Southeast Asian Bulletin of Mathematics (2023) 47: 625–636

Southeast Asian
Bulletin of
Mathematics

A Note on a System of General Mixed Variational Inequalities

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Received 26 March 2021
Accepted 13 November 2021

Communicated by W.S. Cheung

AMS Mathematics Subject Classification(2020): 47J20, 46E20, 54A20

Abstract. In this paper, we show that the results obtained in [7]: On a system of general mixed variational inequalities are incorrect. Also we suggest and analyze new approximation schemes 4.1 for solving the (SGMVID) which were introduced by M.A. Noor in [7].

Keywords: General explicit iteration algorithms; System of general variational inequalities with different mappings; Relaxed (α, β)-cocoercive mappings; Lipschitzian continuous; Hilbert spaces.

1. Introduction

It is known that variational inequality theory and complementarity problem are very powerful tools of the current mathematical technology. In recent years, classical variational inequalities and complementarity problems have been extended and generalized to study a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. (see [9, 5, 4, 2, 1] and the reference therein). In this paper we introduce and study new approximation schemes to correct the main result of [7].

Throughout this paper, let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively, a surjective map $g : H \to H$, and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function.

For given nonlinear operators T_1, T_2: $H \to H$, we consider the problem of finding $(x^*, y^*) \in H \times H$ such that:

$$\begin{cases} \langle \rho T_1 (y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq \rho\varphi (x^*) - \rho\varphi (g(v)), \\ \langle \eta T_2 (x^*, y^*) + y^* - g(x^*), g(v) - y^* \rangle \geq \eta\varphi (y^*) - \eta\varphi (g(v)), \end{cases}$$

for all $v \in H, \rho > 0, \eta > 0$, which is called the system of general mixed variational inequalities involving three different nonlinear operators (SGMVID). In this paper new approximation schemes 4.1 are discussed for solving the problem (SGMVID).

2. Preliminaries

We recall the following notations.

Definition 2.1. *A mapping $T : H \to H$ is called λ-Lipschitz continous if there exist constant $\lambda > 0$, such that:*

$$\forall x, y \in H : \|T(x) - T(y)\| \leq \lambda \|x - y\|.$$

Definition 2.2. *A mapping $T : H \to H$ is called rlaxed (α, β)-cocoercive if there exist constants $\alpha > 0$, $\beta > 0$ such that:*

$$\forall x, y \in H : \langle T(x) - T(y), x - y \rangle \geq -\alpha \|T(x) - T(y)\|^2 + \beta \|x - y\|^2.$$

Proposition 2.3. [4, 9] *For given an element $u \in H$, $z \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle + \rho\varphi (v) - \rho\varphi (u) \geq 0, \forall v \in H,$$

if and only if

$$u = J_\varphi^\rho (z),$$

where $J_\varphi^\rho = (I + \rho\partial\varphi)^{-1}$, is the resolvent operator and $\partial\varphi$ denotes the subdifferential of a proper convex lower semcontinuous function.

It is known that J_φ^ρ is a nonexpansive mapping, i.e.

$$\left\|J_\varphi^\rho (x) - J_\varphi^\rho (y)\right\| \leq \|x - y\|, \forall x, y \in H.$$

Using Proposition 2.3, we can easily show that, finding the solution $x^*, y^* \in H$ of SGMVID is equivalent to finding $x^*, y^* \in H$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n J_\varphi^\rho [g(y^*) - \rho T_1 (y^*, x^*)], \\ y^* = (1 - \alpha_n) y^* + \alpha_n J_\varphi^\eta [g(x^*) - \eta T_2 (x^*, y^*)], \end{cases}$$

where $\alpha_n \in [0,1]$, for all $n \geq 0$.

In the following section, we show that the proof of M.A. Noor in [7] is incorrect.

3. Error in [7]

M.A. Noor used the following iterative algorithm for solving the problem (SGMVID)

Algorithm 3.1. [7, Algorithm 3.1] *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n J_\varphi \left[g(y_n) - \rho T_1(y_n, x_n) \right], \\ y_{n+1} = J_\varphi \left[g(x_{n+1}) - \eta T_2(x_{n+1}, y_n) \right], \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Theorem 3.2. [7, Theorem 4.1] *Let (x^*, y^*) be the solution of SGMVID. If $T_1 : H \times H \to H$ is relaxed $(\gamma_1, r_1)-$cocoercive and μ_1 Lipschitzian in the first variable, and $T_2 : H \times H \to H$ is relaxed $(\gamma_2, r_2)-$cocoercive and μ_2 Lipschitzian in the first variable. Let g be a relaxed $(\gamma_3, r_3)-$cocoercive and μ_3 Lipschitzian. If*

$$\left| \rho - \frac{r_1 - \gamma_1 \mu_1^2}{\mu_1^2} \right| < \frac{\sqrt{\left(r_1 - \gamma_1 \mu_1^2\right)^2 - \mu_1^2 k\,(2-k)}}{\mu_1^2}, \tag{1}$$

$$r_1 > \gamma_1 \mu_1^2 + \mu_1 \sqrt{k\,(2-k)}, k < 1, \tag{2}$$

$$\left| \eta - \frac{r_2 - \gamma_2 \mu_2^2}{\mu_2^2} \right| < \frac{\sqrt{\left(r_2 - \gamma_2 \mu_2^2\right)^2 - \mu_2^2 k\,(2-k)}}{\mu_2^2}, \tag{3}$$

$$r_2 > \gamma_2 \mu_2^2 + \mu_2 \sqrt{k\,(2-k)}, k < 1, \tag{4}$$

where

$$k = \sqrt{1 - 2\,(r_3 - \gamma_3 \mu_3^2) + \mu_3^2},$$

and $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in H$, x_n and y_n obtained from Algorithm 4.1 converge strongly to x^* and y^* respectively.

Next we will prove that his proof of this theorem is incorrect. Let us consider the following text quoted from the proof of Theorem 4.1 in [7].

Proof of Theorem 3.2. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$. From (1), (3), and the nonexpansive property of the resolvent operator J_φ, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\| \\
= & \ \| (1-\alpha_n) x_n + \alpha_n J_\varphi [g(y_n) - \rho T_1(y_n, x_n)] - (1-\alpha_n) x^* \\
- & \ \alpha_n J_\varphi [g(y^*) - \rho T_1(y^*, x^*)] \| \\
\leq & \ (1-\alpha_n) \|x_n - x^*\| \\
+ & \ \alpha_n \|J_\varphi [g(y_n) - \rho T_1(y_n, x_n)] - J_\varphi [g(y^*) - \rho T_1(y^*, x^*)]\| \\
\leq & \ (1-\alpha_n) \|x_n - x^*\| + \alpha_n \|[g(y_n) - \rho T_1(y_n, x_n)] - [g(y^*) - \rho T_1(y^*, x^*)]\| \\
= & \ (1-\alpha_n) \|x_n - x^*\| + \alpha_n \|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x^*)]\| \\
& + \alpha_n \|y_n - y^* - [g(y_n) - g(y^*)]\| .
\end{aligned}$$

From the relaxed (γ_1, r_1) −cocoercive and μ_1 Lipschitzian definition in the first variable on T_1, we have:

$$\begin{aligned}
& \|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x^*)]\|^2 \\
= & \ \|y_n - y^*\|^2 - 2\rho \langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^* \rangle \\
& + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
\leq & \ \|y_n - y^*\|^2 + 2\rho\gamma_1 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 - 2\rho r_1 \|y_n - y^*\|^2 \\
& + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
= & \ \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right] \|y_n - y^*\|^2 .
\end{aligned}$$

■

Question 3.3. How do the sequence x_n and x^* disappear from the above inequality? *Answer:*

(i) The author misuse the concept of relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition in the first variable on T_1.

(ii) There are a clear mistakes in the above formulation so it is not true because:

- (a) We cannot apply the relaxed (γ_1, r_1)-cocoercive definition for the first variable on T_1, (the second variable of T_1 in $\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^* \rangle$ is not equal).
- (b) Also with $\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2$, we cannot apply the Lipschitz continuity definition for the first variable on T_1.

(iii) The same error used when he evaluated $\|x_{n+1} - x^* - \eta[T_2(x_{n+1}, y_n) - T_2(x^*, y^*)]\|^2$.

Clarification 3.4. for n fixed, let f and h define as follows:

$$f : \begin{array}{l} H \to H \\ x \to f(x) = T_1(x, x_n), \end{array} \qquad h : \begin{array}{l} H \to H \\ x \to h(x) = T_1(x, x^*). \end{array}$$

It is clear that $f \neq h$, h is another function that defers to f.

So we cannot apply the relaxed (γ_1, r_1) −cocoercive definition for the first variable on T_1, because:

$$\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^* \rangle = \langle f(y_n) - h(y^*), y_n - y^* \rangle.$$

Also we cannot apply the Lipschitz continuity definition for the first variable on T_1, because:

$$\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 = \|f(y_n) - h(y^*)\|^2.$$

For this reason we propose another algorithm to correct this error.

4. Main Result

Now we suggest and analyze the following iterative method for solving the SG-MVID.

Algorithm 4.1. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_\varphi^\rho [g(y_n) - \rho T_1(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n J_\varphi^\eta [h(x_n) - \eta T_2(x_n, y_n)], \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Special Case. For $T_1 = T_2 = T$ in Algorithm 4.1, we arrive at

Algorithm 4.2. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_\varphi^\rho [g(y_n) - \rho T(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n J_\varphi^\eta [g(x_n) - \eta T(x_n, y_n)], \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Which is the aproximate solvability of the following system:

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq \rho\varphi(x^*) - \rho\varphi(g(x)), \forall v \in H, \rho > 0, \\ \langle \eta T(x^*, y^*) + y^* - g(x^*), g(v) - y^* \rangle \geq \eta\varphi(y^*) - \eta\varphi(g(x)), \forall v \in H, \eta > 0. \end{cases}$$

Now we present the convergence criteria of Algorithm 4.1 under some suitable conditions and this is the main result of this paper.

Theorem 4.3. *Let (x^*, y^*) be the solution of SGMVID. Suppose that $T_1 : H \times H \to H$ is relaxed (γ_1, r_1) −cocoercive and μ_1-Lipschitzian in the first* *variable and let T_1 be λ_1-Lipschitz continous in the second variable. Let $T_2 : H \times H \to H$ be relaxed $(\gamma_2, r_2)-$cocoercive and μ_2 Lipschitzian in the first variable and let T_2 be λ_2-Lipschitz continous in the second variable. Let g be relaxed $(\gamma_3, r_3)-$cocoercive and μ_3 Lipschitz. If*

$$\begin{cases} k < \dfrac{1}{2}, r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{\dfrac{3}{4} - k^2 + k}, \\ \left|\rho - \dfrac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \dfrac{\sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2\left[\dfrac{3}{4} - k^2 + k\right]}}{\mu_1^2}, \rho < \dfrac{1}{2\lambda_1}, \end{cases} \tag{5}$$

$$\begin{cases} k < \dfrac{1}{2}, r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{\dfrac{3}{4} - k^2 + k}, \\ \left|\eta - \dfrac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \dfrac{\sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \mu_2^2\left[\dfrac{3}{4} - k^2 + k\right]}}{\mu_2^2}, \eta < \dfrac{1}{2\lambda_2}, \end{cases} \tag{6}$$

where

$$k = \sqrt{1 - 2\left(r_3 - \gamma_3\mu_3^2\right) + \mu_3^2},$$

and $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm 4.1 converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
= &\ \| \left(1 - \alpha_n\right)\left(x_n - x^*\right) + \alpha_n (J_\varphi^\rho \left[g\left(y_n\right) - \rho T_1\left(y_n, x_n\right)\right] \\
&-J_\varphi^\rho \left[g\left(y^*\right) - \rho T_1\left(y^*, x^*\right)\right])\| \\
\leq &\ \left(1 - \alpha_n\right)\|x_n - x^*\| + \alpha_n \|J_\varphi^\rho \left[g\left(y_n\right) - \rho T_1\left(y_n, x_n\right)\right] \\
&-J_\varphi^\rho \left[g\left(y^*\right) - \rho T_1\left(y^*, x^*\right)\right]\| \\
\leq &\ \left(1 - \alpha_n\right)\|x_n - x^*\| + \alpha_n \left\|\left[g\left(y_n\right) - \rho T_1\left(y_n, x_n\right)\right] - \left[g\left(y^*\right) - \rho T_1\left(y^*, x^*\right)\right]\right\| \\
\leq &\ \left(1 - \alpha_n\right)\|x_n - x^*\| + \alpha_n \left\|y_n - y^* - \rho\left[T_1\left(y_n, x_n\right) - T_1\left(y^*, x^*\right)\right]\right\| \\
&+ \alpha_n \left\|y_n - y^* - \left[g\left(y_n\right) - g\left(y^*\right)\right]\right\| \\
\leq &\ \alpha_n \left\|y_n - y^* - \rho\left[T_1\left(y_n, x_n\right) - T_1\left(y^*, x_n\right) + T_1\left(y^*, x_n\right) - T_1\left(y^*, x^*\right)\right]\right\| \\
&+ \alpha_n \left\|y_n - y^* - \left[g\left(y_n\right) - g\left(y^*\right)\right]\right\| + \left(1 - \alpha_n\right)\|x_n - x^*\| \\
\leq &\ \left(1 - \alpha_n\right)\|x_n - x^*\| + \alpha_n \left\|y_n - y^* - \rho\left[T_1\left(y_n, x_n\right) - T_1\left(y^*, x_n\right)\right]\right\| \\
&+ \rho\alpha_n \left\|T_1\left(y^*, x_n\right) - T_1\left(y^*, x^*\right)\right\| + \alpha_n \left\|y_n - y^* - \left[g\left(y_n\right) - g\left(y^*\right)\right]\right\|.
\end{aligned}$$

From the relaxed (γ_1, r_1) −cocoercive for the first variable on T_1, we have

$$\begin{aligned}
&\|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x_n)]\|^2 \\
&= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x_n), y_n - y^*\rangle \\
&\quad + \rho^2 \|T_1(y_n, x_n) - T_1(x_n, y^*)\|^2 \\
&\le -2\rho[-\gamma_1 \|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 + r_1 \|y_n - y^*\|^2] \\
&\quad + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 \\
&\le 2\rho\gamma_1 \|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 - 2\rho r_1 \|y_n - y^*\|^2 \\
&\quad + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 .
\end{aligned}$$

From the μ_1-Lipschitzian definition for the first variable on T_1, we have:

$$\begin{aligned}
&\|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x_n)]\|^2 \\
&\le \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right] \|y_n - y^*\|^2 .
\end{aligned}$$

In a similar way, using the (γ_3, r_3) −cocoercivity and μ_3-Lipschitz continuity of the operator g; we have:

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \le k \|y_n - y^*\| .$$

From the λ_1-Lipschitzian definition for the second variable on T_1, we have:

$$\|T_1(y^*, x_n) - T_1(y^*, x^*)\| \le \lambda_1 \|x_n - x^*\| .$$

As a result, we have:

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1 \|y_n - y^*\| + \alpha_n\rho\lambda_1 \|x_n - x^*\| , \quad (7)$$

where,

$$\theta_1 = k + \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right]^{\frac{1}{2}} .$$

Similarly we have:

$$\|y_{n+1} - y^*\| \le (1 - \alpha_n) \|y_n - y^*\| + \alpha_n\theta_2 \|x_n - x^*\| + \alpha_n\eta\lambda_2 \|y_n - y^*\| , \quad (8)$$

where,

$$\theta_2 = k + \left[1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2\right]^{\frac{1}{2}} .$$

It is clear from the conditions (5) and (6) that,

$$\theta_1 + \eta\lambda_2 < 1 \quad \text{and} \quad \theta_2 + \rho\lambda_1 < 1.$$

Then from (7) and (8),

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\le (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1 \|y_n - y^*\| + \alpha_n\rho\lambda_1 \|x_n - x^*\| \\
&\quad + (1 - \alpha_n) \|y_n - y^*\| + \alpha_n\theta_2 \|x_n - x^*\| + \alpha_n\eta\lambda_2 \|y_n - y^*\| \\
&\le (1 - \alpha_n) [\|x_n - x^*\| + \|y_n - y^*\|] + \sigma\alpha_n [\|x_n - x^*\| + \|y_n - y^*\|] ,
\end{aligned}$$

where,

$$\sigma = \max\left(\theta_1 + \eta\lambda_2, \theta_2 + \rho\lambda_1\right) < 1.$$

Set

$$z_n = \|x_n - x^*\| + \|y_n - y^*\|.$$

So,

$$z_{n+1} \leq \left(1 - \left(1 - \sigma\right)\alpha_n\right) z_n,$$

which implies that:

$$z_{n+1} \leq \prod_{k=0}^{k=n} \left(1 - \left(1 - \sigma\right)\alpha_k\right) z_0.$$

Since $0 < \sigma < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, it implies in light of [8] that $\lim_{n\to+\infty} \prod_{k=0}^{k=n}\left(\left(1 - \left(1 - \sigma\right)\alpha_k\right)\right) = 0$, therefore $x_n \to x^*$ and $y_n \to y^*$. ■

Corollary 4.4. *We can replace the conditions* (5) *and* (6) *by* (9) *and* (10) *where,* $0 < p < 1$

$$\begin{cases} k < p, r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{-k^2 + 2pk + 1 - p^2}, \\ \left|\rho - \dfrac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \dfrac{\sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2\left[-k^2 + 2pk + 1 - p^2\right]}}{\mu_1^2}, \\ \rho < \dfrac{1-p}{\lambda_1}, \end{cases} \tag{9}$$

$$\begin{cases} k < p, r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{-k^2 + 2pk + 1 - p^2}, \\ \left|\eta - \dfrac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \dfrac{\sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \mu_2^2\left[-k^2 + 2pk + 1 - p^2\right]}}{\mu_2^2}, \\ \eta < \dfrac{1-p}{\lambda_2}. \end{cases} \tag{10}$$

Remark 4.5. If $T_1, T_2 : H \to H$ are univariate operators, then Algorithm 4.1 can be replaced by the following Algorithm.

Algorithm 4.6. *For arbitrary chosen initial points* $x_0 \in K$, *compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *using*

$$\begin{cases} x_{n+1} = \left(1 - \alpha_n\right)x_n + \alpha_n J_\varphi^\rho\left[g\left(y_n\right) - \rho T_1\left(y_n\right)\right], \\ y_n = J_\varphi^\eta\left[g\left(x_n\right) - \eta T_2\left(x_n\right)\right], \end{cases}$$

where $\alpha_n \in [0,1]$ *for all* $n \geq 0$.

Which is the approximate solvability of the system (11):

$$\begin{cases} \left\langle \rho T_1\left(y^*\right) + x^* - g\left(y^*\right), g\left(v\right) - x^*\right\rangle \geq \rho\varphi\left(x^*\right) - \rho\varphi\left(g\left(x\right)\right), \\ \left\langle \eta T_2\left(x^*\right) + y^* - g\left(x^*\right), g\left(v\right) - y^*\right\rangle \geq \eta\varphi\left(y^*\right) - \eta\varphi\left(g\left(x\right)\right), \end{cases} \tag{11}$$

for all $v \in H, \rho > 0, \eta > 0$.

For the system (11), we use Algorithm 4.6 and present the following theorem which uses less conditions than the previous theorem.

Theorem 4.7. *Let (x^*, y^*) be the solution of (11). Suppose that $T_1, T_2, g : H \to H$ are both relaxed-cocoercive with constants (γ_1, r_1), (γ_2, r_2), (γ_3, r_3) and Lipschitz continuous with constants μ_1, μ_2, μ_3 respectively. If*

$$\begin{cases} k < 1, r_1 > \gamma_1 \mu_1^2 + \mu_1 \sqrt{-k^2 + 2k}, \\ \left| \rho - \dfrac{r_1 - \gamma_1 \mu_1^2}{\mu_1^2} \right| < \dfrac{\sqrt{\left(r_1 - \gamma_1 \mu_1^2\right)^2 - \mu_1^2 \left[-k^2 + 2k\right]}}{\mu_1^2}, \end{cases} \tag{12}$$

$$\begin{cases} k < 1, r_2 > \gamma_2 \mu_2^2 + \mu_2 \sqrt{-k^2 + 2k}, \\ \left| \eta - \dfrac{r_2 - \gamma_2 \mu_2^2}{\mu_2^2} \right| < \dfrac{\sqrt{\left(r_2 - \gamma_2 \mu_2^2\right)^2 - \mu_2^2 \left[-k^2 + 2k\right]}}{\mu_2^2}, \end{cases} \tag{13}$$

where

$$k = \sqrt{1 - 2\left(r_3 - \gamma_3 \mu_3^2\right) + \mu_3^2},$$

and $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0 \in K$, x_n and y_n obtained from Algorithm 4.6 converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ = & \left\| (1-\alpha_n) x_n + \alpha_n J_\varphi^\rho \left[g(y_n) - \rho T_1(y_n)\right] - (1-\alpha_n) x^* \right. \\ & \left. + \alpha_n J_\varphi^\rho \left[g(y^*) - \rho T_1(y^*)\right] \right\| \\ \leq & (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\| J_\varphi^\rho \left[g(y_n) - \rho T_1(y_n)\right] - J_\varphi^\rho \left[g(y^*) - \rho T_1(y^*)\right] \right\| \\ \leq & (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\| \left[g(y_n) - \rho T_1(y_n)\right] - \left[g(y^*) - \rho T_1(y^*)\right] \right\| \\ \leq & (1-\alpha_n)\|x_n - x^*\| + \alpha_n \left\| y_n - y^* - \rho\left[T_1(y_n) - T_1(y^*)\right] \right\| \\ & + \left\| y_n - y^* - \left[g(y_n) - g(y^*)\right] \right\|. \end{aligned}$$

From the relaxed (γ_1, r_1) $-$cocoercive on T_1, we have:

$$\begin{aligned} & \left\| y_n - y^* - \rho\left[T_1(y_n) - T_1(y^*)\right] \right\|^2 \\ = & \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n) - T_1(y^*), y_n - y^*\rangle + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2 \\ \leq & -2\rho\left[-\gamma_1 \|T_1(y_n) - T_1(y^*)\|^2 + r_1 \|y_n - y^*\|^2\right] \\ & + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2 \\ \leq & 2\rho\gamma_1 \|T_1(y_n) - T_1(y^*)\|^2 - 2\rho r_1 \|y_n - y^*\|^2 \\ & + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2. \end{aligned}$$

From the μ_1-Lipschitzian definition on T_1, we have

$$\|y_n - y^* - \rho [T_1(y_n) - T_1(y^*)]\|^2 \leq \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right] \|y_n - y^*\|^2 .$$

In a similar way, using the (γ_3, r_3) −cocoercivity and μ_3-Lipschitz continuity of the operator g; we have

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \leq k \|y_n - y^*\| .$$

As a result, we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1 \|y_n - y^*\| , \tag{14}$$

where,

$$\theta_1 = k + \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right]^{\frac{1}{2}} .$$

Now we evaluate $\|y_n - y^*\|$, for all $n \geq 0$.

$$\begin{aligned}
\|y_n - y^*\| &= \left\|J_\varphi^\eta [g(x_n) - \eta T_2(x_n)] - J_\varphi^\eta [g(x^*) - \eta T_1(x^*)]\right\| \\
&\leq \|[g(x_n) - \eta T_2(x_n)] - [g(x^*) - \eta T_2(x^*)]\| \\
&\leq \|x_n - x^* - \eta [T_2(x_n) - T_2(x^*)]\| + \|x_n - x^* - [g(x_n) - g(x^*)]\| .
\end{aligned}$$

From the relaxed (γ_2, r_2) −cocoercive on T_2, we have

$$\begin{aligned}
&\|x_n - x^* - \eta [T_2(x_n) - T_2(x^*)]\|^2 \\
= &\|x_n - x^*\|^2 - 2\eta\langle T_2(x_n) - T_2(x^*), x_n - x^*\rangle + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 \\
\leq &-2\eta \left[-\gamma_2 \|T_2(x_n) - T_2(x^*)\|^2 + r_2 \|x_n - x^*\|^2\right] \\
&+ \|x_n - x^*\|^2 + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 \\
\leq &2\eta\gamma_2 \|T_2(x_n) - T_2(x^*)\|^2 - 2\eta r_2 \|x_n - x^*\|^2 \\
&+ \|x_n - x^*\|^2 + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 .
\end{aligned}$$

By using the (γ_4, r_4) −cocoercivity and μ_4-Lipschitz continuity of the operator g, we have:

$$\|x_n - x^* - [g(x_n) - g(x^*)]\| \leq k \|x_n - x^*\| .$$

As a result, we have:

$$\|y_n - y^*\| \leq \theta_2 \|x_n - x^*\| , \tag{15}$$

where,

$$\theta_2 = k + \left[1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2\right]^{\frac{1}{2}} .$$

It is clear from the condition (12) and (13) that

$$\theta_1 < 1 \quad \text{and} \quad \theta_2 < 1.$$

It follow that from (14) and (15),

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1\theta_2 \|x_n - x^*\| ,$$

which implies that:

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{k=n} \left(1 - \left(1 - \theta_1\theta_2\right)\alpha_k\right) \|x_0 - x^*\| .$$

Since $0 < \theta_1\theta_2 < 1$, and $\sum_{k=0}^{\infty} \alpha_k = \infty$, it implies in light of [8] that $\lim_{n\to+\infty} \prod_{k=0}^{k=n} \left(\left(1 - \left(1 - \theta_1\theta_2\right)\alpha_k\right)\right) = 0$, therefore $x_n \to x^*$ and $y_n \to y^*$. ■

For $g = I$, in Algorithm 4.6 we get the following algorithm.

Algorithm 4.8. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = \left(1 - \alpha_n\right) x_n + \alpha_n J_\varphi^\rho \left[y_n - \rho T_1\left(y_n\right)\right], \\ y_n = J_\varphi^\eta \left[x_n - \eta T_2\left(x_n\right)\right], \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Which is the approximate solvability of the following system:

$$\begin{cases} \langle \rho T_1\left(y^*\right) + x^* - y^*, v - x^* \rangle \geq \rho\varphi\left(x^*\right) - \rho\varphi\left(x\right), \forall v \in H, \rho > 0, \\ \langle \eta T_2\left(x^*\right) + y^* - x^*, v - y^* \rangle \geq \eta\varphi\left(y^*\right) - \eta\varphi\left(x\right), \forall v \in H, \eta > 0, \end{cases}$$

which has been studied by J.K. Kim and D.S. Kim [6] as a special case of their work (see the first case).

Acknowledgement. The authors would like to express their thanks to the editor (Professor Hanchun Yang) and the anonymous referees for their useful comments and suggestions, which helped to improve the presentation of this paper.

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