal of Algebraic Systems* **10** (1) (2022) 31–47.

[5] S. Celani, A note on homomorphisms of Hilbert algebras, *International Journal of Mathematics and Mathematical Sciences* **29** (1) (2002) 55–61. DOI: 10.1155/S0161171202011134.

[6] I. Chajda, R. Halas, Y.B. Jun, Annihilators and deductive systems in commutative Hilbert algebras, *Commentationes Mathematicae Universitatis Carolinae* **43** (3) (2002) 407–417.

[7] A. Diego, *Sur les Algebres de Hilbert*, Collection de Logique Mathematique, Serie A, **XXI**, Gauthier-Villars Publisher, Paris, 1966.

[8] S.M. Hong and Y.B. Jun, On deductive systems of Hilbert algebras, *Communications of the Korean Mathematical Society* **11** (3) (1996) 595–600.

[9] Y. Imai and K. Iséki, On axiom system of propositional calculi, XIV, *Proceedings of the Japan Academy, Ser. A* **42** (1) (1966) 19–22.

[10] K. Iséki, An algebra related with a propositional calculus, *Proceedings of the Japan Academy, Ser. A* **42** (1) (1966) 26–29.

[11] Y.B. Jun, Commutative Hilbert algebras, *Soochow Journal of Mathematics* **22** (4) (1996) 477–484.

[12] Y.B. Jun, S.M. Hong, X.L. Xin, E.H. Roh, Chinese remainder theorems in BCI-algebras, *Soochow Jorunal of Mathematics* **24** (3) 1998 219–230.

[13] Y.B. Jun and K.H. Kim, H-filters of Hilbert algebras, *Scientiae Mathematicae Japonicae Online* **e-2005** 231–236.

[14] H.S. Kim and Y.H. Kim, On BE-algebras, *Scientiae Mathematicae Japonicae Online* **e-2006** 1299–1302.

[15] M.A. Ozturk, J.G. Lee, R.K. Bandaru, Y.B. Jun, Strong GE-filters and GE-ideals of bordered GE-algebras, *Journal of Mathematics* **2021**, Art. ID 5520023, 9 pages. https://doi.org/10.1155/2021/5520023.

[16] A. Rezaei, A. Borumand Saeid, R.A. Borzooei, Relation between Hilbert algebras and BE-algebras, *Applications and Applied Mathematics* **8** (2) (2013) 573–584.

[17] A. Rezaei, R.K. Bandaru, A. Borumand Saeid, Y.B. Jun, Prominent GE-filters and GE-morphisms in GE-algebras, *Afrika Matematika* **32** (2021) 1121–1136. https://doi.org/10.1007/s13370-021-00886-6

[18] S.Z. Song, R.K. Bandaru, Y.B. Jun, Imploring GE-filters of GE-algebras, *Journal of Mathematics* **2021**, Art. ID 6651531, 7 pages. https://doi.org/10.1155/2021/6651531.