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# Spectral and Distribution of Eigenvalues of Non-self-adjoint Differential Operators*

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**Abstract.** Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\partial\Omega$. In this article, in view of our earlier paper [12], let the conditions made on this paper be sufficiently more general than [12]. We investigate the spectral properties of a non-selfadjoint elliptic differential operator $(Pu)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x)a_{ij}(x)Q(x)u'_{x_i}(x)\right)'_{x_j}$, acting on Hilbert space $H_\ell = L^2(\Omega)^\ell$. Here $c|s|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j}$ $(s = (s_1, \ldots, s_n) \in \mathbf{C}^n,\ x \in \Omega)$, $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$, $a_{ij}(x) = \overline{a_{ji}(x)} \in C^2(\overline{\Omega})$, $0 \leq \alpha < 1$. Furthermore, suppose that $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega})$ $(1 \leq j \leq \ell)$ lie in the $\mathbf{C}\backslash\Phi$, where $\Phi = \{z \in \mathbf{C} :\ |arg\ z| \leq \varphi\}$, $\varphi \in (0, \pi)$.

**Keywords:** Resolvent; Asymptotic spectrum; Eigenvalues; Non-self-adjoint elliptic differential operators.

## 1. Introduction

Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^\infty$). We introduce the weighted Sobolev space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ as the space of complex

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value functions $u(x)$ defined on $\Omega$ with the finite norm:

$$|u|_+ = \left( \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x) |u'_{x_i}(x)|^2 dx + \int_\Omega |u(x)|^2 dx \right)^{1/2}.$$

We denote by $\overset{\circ}{\mathcal{H}}$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}$ with respect to the above norm. i.e., $\overset{\circ}{\mathcal{H}}$ is the closure of $C_0^\infty(\Omega)$ in $W^2_{2,\alpha}(\Omega)$. The notion $C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in $\Omega$. In this paper, we investigate the spectral properties, in particular we estimate the resolvent of a non-selfadjoint elliptic differential operator of type

$$(Pu)(x) = -\sum_{i,j=1}^{n} \left( \rho^{2\alpha}(x) a_{ij}(x) Q(x) u'_{x_i}(x) \right)'_{x_j} \tag{1}$$

acting on Hilbert space $H_\ell = L^2(\Omega)^\ell$ with Dirichlet-type boundary conditions. Here $\rho(x) = \text{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $a_{ij}(x) = \overline{a_{ji}(x)}$ $(i, j = 1, \ldots, n)$, $a_{ij}(x) \in C^2(\overline{\Omega})$ $(i, j = 1, \ldots, n)$, and the functions $a_{ij}(x)$ satisfies the uniformly elliptic condition, i.e., there exists $c > 0$ such that:

$$c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) s_i \overline{s_j} \quad (s = (s_1, \ldots, s_n) \in \mathbf{C}^n, \ \ x \in \Omega).$$

Furthermore, suppose that $Q(x) \in C^2(\overline{\Omega},\ End\ \mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega})$ $(1 \le j \le \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x), \ldots, \mu_\ell(x) \in \mathbf{C} \backslash \Phi,$$

where $\Phi = \{z \in \mathbf{C} : \ |arg\ z| \le \varphi\}$, $\varphi \in (0, \pi)$. (i.e., the eigenvalues $\mu_j(x)$ of $Q(x)$ lie on the complex plane and outside of the closed angle $\Phi$).

For a closed extension of the operator $A$ with respect to space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ above, we need to extend its domain to the closed domain

$$D(P) = \{y \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\ loc}(\Omega) : \ \sum_{i,j=1}^{n} \rho^{2\alpha} a_{ij} Q y'_{x_i})'_{x_j} \in H\},$$

(see [8]) where the local space $W^2_{2,\ loc}(\Omega)$ is the functions $u(x)$ $(x \in \Omega)$ in this form $W^2_{2,loc}(\Omega) = \{u(x) : \ \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty, \quad J \subset \Omega, \quad \text{open}\}$. Here, and in the sequel the value of the function arg $z \in (-\pi, \pi]$, and $\|T\|$ denotes the norm of the bounded operator $T : H \longrightarrow H$.

To get a feeling for the history of the subject under study, refer to our earlier papers [12, 13]. Indeed this paper was written in continuing on earlier our papers, the paper is sufficiently more general than earlier our papers, which here, we obtain the resolvent estimate of the operator $P$, that satisfying the special and general conditions, The paper consists of four sections. Section 1 is devoted to introduction. In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator $A$, acting on $H$ in the certain case (i.e., in this case, we will study Theorem 2.1 under assumption (3)). In Section 3, we have Theorem 3.1 on the resolvent estimate of the differential operator $A$, acting on $H$ in the general case (i.e., in this case, we will study Theorem 3.1 in contrast to Theorem 2.1. in other words, Theorem 3.1 does not include assumption (3) of Theorem 2.1). It is necessary to take note some remarks regarding Theorem 2.1 and Theorem 3.1: Theorem 3.1 follows from Theorem 2.1 by dropping assumption (3) from Theorem 2.1, and so another comment regarding the assertion of these two theorems: We will see that Theorem 2.1 under the assumption (3) leads to its assertion that includes two estimates (4) and (5). Meanwhile, Theorem 3.1 without including assumption (3) of Theorem 2.1, leads to its assertion that is similar to the assertion of Theorem 2.1, but asserts only statement (4) of Theorem 2.1, which becomes (11) (in other words now here, it is an open question arises for us, i.e., whether we can prove a theorem the same Theorem 2.1 for general case ? i.e., without condition (3), which its assertion includes two estimates (4) and (5)?). In Section 4, we have a general Theorem 4.1, i.e. in this theorem we let the operator $P$ in (1) acting on the general space $H_\ell = L^2(\Omega)^\ell$ and then by using the result of Theorem 2.1 we prove Theorem 4.1.

## 2. The Resolvent Estimate of Degenerate Elliptic Differential Operators on $H$ in Some Special Case

**Theorem 2.1.** *Let $A = P$ in (1), i.e., assume that the operator $A$ acting on Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions, and the sector $\Phi$ be defined as in Section 1. Let the complex function $q(x)$ satisfy the following conditions*

$$q(x) \in C^1(\overline{\Omega}),\ q(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}), \tag{2}$$

$$|arg\{q(x_1)q^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall\, x_1, x_2 \in \overline{\Omega}). \tag{3}$$

*Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in $H$, and the following estimates are valid*

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}\ (\lambda \in \Phi,\ |\lambda| > C_\Phi), \tag{4}$$

$$\|\rho^\alpha \frac{\partial}{\partial x_i}(A - \lambda I)^{-1}\| \leq M'_\Phi |\lambda|^{-\frac{1}{2}}\ (\lambda \in \Phi,\ |\lambda| > C_\Phi), \tag{5}$$

*for $i = 1, \ldots, n$ where $M_\Phi$, $C_\Phi > 0$ are sufficiently large numbers depending on $S$. The symbol $\|.\|$ stands for the norm of a bounded arbitrary operator $T$ in $H$.*

*Proof.* Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (4). So, as in Section 1 for a closed extension the operator $A$ (for more explain, see [8, Chap. 6]), we need to extend its domain to the closed set

$$D(A)=\left\{v\in \overset{\circ}{\mathcal{H}}\cap W^2_{2,\mathrm{loc}}(\Omega):\ \ hu'\in H,\quad (hqv')'\in H\right\}.$$

Let the operator $A$, now satisfy (2), (3). Then there exists a complex number $Z\in C$ (noticed that we can take $Z=e^{i\gamma}$, for a fix real $\gamma\in(-\pi,\pi]$), such that: $|Z=e^{i\gamma}|=1$, and so

$$c'\le Re\{Zq(x)\},\ \ c'|\lambda|\le -Re\{Z\lambda\},\quad c'>0\ (\forall\ x\in\overline{\Omega},\ \lambda\in\Phi). \tag{6}$$

In view of the uniformly elliptic condition, we have

$$c|s|^2=c\sum_{i=1}^n|s_i|^2\le\sum_{i,j=1}^n a_{ij}(x)s_i\overline{s_j},\ (c>0,\ s=(s_1,\ldots,s_n)\in\mathbf{C}^n,\ x\in\Omega)$$

Taking $s_i=y'_{x_i}$, we have $c\sum_{i=1}^n|y'_{x_i}(x)|^2\le\sum_{i,j=1}^n a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}$. From this, and according to $c'\le Re\{Zq(x)\}$ in (5), we then multiply these two positive relations with each other implies that

$$c_1\sum_{i=1}^n|y'_{x_i}(x)|^2\le ReZq(x)\sum_{i,j=1}^n a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)},\quad \text{for}\ \ y\in D(A)$$

Multiply both sides of the latter relation by the positive term $\rho^{2\alpha}(x)$, and then integrate from both sides, we will have

$$c_1\sum_{i=1}^n\int_\Omega\rho^{2\alpha}(x)|y'_{x_i}(x)|^2dx\le ReZ\sum_{i,j=1}^n\int_\Omega\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx.$$

Now by applying the integration by parts, and using Dirichlet-type condition, then the right sides of the latter relation without multiple $ReZ$ becomes:

$$\begin{aligned}&\sum_{i,j=1}^n\int_\Omega\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx\\ =&-\sum_{i,j=1}^n\int_\Omega(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j}\overline{y}(x)dx\\ =&\left(-\sum_{i,j=1}^n(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j}\ ,\ y(x)\right)=(Ay,y).\end{aligned} \tag{7}$$

Since $(Ay)(x)=-\sum_{i,j=1}^n\left(\rho^{2\alpha}(x)a_{ij}(x)q(x)u'_{x_i}(x)\right)'_{x_j}$.

Here, the the symbol (,) denotes the inner product in $H$.

Notice that the above equality in (7) obtains by the well known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation, see the well known Theorem 2.1, [8, Chap. 6]). This why we extend the domain of the operator $A$ to the closed domain in space $\mathcal{H}$ above. Therefore

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \le ReZ(Ay\ ,\ y)$$

from (5) we have: $c'|\lambda| \le -Re\{Z\lambda\},\ c' > 0,\ \forall\lambda \in \Phi$. Multiply this inequality by $\int_\Omega |y(x)|^2\, dx = (y\ ,\ y) = \|y\|^2 > 0$. It follows that

$$c'|\lambda| \int_\Omega |y(x)|^2\, dt \le -Re\{Z\lambda\}(y\ ,\ y).$$

From this and the above inequality we will have

$$\begin{aligned} & c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 dx \\ \le\ & Re\{Z(Ay, y) - Z\lambda(y, y)\} \\ =\ & Re\{Z((A - \lambda I)y, y)\} \\ \le\ & \|Z\|\|y\|\|(A - \lambda I)y\| \\ =\ & \|y\|\|(A - \lambda I)y\|; \end{aligned} \tag{8}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2\, dx \le \|y\|\ \|(A\ -\ \lambda\ I)y\|.$$

Since $c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx$ is positive, we will have either $c'|\lambda|\|y(x)\|^2 = |\lambda| \int_\Omega |y(x)|^2\, dx \le \|y\|\|(A\ \ -\lambda\ I)y\|$ or

$$|\lambda|\|y(x)\| \le M_\Phi\|(A\ \ -\lambda\ I)y\|. \tag{9}$$

This inequality ensures that the operator $(A\ -\ \lambda\ I)$ is one to one, which implies that $ker(A\ -\ \lambda\ I) = 0$. Therefore the inverse operator $(A -\ \ \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (4) of Theorem 2.1. To prove (4), we set $v = (A - \lambda I)^{-1} f,\ f \in H$ in (8) implies that

$$|\lambda| \int_\Omega |(A - \lambda I)^{-1} f|^2\, dx \le M_\Phi\|(A -\ \ \lambda I)^{-1} f\|\|(A\ \ -\lambda I)(A - \lambda I)^{-1} f\|.$$

Since $(A\ -\ \lambda\ I)(A - \lambda I)^{-1} f = I(f) = f$, we have

$$|\lambda| \int_\Omega |(A\ \ -\lambda I)^{-1} f|^2\, dx \le M_\Phi\|(A\ \ -\lambda I)^{-1} f\||f|.$$

So

$$|\lambda|\|(A\ \ -\lambda I)^{-1}(f)\|^2 \le M_\Phi\|(A\ \ -\lambda I)^{-1}(f)\||f|.$$

This implies that $|\lambda|\|(A \quad - \lambda I)^{-1}(f)\| \leq M_\Phi |f|$. Since $\lambda \neq 0$, we have $\|(A-\lambda I)^{-1}(f)\| \leq M_\Phi |\lambda|^{-1}|f|$; i.e., $\|(A-\lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}$. This estimate completes the proof of the assertion of Theorem 2.1 together with the estimate (4). Now, we start to prove the estimate (5) of Theorem 2.1 As in the above argument, we drop the positive term $c'|\lambda| \int_\Omega |y(x)|^2\, dx$ from

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2\, dx \leq \|y\|\, \|(A \ - \ \lambda\, I)y\|.$$

It follows that

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \leq \|y\|\, \|(A \ - \ \lambda\, I)y\|.$$

Equivalently

$$c_1 \left\| \rho^\alpha \frac{\partial}{\partial x_i}(A-\lambda I)^{-1} f \right\|^2 \leq \|y\|\, \|(A \ - \ \lambda\, I)y\|.$$

Setting $y = (A-\lambda I)^{-1} f,\ f \in H$ in the latter relation, and proceeding by similar calculation as in the proof (4) we then obtain:

$$c_1 \left\| \rho^\alpha \frac{\partial}{\partial x_i}(A-\lambda I)^{-1} f \right\|^2 \leq \|(A-\lambda I)^{-1} f\|\, \|(A \ - \ \lambda\, I)(A-\lambda I)^{-1} f\|.$$

Since $(A \ - \ \lambda\, I)(A-\lambda I)^{-1} f = I(f) = f$, we have

$$c_1 \left\| \rho^\alpha \frac{\partial}{\partial x_i}(A-\lambda I)^{-1} f \right\|^2 \leq \|(A-\lambda I)^{-1}\| \|f\|^2,$$

consequently, by (4) this implies that

$$c_1 \left\| \rho^\alpha \frac{\partial}{\partial x_i}(A-\lambda I)^{-1} f \right\|^2 \leq M_\Phi |\lambda|^{-1} \|f\|^2$$

to this end we will have

$$\|\rho^\alpha \frac{\partial}{\partial x_i}(A-\lambda I)^{-1}\| \leq M'_\Phi |\lambda|^{-\frac{1}{2}}.$$

Thus, here the proof of the estimate (5) is finished; i.e., this completes the proof of Theorem 2.1. ∎

Now let the condition (3) does not hold. Then we will have the following statement:

## 3. The Resolvent Estimate of Some Classes of Degenerate Elliptic Differential Operators on *H*

In this section, we will derive a new general theorem by dropping the assumption (3) from Theorem 2.1 in Section 2.

**Theorem 3.1.** *As in Section 1, let $\Phi$ be some closed sector with vertex at 0 in the complex plane (for more explain see [8]), and let the complex function $q(x)$ satisfy*

$$q(x) \in C^1(\overline{\Omega}),\ q(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}). \tag{10}$$

*Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse operator $(A-\lambda I)^{-1}$ exists and is continuous in $H$, and the following estimates holds:*

$$\|(A-\lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1},\ (\lambda \in \Phi,\ |\lambda| > C_\Phi) \tag{11}$$

*where $M_\Phi,\ C_\Phi > 0$ are sufficiently large numbers depending on $\Phi$.*

*Proof.* Suppose that (4) does not satisfy. To prove the assertion of Theorem 3.1 together with (11), we construct the functions $\varphi_1(x), \ldots, \varphi_m(x)$, $q_1(x), \ldots, q_m(x)$ so that each one of the functions $q_1(x), \ldots, q_m(x)$ $(x \in \overline{\Omega})$, as the function $q(x)$ in Theorem 2.1 satisfies (3). Therefore, let

$$\varphi_1(x), \ldots, \varphi_m(x),\ \ q_1(x), \ldots, q_m(x) \in C_0^\infty(\Omega),$$

satisfy

$$0 \leq \varphi_r(x),\ \ r = 1, \ldots, m,\ \ \varphi_1^2(x) + \ldots + \varphi_m^2(x) \equiv 1 \quad (x \in \overline{\Omega})$$

$$\frac{d}{dt}\varphi_r(x) \in C_0^\infty(\Omega),\ q_r(x) = q(x),\ \forall x \in supp\ \varphi_r$$

$$q_r(x) \in \mathbf{C}\backslash\Phi,\ (\forall x \in \overline{\Omega}),\ r = 1, \ldots, m.$$

$$|\arg\{q_r(x_1)q_r^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall\, x_1, x_2 \in supp\ \varphi_r), \quad r = 1, \ldots, m.$$

In view of Theorem 2.1, and by (4), and (5), setting $A_r = A$ in the definition of the differential operator, we have

$$A_r u(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x) a_{ij}(x) q_r(x) u'_{x_i}(x)\right)'_{x_j} \text{ acting on } H$$

where

$$D(A_r) = \left\{ u \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\ loc}(\Omega) :\ \sum_{i,j=1}^{n} (\rho^{2\alpha} a_{ij} q_r u'_{x_i})'_{x_j} \in H \right\}.$$

Due to the assertion of Theorem 2.1, for $0 \neq \lambda \in \Phi$ the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(\Omega)$, and satisfies

$$\|(A_r - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}, \qquad \left\|\rho^\alpha \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\right\| \leq M'_\Phi |\lambda|^{-\frac{1}{2}} \tag{12}$$

for $\lambda \in \Phi,\ |\lambda| > C_\Phi$, where $0 \neq \lambda \in \Phi$. Let us introduce

$$G(\lambda) = \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r, \tag{13}$$

Here $\varphi_r$ is the multiplication operator in $H$ by the function $\varphi_r(x)$. Consequently, it is easily verified that

$$\begin{aligned}(A - \lambda I)G(\lambda) \,=\, & I + \rho^{2\alpha-1}(x) \sum_{r=1}^{m} \beta_r(x)(A_r - \lambda I)^{-1}\varphi_r \\ & + \rho^{2\alpha}(x) \sum_{i=1}^{n} \sum_{r=1}^{m} \gamma_{i_r}(x) \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\varphi_r \end{aligned} \tag{14}$$

where $\beta_r, \gamma_{i_r} \in L_\infty(\Omega), supp\beta_r$ and supp $\gamma_{i_r}$ are contained in supp $\varphi_r$.

Let us take the right side of (14) equals to $I + T(\lambda)$ Thus, we will have

$$(A - \lambda I)G(\lambda) = I + T(\lambda). \tag{15}$$

Now according to Section 2 if we put $A = A_r$ for $r = 1, \ldots, m$ in (3) we will have

$$\|(A_r - \lambda I)^{-1}\| \leq M1_S |\lambda|^{-1}, \ \left\|\rho^\alpha \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\right\| \leq M'_\Phi \|\lambda\|^{-\frac{1}{2}}.$$

Owing to the definition of $T(\lambda)$ in the (14), it easily follows that

$$\|T(\lambda)\| \leq M_\Phi |\lambda|^{-\frac{1}{2}} (\lambda \in \Phi, \ \ |\lambda| > 1). \tag{16}$$

Since $|\lambda|$ is sufficiently large number easily implies that $\|T(\lambda)\| < \frac{1}{2} < 1$, from this and using the well known theorem in the operator theory we conclude that $I + T(\lambda)$ and so $(A - \lambda I)G(\lambda)$ are invertible. Hence, $((A - \lambda I)G(\lambda))^{-1}$ exists and equals to

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I \ + \ T(\lambda))^{-1}, \tag{17}$$

By adding $+I$ and $-I$ to the right side of the (16) it follows that

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I \ + \ T(\lambda))^{-1} \ \ - I \ \ + I.$$

We now set

$$F(\lambda) = (I \ + \ T(\lambda))^{-1} \ - \ I.$$

Then

$$(G(\lambda))^{-1}(A-\lambda I)^{-1} = I \ + \ F(\lambda).$$

In view of $\|T(\lambda)\| < 1$ and (16), we now estimate $F(\lambda)$ by the following geometric series:

$$\begin{aligned}\|F(\lambda)\| \ &\leq \sum_{i=2}^{+\infty} \|T^k(\lambda)\| \leq \|T(\lambda)\|^2(1+\|T(\lambda)\|+\|T(\lambda)\|^2+\ldots) \\ &\leq \|T(\lambda)\|^2 M_\Phi(1+1/2+\ldots) \leq 2M_\Phi(M'_\Phi|\lambda|^{-1/2})^2\end{aligned}$$

i.e., $\|F(\lambda)\| \leq 2M1_\Phi|\lambda|^{-1}$. By $\|(A_r-\lambda I)^{-1}\| \leq M1_\Phi|\lambda|^{-1}$, for we will have

$$\|G(\lambda)\| = \left\|\sum_{r=1}^{m}\varphi_r(A_r-\lambda I)^{-1}\varphi_r\right\| \leq M''_\Phi\|(A_r-\lambda I)^{-1}\| \leq M''_\Phi M1_\Phi|\lambda|^{-1};$$

i.e., $\|G(\lambda)\| \leq M2_\Phi|\lambda|^{-1}$. Now from (17) we have

$$(A-\lambda I)^{-1} = G(\lambda)(I \ + \ T(\lambda))^{-1} = G(\lambda)(I+F(\lambda)).$$

Therefore

$$\|(A-\lambda I)^{-1}\| = \|G(\lambda)\|\|(I \ + \ F(\lambda))\| \leq M2_\Phi|\lambda|^{-1}\|(1+2M1_\Phi|\lambda|^{-1});$$

i.e., here the assertion of Theorem 3.1 is proved. Therefore to complete the proof Theorem 3.1 we must prove the estimate in (11). Finally, according to latter inequality, we have

$$\|(A-\lambda I)^{-1}\| \leq M2_\Phi|\lambda|^{-1} + 2M2_\Phi M1_\Phi|\lambda|^{-1}|\lambda|^{-1},$$

and since $|\lambda|^{-1}|\lambda|^{-1} = |\lambda|^{-2} \leq |\lambda|^{-1}$, it follows that

$$\|(A-\lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}, \ \ (|\lambda| \geq C, \qquad \lambda \in \Phi).$$

This complete the proof. ∎

## 4. On the Resolvent Estimate of the Differential Operator in $H_\ell$

As in Section 1, let the differential operator

$$(Pu)(x) = -\sum_{i,j=1}^{n}\left(\rho^{2\alpha}(x)a_{ij}(x)Q(x)u'_{x_i}(x)\right)'_{x_j}$$

acting on Hilbert space $H_\ell = L^2(\Omega)^\ell$ with Dirichlet-type boundary conditions, and suppose that $Q(x) \in C^2(\overline{\Omega}, \ End\ \mathbf{C}^\ell)$ such that for each $x \in \overline{\Omega}$ the matrix function $Q(x)$ has non-zero simple eigenvalues $\mu_j(x) \in C^2(\overline{\Omega}) \ \ (1 \leq j \leq \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x),\ldots,\mu_\ell(x) \in \mathbf{C}\backslash\Phi, \tag{18}$$

where

$$\Phi = \{z \in \mathbf{C}: \ |arg\ z| \leq \varphi\}, \ \varphi \in (0, \pi).$$

Furthermore suppose that for $j = 1, \ldots, \ell$ we have

$$\mu_j(x) \in C^1(\overline{\Omega}), \ \mu_j(x) \in \mathbf{C} \backslash \Phi, \ (\forall x \in \overline{\Omega}), \tag{19}$$

$$|arg\{\mu_j(x_1)\mu_j^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall \, x_1, x_2 \in \overline{\Omega}). \tag{20}$$

Now according to Theorem 2.1, but here instead of the operator $A$ which acting on the space $H = L^2(\Omega)$, let the operator $P$ acting on the space $H_\ell = L^2(\Omega)^\ell$, now by the assumption of Section 1, we will have the following theorem in the general case:

**Theorem 4.1.** *Let* (19), (20) *and the assumptions of Section* 1 *hold for the operator* $P$ *as in* $(1,1)$*, then for sufficiently large in modulus* $\lambda \in \Phi$*, the inverse operator* $(P - \lambda I)^{-1}$ *exists and is continuous in the space* $H_\ell = L^2(\Omega)^\ell$ *and the following estimate holds:*

$$\|(P - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \tag{21}$$

*where* $M_\Phi, \ C_\Phi > 0$ *is sufficiently large number depending on* $\Phi$ *and* $|\lambda| > C_\Phi$*.*

*Proof.* Now by applying the eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ of the matrix function $Q(x)$ we defined the operators $P_1, \ldots, P_\ell$ such that

$$(P_j u)(x) = -\sum_{i,j=1}^{n} \left(\rho^{2\alpha}(x) a_{ij}(x) \mu_j(x) u'_{x_i}(x)\right)'_{x_j} \quad (j = 1, \ldots, \ell),$$

where its extension domains are

$$D(P_j) = \left\{ y \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,\,loc}(\Omega) : \ \sum_{i,j=1}^{n} \rho^{2\alpha} a_{ij} \mu_j(x) y'_{x_i})'_{x_j} \in H \right\},$$

which as the operator $A$ in Theorem 2.1, the operators $P_j, \ \ j = 1, \ldots, \ell$, acting on space $H = L^2(\Omega)$ (Notice that here the operators $P_j$ are the same of the operator $A$ in Section 2, i.e., to define the operators $P_j$, we just change the function $q(x)$ in the operator $A$ by the eigenvalues functions $\mu_j(x), \ \ j = 1, \ldots, \ell$ of matrix $Q(x)$).

The conditions which we consider on the eigenvalues $\mu_j(x)$ of the matrix function $Q(x)$ in Section 1 guarantee that one can convert the matrix $Q(x)$ to the diagonal form

$$Q(x) = U(x)\Lambda(x)U^{-1}(x), \quad \text{for} \quad U(x), U^{-1}(x) \in C^2([0,1], End\, \mathbf{C}^\ell)$$

where

$$\Lambda(x) = diag\{\mu_1(x), \ldots, \mu_\ell(x)\}.$$

Consider space $H_\ell = H \oplus \cdots \oplus H \quad (\ell\text{-times})$. Put

$$\Gamma(\lambda) = UB(\lambda)U^{-1},$$

where the operator

$$B(\lambda) = diag\{(P_1 - \lambda I)^{-1}, \ldots, (P_\ell - \lambda I)^{-1}\}.$$

acting on the direct sum

$$H_\ell = H \oplus \cdots \oplus H \quad (\ell\text{-times})$$

where $\lambda \in \overline{\Phi}\backslash R_+,\ |\lambda| \geq C_0$ and $(Uu)(x) = U(x)u(x),\ (u \in H_\ell)$.

Consequently, as the relation (14) in Section 3, we will have

$$(A - \lambda I)\Gamma(\lambda) = I + \rho^{2\alpha-1}(x)q_0(x)\beta(\lambda)U^{-1} + \rho^{2\alpha}(x)\sum_{i=1}^{n} q_i(x)\frac{\partial}{\partial x_i}\beta(\lambda)U^{-1}; \quad (22)$$

for $q_i \in C(\bar{\Omega}; End\mathbf{C}^l),\ \ i = 0, 1, \ldots, n$. Now as in section 3, let us take the right side of (22) equals to $I + T'(\lambda)$ Thus, we will have

$$(P - \lambda I)\Gamma(\lambda) = I + T'(\lambda). \quad (23)$$

Now according to Section 2 if we put $P_j = A$ for $j = 1, \ldots, \ell$ in (4) and (5) we will have

$$\|(P_j - \lambda I)^{-1}\| \leq M1_\Phi|\lambda|^{-1}, \ \left\|\rho^\alpha \frac{\partial}{\partial x_i}(P_j - \lambda I)^{-1}\right\| \leq M'_\Phi\|\lambda\|^{-\frac{1}{2}}.$$

Owing to the latter relations and the definition of $T'(\lambda)$ in (22) easily it follows that

$$\|T'(\lambda)\| \leq M_\Phi|\lambda|^{-\frac{1}{2}}(\lambda \in \Phi,\ \ |\lambda| > 1). \quad (24)$$

Since $|\lambda|$ is sufficiently large number easily implies that $\|T'(\lambda)\| < \frac{1}{2} < 1$, from this and using the well known theorem in the operator theory we conclude that $I + T'(\lambda)$ and so $(P - \lambda I)\Gamma(\lambda)$ are invertible. Hence, $((A - \lambda I)\Gamma(\lambda))^{-1}$ exists and equals to

$$(\Gamma(\lambda))^{-1}(P - \lambda I)^{-1} = (I\ +\ T'(\lambda))^{-1}, \quad (25)$$

By adding $+I$ and $-I$ to the right side of the (24) it follows that

$$(\Gamma(\lambda))^{-1}(P - \lambda I)^{-1} = (I\ +\ T'(\lambda))^{-1}\ \ -I\ \ +I.$$

We now set

$$F'(\lambda) = (I\ +\ T'(\lambda))^{-1}\ -\ I.$$

Then

$$(\Gamma(\lambda))^{-1}(A - \lambda I)^{-1} = I\ +\ F'(\lambda).$$

In view of $\|T'(\lambda)\| < 1$ and (24), we now estimate $F'(\lambda)$ by the following geometric series:

$$\|F'(\lambda)\| \leq \sum_{i=2}^{+\infty} \|T'^k(\lambda)\| \leq \|T'(\lambda)\|^2(1 + \|T'(\lambda)\| + \|T'(\lambda)\|^2 + \ldots)$$
$$\leq \|T'(\lambda)\|^2 M_\Phi(1 + 1/2 + \ldots) \leq 2M_\Phi(M'_\Phi|\lambda|^{-1/2})^2$$

i.e., $\|F'(\lambda)\| \leq 2M1_\Phi|\lambda|^{-1}$. By $\|(P_j - \lambda I)^{-1}\| \leq M1_\Phi|\lambda|^{-1}$, $j = 1, \ldots, \ell$ in view of definition $B(\lambda)$ and $\Gamma(\lambda)$ we will have

$$\|(\Gamma(\lambda)\| \leq M2_\Phi|\lambda|^{-1}.$$

Now from (25) we have

$$\|(P - \lambda I)^{-1}\| = \|(\Gamma(\lambda)\|(I \ + \ T'(\lambda))^{-1} = \|(\Gamma(\lambda)\|(I + F'(\lambda))\|.$$

Therefore

$$\|(P - \lambda I)^{-1}\| = \|(\Gamma(\lambda)\|\|(I \ + \ F'(\lambda))\| \leq M2_\Phi|\lambda|^{-1}\|(1 + 2M1_\Phi|\lambda|^{-1}).$$

To the end according to latter inequality we have

$$\|(P - \lambda I)^{-1}\| \leq M2_\Phi|\lambda|^{-1} + 2M2_\Phi M1_\Phi|\lambda|^{-1}|\lambda|^{-1},$$

and since $|\lambda|^{-1}|\lambda|^{-1} = |\lambda|^{-2} \leq |\lambda|^{-1}$, it follows that

$$\|(P - \lambda I)^{-1}\| \leq M_\Phi|\lambda|^{-1}, \ \ (|\lambda| \geq C, \quad \lambda \in \Phi).$$

This completes the proof. ∎

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