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On Concave Univalent Functions of Order α

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Abstract. Let $S(p)$ be the class of meromorphic univalent functions f in the unit disk $\mathbb{D}$ with a simple pole at $p \in (0, 1)$, $C_O(p, \alpha)$ be the subclass of $S(p)$ such that $\hat{\mathbb{C}} \setminus f(\mathbb{D})$ is a convex domain of order α. In this paper, some characterizations of functions in $C_O(p, \alpha)$ are given and the Livingston conjecture of $f \in C_O(p, \alpha)$ is considered.

Keywords: Meromorphic functions; Livingston conjecture; Concave functions.

1. Introduction

Let S be the class of analytic univalent functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. For $f \in S$, it has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, z \in \mathbb{D}.$$

The famous Bieberbach conjecture, which was proposed by Bieberbach in 1916, claimed that $|a_n(f)| \leq n$ for $n \in \mathbb{N}$, strict inequality holds for all n unless f is the Koebe function or one of its rotation. Since then, many mathematicians have devoted to this conjecture (for example [4, 7, 11, 15]). As we know, the conjecture was finally proved by de Branges in [5].

During the study of Bieberbach conjecture, many important subclasses of S have been considered, such as convex functions, starlike functions, close-to-convex functions and so on. For more details about these subclasses, we refer to the monographs of Duren [6] and Pommerenke [18]. Following [18], we call $f \in S$ convex function if the image $f(\mathbb{D})$ is a convex domain in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We denote by $K \subset S$ the class of convex functions. It is well known that $f \in K$ if and only if

$$Re \left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, z \in \mathbb{D}. \tag{1}$$

Let $\alpha \in [0, 1)$. We call $f \in S$ convex function of order α if

$$Re \left(1 + z\frac{f''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{D}. \tag{2}$$

We denote by $K(\alpha)$ the class of convex functions of order α. The class $K(\alpha)$ was introduced by Robertson in [19], and was further studied by many scholars, such as Jack [9], Pinchuk [17], Sugawa and Wang [20]. We call Ω convex domain of order α if there exists $f \in K(\alpha)$ and suitable constants a and b such that $\Omega = \tilde{f}(\mathbb{D})$, where $\tilde{f} = af + b$. Let f be a analytic univalent function in $\mathbb{D}$ (f is not necessary in S) and $f(\mathbb{D}) = \Omega$, since

$$Re \left(1 + z\frac{f''(z)}{f'(z)}\right) = Re \left(1 + z\frac{\tilde{f}''(z)}{\tilde{f}'(z)}\right),$$

then Ω is a convex domain of order α if and only if $Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{D}$.

The class $K \subset S$ was proposed to solve Bieberbach conjecture. Another way to attack Bieberbach conjecture was thought to be the class Σ, mapping the outside of the unit circle conformally onto a simply connected domain in $\hat{\mathbb{C}}$. This class was considered to be the counterpart to the class S and therefore lead to a new angle on the problem asserted for the class S. Although Bieberbach conjecture was proved, many of the problems which arouse during the time were still left open. Such as the class Σ, its coefficient estimate is incomplete. As it was the case with the class S, subclasses of Σ with especial geometry were considered to get closer to functions of the class. Types like meromorphically starlike functions, concave functions were considered. Originally concave functions were defined to map the the outside of the unit circle conformally to the outside of a convex domain, therefore giving the counterpart to the class of convex functions in the class Σ, fixing the point at infinity. However, it turned out to be more convenient to analyze meromorphic univalent functions defined in the unit disk $\mathbb{D}$, having a simple at some point in $\mathbb{D}$. In the early time, considerations were made by Goodman [8] in 1956, Miller [12, 13] in 1970 and 1980. They considered the geometry of a function being concave and deduced several analytic characterizations. The concave function was further studied by Livingston [10] in 1994, where he considered a simple pole at $p \in (0, 1)$.

Throughout this paper, we restrict $p \in (0, 1)$ and denote by $S(p)$ the class of meromorphic univalent functions f in $\mathbb{D}$ with a simple pole at $z = p$ and the normalization $f(0) = f'(0) - 1 = 0$. Following [1], [3] and [21], we call $f \in S(p)$ concave function if $\hat{\mathbb{C}} \backslash f(\mathbb{D})$ is a convex domain. We denote by $C_O(p)$ the class of concave functions.

Similar to the consideration of (1), Livingston obtained the following characterization of $f \in C_O(p)$.

Lemma 1.1. [10, Theorem 1] *Let $f \in S(p)$. Then $f \in C_O(p)$ if and only if*

$$Re\ \left(1 + p^2 - 2pz + \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right) < 0, z \in \mathbb{D}. \tag{3}$$

For $f \in C_O(p)$, it has the following Taylor expansion,

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, |z| < p. \tag{4}$$

In [10], for $f \in C_O(p)$, Livingston conjectured that

$$Re\ (a_n(f)) \geq \frac{1+p^{2n}}{p^{n-1}(1+p^2)}, n \geq 2. \tag{5}$$

Furthermore, Avkhadiev, Pommerenke and Wirths [1] conjectured that

$$\left| a_n(f) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} \right| \leq \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}, n \geq 2. \tag{6}$$

Parallel to Bieberbach conjecture, (5) and (6) are named as Livingston conjecture. As we know, the conjecture was proved by Avkhadiev and Wirths as follows.

Lemma 1.2. [2] *For $f \in C_O(p)$, the Taylor coefficient $a_n(f)$ in* (4) *is determined by the inequality*

$$\left| a_n(f) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} \right| \leq \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}, n \geq 2.$$

When $f \in C_O(p)$, we know that $\hat{\mathbb{C}} \backslash f(\mathbb{D})$ is a convex domain. Naturally, based on the relationship between K and $K(\alpha)$, we define $f \in S(p)$ concave function of order α if $\hat{\mathbb{C}} \backslash f(\mathbb{D})$ is a convex domain of order α and denote by $C_O(p, \alpha)$ the class of concave functions of order α. In this paper, we will study characterizations and Livingston conjecture of $f \in C_O(p, \alpha)$.

Our arrangements are as follows, we give the proof of Theorem 2.2 in Section 2 and two other characterizations in Section 3. Finally, the proof of Theorem 4.6 is given in Section 4.

2. Characterization of Concave Functions of Order α

In this section, we will give the proof of Theorem 2.2.

Lemma 2.1. [14] *Let $\mathbb{D}^* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ and $f\colon \mathbb{D}^* \to \hat{\mathbb{C}}$ be a meromorphic univalent function which maps $\mathbb{D}^*$ onto the outside of a bounded Jordan curve Γ and $f(\infty) = \infty$. The curve Γ is analytic if and only if f is univalent and analytic in $\{z \in \hat{\mathbb{C}} : |z| > r\}$ for some $r < 1$.*

Theorem 2.2. *Let $f \in S(p)$. Then $f \in C_O(p, \alpha)$ if and only if*

$$Re\ \left(1 + p^2 - 2pz + \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right) < -\alpha(1-p^2), z \in \mathbb{D}. \tag{7}$$

Proof. We divide the proof into the following two steps.

Step 1: Let f be a concave function of order α. We will prove that f satisfies (7).

By assumption, we know that $f(\mathbb{D}) = \Omega^* = \hat{\mathbb{C}}\backslash\Omega$ and Ω is a convex domain of order α, we denote by $\Gamma = \partial\Omega$. Let $u(z) = \frac{1+zp}{z+p}$, which maps $\mathbb{D}^*$ onto $\mathbb{D}$. Choosing suitable θ, we have $g(z) = e^{i\theta} \cdot f \circ u(z)$ as a meromorphic univalent function from $\mathbb{D}^*$ onto Ω^* with $g(\infty) = \infty, g'(\infty) > 0$.

Let $h(z)$ be a analytic univalent function from $\mathbb{D}$ onto Ω, and Γ_k=$\{h(z) : |z| = 1 - \frac{1}{k}\}$ for $k = 2, 3\cdots$. Then Γ_k are analytic by Lemma 2.1. We denote by Ω_k the interior domain of Γ_k and Ω_k^* the outer domain of Γ_k. Let g_k be the meromorphic univalent function from $\mathbb{D}^*$ onto Ω_k^* with $g_k(\infty) = \infty$ and $g_k'(\infty) > 0$, h_k be the analytic univalent function from $\mathbb{D}$ onto Ω_k.

By the definition, Ω_k is a convex domain of order α. When $z \in \mathbb{D}$, we have

$$Re\ \left(1 + \frac{zh_k''(z)}{h_k'(z)}\right) > \alpha. \tag{8}$$

Since $h_k(\partial\mathbb{D}) = g_k(\partial\mathbb{D}^*)$, h_k and g_k can be continuously extended to the boundary $|z| = 1$. Following (8), we have

$$Re\ \left(1 + \frac{zg_k''(z)}{g_k'(z)}\right) > \alpha, |z| = 1. \tag{9}$$

Since g_k is a meromorphic univalent function from $\mathbb{D}^*$ onto Ω_k^* with $g_k(\infty) = \infty$, we have

$$\lim_{z\to\infty} Re\ \left(1 + \frac{zg_k''(z)}{g_k'(z)}\right) = 1. \tag{10}$$

By (9), (10) and the maximum principle of harmonic function $Re\left(1 + \frac{zg_k''(z)}{g_k'(z)}\right)$, we have

$$Re\ \left(1 + \frac{zg_k''(z)}{g_k'(z)}\right) > \alpha, |z| > 1. \tag{11}$$

Let $w_0 \in \Omega_2$, $\hat{g}(z) = \frac{1}{g(1/z)-w_0}$ and $\hat{g_k}(z) = \frac{1}{g_k(1/z)-w_0}$. Then $\hat{g}(z)$ and $\hat{g_k}(z)$ are defined in $\mathbb{D}$ with $\hat{g}(0) = \hat{g_k}(0) = 0$, $\hat{g}'(0) > 0$ and $\hat{g_k}'(0) > 0$. Since Γ_k converge to Γ in the sense of Carathéodory, applying the Carathéodory Convergence Theorem [6, pp. 78] to $\hat{g}(z)$ and $\hat{g_k}(z)$, we know that $\hat{g}_k$ converge locally uniformly to $\hat{g}$ in $\mathbb{D}$. That means g_k converge locally uniformly to g in $\mathbb{D}^*$. Following (11), we have

$$Re\ \left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha, |z| > 1. \tag{12}$$

When $z \in \mathbb{D}^*$, since $u(z) = \frac{1+zp}{z+p} \in \mathbb{D}$, $g(z) = e^{i\theta} \cdot f \circ u(z)$ and (12), we have

$$\begin{aligned} \alpha &< Re\ \left(1 + \frac{zg''(z)}{g'(z)}\right) \\ &= Re\ \left(1 + \frac{z(p^2-1)f''(u)}{(z+p)^2 f'(u)} - \frac{2z}{z+p}\right) \\ &= Re\ \left(1 - \frac{2(1-up)}{1-p^2} - \frac{(1-up)(u-p)f''(u)}{(1-p^2)f'(u)}\right). \end{aligned} \tag{13}$$

It is easy to check that (13) is equivalent to, when $u \in \mathbb{D}$,

$$Re\ \left(1 + p^2 - 2pu + \frac{(1-up)(u-p)f''(u)}{f'(u)}\right) < -\alpha(1-p^2). \tag{14}$$

So (7) is proved.

Step 2: If f satisfies (7), we will prove that f is a concave function of order α.

Let $\Omega^* = f(\mathbb{D})$ and $\Omega = \hat{\mathbb{C}}\backslash\Omega^*$. We will show that Ω is convex domain of order α. Let $u(z) = \frac{1+zp}{z+p}$, which maps $\mathbb{D}^*$ onto $\mathbb{D}$. We can choose suitable θ such that $g(\mathbb{D}^*) = e^{i\theta} \cdot f \circ u(\mathbb{D}^*) = \Omega^*$ and $g(\infty) = \infty, g'(\infty) > 0$. By the same computation in (13), we have

$$Re\ \left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha, z \in \mathbb{D}^*. \tag{15}$$

Let h be a analytic univalent function in $\mathbb{D}$ with $h(\mathbb{D}) = \Omega$. Following the same arguments as them in Step 1, when $z \in \mathbb{D}$, we have,

$$Re\ \left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha.$$

Hence, by the definition of convex domain of order α, we know that Ω is convex domain of order α and f is concave function of order α. ∎

3. Some Necessary Characterizations of Concave Functions of Order α

For the class $C_O(p)$, some characterizations were obtained as follows.

Lemma 3.1. [16, Theorem 9.2] *Let $f \in S(p)$. Then $f \in C_O(p)$ if and only if*

$$Re\left(1+z\frac{f''(z)}{f'(z)}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}\right)<0, z\in\mathbb{D}. \tag{16}$$

Lemma 3.2. [14] *For $f \in C_O(p)$, there exists an analytic function φ: $\mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$ such that*

$$f'(z)=\frac{p^2}{(z-p)^2(1-pz)^2}\exp\int_0^z\frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \tag{17}$$

Conversely, for any analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(p) = p$, the function $f \in S(p)$ determined by (17) *belongs to $C_O(p)$.*

Parallel to Lemmas 3.1 and 3.2, in this section, we will give other characterizations of $f \in C_O(p, \alpha)$.

Theorem 3.3. *Let $f \in C_O(p, \alpha)$. Then*

$$Re\left(1+\frac{zf''(z)}{f'(z)}-\frac{1+pz}{1-pz}+\frac{z+p}{z-p}\right)<-\frac{\alpha(1-p)}{(1+p)}, z\in\mathbb{D}. \tag{18}$$

Proof. When $p < r < 1$, we let $\sigma = (r-1)p/(r-p^2) \in \mathbb{D}$ and $L_r(z) = r(z-\sigma)/(1-z\bar{\sigma})$. It is easy check that $L_r(p) = p$ and $L_r(\mathbb{D}) = \{z : |z| < r\}$.

For $f \in C_O(p, \alpha)$, we let

$$P(z)=2pz-1-p^2-\frac{(z-p)(1-pz)f''(z)}{f'(z)} \tag{19}$$

and

$$Q_r(z)=\frac{zP(L_r(z))-p+pz^2}{(z-p)(1-pz)}. \tag{20}$$

Then $P(z)$ and $Q_r(z)$ are analytic in $\mathbb{D}$.

When $|z| = 1$, it is easy to check that

$$Re\left(\frac{pz(z-\frac{1}{z})}{(z-p)(1-pz)}\right)=Re\left(\frac{p(z-\bar{z})}{|1-pz|^2}\right)=0, \tag{21}$$

$$\frac{z}{z-p}=\frac{1}{\overline{1-zp}}. \tag{22}$$

Since $L_r(z) \in \mathbb{D}$, (21), (22) and (7), when $|z| = 1$, we have

$$\begin{aligned} Re\ (Q_r(z)) &= Re\ \left(\frac{zP(L_r(z))}{(z-p)(1-pz)} + \frac{pz(z-\frac{1}{z})}{(z-p)(1-pz)} \right) \\ &= Re\ \left(\frac{P(L_r(z))}{|1-pz|^2} \right) > \frac{\alpha(1-p^2)}{|1-pz|^2} \geq \frac{\alpha(1-p)}{1+p}. \end{aligned} \tag{23}$$

Since $Q_r(z)$ is analytic for $|z| \leq 1$, $L_r(z) \to z$ as $r \to 1$ and (23), letting $r \to 1$, we have

$$Re\ \left(\frac{zP(z) - p + pz^2}{(z-p)(1-pz)} \right) \geq \frac{\alpha(1-p)}{1+p}, |z| = 1. \tag{24}$$

By the maximum principle of harmonic function $Re\ \left(\frac{zP(z)-p+pz^2}{(z-p)(1-pz)} \right)$, we have

$$Re\ \left(\frac{zP(z) - p + pz^2}{(z-p)(1-pz)} \right) > \frac{\alpha(1-p)}{1+p}, z \in \mathbb{D}. \tag{25}$$

A straightforward computation gives

$$\begin{aligned} -1 - \frac{zf''(z)}{f'(z)} + \frac{1+pz}{1-pz} - \frac{z+p}{z-p} &= \tfrac{2pz^2 - z - p^2z - p + pz^2}{(z-p)(1-pz)} - \tfrac{zf''(z)}{f'(z)} \\ &= \tfrac{zP(z) - p + pz^2}{(z-p)(1-pz)}. \end{aligned} \tag{26}$$

By (25) and (26), we have

$$Re\ \left(-1 - \frac{zf''(z)}{f'(z)} + \frac{1+pz}{1-pz} - \frac{z+p}{z-p} \right) > \frac{\alpha(1-p)}{1+p}.$$

Then, we complete the proof. ■

Theorem 3.4. *Let $f \in C_O(p, \alpha)$. Then there exists an analytic function φ: $\mathbb{D} \to \mathbb{D}$ such that*

$$f'(z) = \frac{p^2}{(z-p)^2(1-pz)^2} \exp \int_0^z \frac{-2\left(1 - \frac{\alpha(1-p)}{(1+p)}\right)\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta. \tag{27}$$

Proof. From [18, pp. 39], we know that for an analytic function $\psi(z)$ in $\mathbb{D}$ with $Re\ (\psi(z)) > 0$ and $\psi(0) = 1$, there exists an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\psi(z) = \frac{1+z\varphi(z)}{1-z\varphi(z)}$. By Theorem 3.3, we have an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ such that

$$\frac{1}{1 - \frac{\alpha(1-p)}{(1+p)}} \left(-1 - \frac{zf''(z)}{f'(z)} - \frac{z+p}{z-p} + \frac{1+zp}{1-zp} - \frac{\alpha(1-p)}{(1+p)} \right) = \frac{1+z\varphi(z)}{1-z\varphi(z)}. \tag{28}$$

The equation (28) is equivalent to

$$1+\frac{zf''(z)}{f'(z)}+\frac{2z-(z-p)}{z-p}-\frac{1-zp+2zp}{1-zp}+\frac{\alpha(1-p)}{(1+p)}$$
$$=-\left(1-\frac{\alpha(1-p)}{(1+p)}\right)\frac{1-z\varphi(z)+2z\varphi(z)}{1-z\varphi(z)}. \tag{29}$$

The left side of (29) is

$$\frac{zf''(z)}{f'(z)}+\frac{2z}{z-p}-\frac{2pz}{1-pz}-1+\frac{\alpha(1-p)}{(1+p)},$$

and the right side of (29) is

$$-1+\frac{\alpha(1-p)}{(1+p)}-\frac{2\left(1-\frac{\alpha(1-p)}{(1+p)}\right)z\varphi(z)}{1-z\varphi(z)}.$$

So we obtain

$$\frac{zf''(z)}{f'(z)}+\frac{2z}{z-p}-\frac{2pz}{1-zp}=-\frac{2\left(1-\frac{\alpha(1-p)}{(1+p)}\right)z\varphi(z)}{1-z\varphi(z)}. \tag{30}$$

Dividing both sides of (30) by z and integrating both sides, we have

$$\log f'(z)(z-p)^2(1-zp)^2-\log p^2=\int_0^z\frac{-2\left(1-\frac{\alpha(1-p)}{(1+p)}\right)\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \tag{31}$$

It is easy to check that (31) is equivalent to

$$f'(z)=\frac{p^2}{(z-p)^2(1-zp)^2}\exp\int_0^z\frac{-2\left(1-\frac{\alpha(1-p)}{(1+p)}\right)\varphi(\zeta)}{1-\zeta\varphi(\zeta)}d\zeta. \tag{32}$$

Then we finish the proof. ∎

Remark 3.5. When $\alpha=0$, Theorems 3.3 and 3.4 correspond to the necessary conditions of Lemmas 3.1 and 3.2 respectively.

4. Coefficient Estimate of Concave Univalent Functions of Order α

In [2], Avkhadiev and Wirths gave the coefficient estimate of $f\in C_O(p)$. Following the idea of proof of the Theorem in [2] and some lemmas, we will give the proof of Theorem 2.2.

Lemma 4.1. *Let $f\in C_O(p,\alpha)$. Then for $|z|<1, |\zeta|<1$, we have*

$$Re\left(\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}\right)<-\frac{\alpha(1-p)}{1+p}. \tag{33}$$

Proof. When $|z|<1, |\zeta|<1$, we define $F(z,\zeta)$ as

$$F(z,\zeta)=\begin{cases}1+\frac{zf''(z)}{f'(z)}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}, & z=\zeta,\\ \frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}, & z\neq\zeta.\end{cases}$$

Then $F(z,\zeta)$ is analytic in $\mathbb{D}\times\mathbb{D}$ for each $f\in S(p)$. The case $z=\zeta$ was proved in Theorem 3.3, here we omit its details. Next, we consider the case $z\neq\zeta$.

We denote by $\Gamma=\{f(z):|z|=r<1\}$, which is the boundary curve of convex domain of order α. Following [9], we know that a convex function of order α is starlike of order at least $\beta(\alpha)\geq\frac{2\alpha-1+\sqrt{9-4\alpha+4\alpha^2}}{4}\geq\alpha$. By the definition of concave function of order α and result in [18, pp. 45], we have

$$Re\ \left(\frac{zf'(z)}{f(z)-f(\zeta)}\right)=\frac{\partial}{\partial t}\arg(f(re^{it})-f(re^{i\theta}))<-\alpha, z=re^{it}\neq re^{i\theta}=\zeta. \quad (34)$$

When $|z|=|\zeta|$, $z\neq\zeta$, we have

$$Re\ \left(\frac{z+\zeta}{z-\zeta}\right)=0. \quad (35)$$

Hence,

$$Re\ \left(\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}\right)<-2\alpha, |z|=|\zeta|, z\neq\zeta.$$

When $|z|=1$, we have

$$Re\left(\frac{z+p}{z-p}-\frac{1+pz}{1-pz}\right)=Re\left(\frac{1+p\bar{z}}{1-p\bar{z}}-\frac{1+pz}{1-pz}\right)=Re\ \left(\frac{2p(\bar{z}-z)}{|1-pz|^2}\right)=0. \quad (36)$$

Letting $r\to 1$, by the maximum principle of harmonic function $Re\ (\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz})$, we have

$$Re\ \left(\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}\right)<-2\alpha, z\neq\zeta, |z|, |\zeta|<1.$$

It's obvious that

$$Re\ \left(\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}+\frac{z+p}{z-p}-\frac{1+pz}{1-pz}\right)<-2\alpha<-\frac{\alpha(1-p)}{1+p}. \quad (37)$$

So we have (33). ∎

Lemma 4.2. [6] *Let $h(z)$ be analytic function in $\mathbb{D}$ with $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ and Re $(h(z))>0$. Then*

$$|c_n|\leq 2, n=1,2\cdots.$$

This inequality is sharp for each n.

Lemma 4.3. *Let $f \in C_O(p,\alpha)$. Then for $z \in \mathbb{D}\backslash\{0\}$, we have*

$$\left|\frac{1}{f(z)} - \frac{1}{z} + \frac{1}{p} + p\right| \leq 1 - \frac{\alpha(1-p)}{1+p}. \tag{38}$$

Proof. The main idea of the proof is similar to Miller [13], his result is the key to get the Taylor coefficient estimate. We show the proof in detail.

For z, ζ $(\zeta \neq 0) \in \mathbb{D}$, let

$$-h(z) = \frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz} + \frac{\alpha(1-p)}{1+p}.$$

By Lemma 4.1, we have $Re\ (h(z)) > 0$. A straightforward computation gives

$$h(0) = 1 - \frac{\alpha(1-p)}{1+p},$$

and

$$h'(z) = -\frac{2f'(z)+2zf''(z)}{f(z)-f(\zeta)} + \frac{2zf'(z)^2}{(f(z)-f(\zeta))^2} - \frac{2\zeta}{(\zeta-z)^2} + \frac{2p}{(z-p)^2} + \frac{2p}{(1-pz)^2}.$$

A simple computation gives

$$h'(0) = \frac{2}{f(\zeta)} - \frac{2}{\zeta} + \frac{2}{p} + 2p.$$

Hence, $h(z)$ has the following Taylor expansion

$$h(z) = 1 - \frac{\alpha(1-p)}{1+p} + \left(\frac{2}{f(\zeta)} - \frac{2}{\zeta} + \frac{2}{p} + 2p\right)z + \cdots. \tag{39}$$

By Lemma 4.2, we obtain

$$\left|\frac{1}{f(\zeta)} - \frac{1}{\zeta} + \frac{1}{p} + p\right| \leq 1 - \frac{\alpha(1-p)}{1+p}. \tag{40}$$

■

Lemma 4.4. *Let $f \in C_O(p,\alpha)$. Then there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$, such that*

$$f(z) = \frac{z\left(1 - \frac{zp(1-p^2\lambda^2)}{1-p^4\lambda^2}\right)}{(1-zp)(1-\frac{z}{p})} - \frac{\frac{\lambda p(1-p^2)}{1-p^4\lambda^2}z^2\omega(z)}{(1-zp)(1-\frac{z}{p})}, z \in \mathbb{D}, \tag{41}$$

where $\gamma = 1 - \frac{\alpha(1-p)}{1+p}$ and $\lambda = \frac{p(1-\gamma^2)+\sqrt{(\gamma^2-p^2)(1-\gamma^2p^2)-p^2(1-\gamma^2)^2}}{1-\gamma^2p^2}$.

Proof. Let

$$W(z) = \frac{1}{f(z)} - \frac{1}{z} + \frac{1}{p} + p. \tag{42}$$

Then $W(p) = p$. By (40), we have $W(\mathbb{D}) \subset \mathbb{D}_{1-\frac{\alpha(1-p)}{1+p}}$, where $\mathbb{D}_{1-\frac{\alpha(1-p)}{1+p}} = \{z \in \mathbb{C} : |z| < 1 - \frac{\alpha(1-p)}{1+p}\}$. For convenience, we denote by $\gamma = 1 - \frac{\alpha(1-p)}{1+p}$.

Let $\varpi = A(z) = \frac{z-p}{1-zp}$. By some direct computations, we have that $A(\mathbb{D}_\gamma)$ is a disk with the center $-\frac{p(1-\gamma^2)}{1-\gamma^2 p^2}$ and radius $\frac{\sqrt{(\gamma^2-p^2)(1-\gamma^2p^2)-p^2(1-\gamma^2)^2}}{(1-\gamma^2p^2)}$. So, $A(\mathbb{D}_\gamma)$ is contained in the disk with the center 0 and radius $\lambda = \frac{p(1-\gamma^2)+\sqrt{(\gamma^2-p^2)(1-\gamma^2p^2)-p^2(1-\gamma^2)^2}}{1-\gamma^2p^2}$.

Let $\tilde{v}(z) = \frac{1}{\lambda} A \circ W$. Then

$$W(z) = \frac{\lambda \tilde{v}(z) + p}{1 + \lambda p \tilde{v}(z)}. \tag{43}$$

Since $\tilde{v}(z)$ maps $\mathbb{D}$ into $\mathbb{D}$ and p to the origin, by Schwarz Lemma, we have

$$|\tilde{v}(z)| \leq |A(z)|, z \in \mathbb{D}.$$

Let analytic function $v(z) = \frac{\tilde{v}(z)}{A(z)}$. By (43), we have

$$W(z) = \frac{\lambda \frac{z-p}{1-zp} v(z) + p}{1 + \lambda p \frac{z-p}{1-zp} v(z)}. \tag{44}$$

Following (42) and (44), we have

$$f(z) = \frac{z[1 - zp + \lambda p(z-p)v(z)]}{(1-zp)(1-\frac{z}{p})(1-\lambda p^2 v(z))}. \tag{45}$$

We could writ $v(z)$ as a suitable form

$$v(z) = \frac{\lambda p^2 - \omega(z)}{1 - \lambda p^2 \omega(z)}, \tag{46}$$

then inserting (46) into (45), we obtain (41). ∎

Up to now, we have obtained the formula representation of the function $f \in C_O(p, \alpha)$. In order to estimate the Taylor coefficient in (4), we need the following result.

Lemma 4.5. [2] *Let $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$ be an analytic function in $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then for any $m \geq 0$, we have*

$$\left| \sum_{k=0}^{m} c_k \frac{1 - p^{2(m-k)+2}}{p^{m-k}} \right| \leq \frac{1 - p^{2m+2}}{p^m}. \tag{47}$$

From Lemma 4.4, we can get the coefficient estimate of $f \in C_O(p,\alpha)$ by estimating the coefficient of $\omega(z)$. By these lemmas, we give the proof of Theorem 4.6.

By Theorem 2.2 and some lemmas, we consider the Livingston conjecture of $f \in C_O(p,\alpha)$ and obtain the following theorem.

Theorem 4.6. *Let $f \in C_O(p,\alpha)$ with the expansion (4). Then*

$$\left| a_n - \frac{1-\lambda^2 p^{2n+2}}{p^{n-1}(1-\lambda^2 p^4)} \right| \leq \frac{\lambda p^2(1-p^{2n-2})}{p^{n-1}(1-\lambda^2 p^4)}, n \geq 2, \tag{48}$$

where $\gamma = 1 - \frac{\alpha(1-p)}{1+p}$, $\lambda = \frac{p(1-\gamma^2)+\sqrt{(\gamma^2-p^2)(1-\gamma^2p^2)-p^2(1-\gamma^2)^2}}{1-\gamma^2p^2}$.

Proof. By Lemma 4.4, for $f \in C_O(p,\alpha)$, we have

$$f(z) = \frac{z(1-\frac{zp(1-p^2\lambda^2)}{1-p^4\lambda^2})}{(1-zp)(1-\frac{z}{p})} - \frac{\frac{\lambda p(1-p^2)}{1-p^4\lambda^2}z^2\omega(z)}{(1-zp)(1-\frac{z}{p})}.$$

We write $f(z)$ as the form

$$f(z) = h(z) + g(z), \tag{49}$$

where $h(z) = \frac{z(1-\frac{zp(1-p^2\lambda^2)}{1-p^4\lambda^2})}{(1-zp)(1-\frac{z}{p})}$, and $g(z) = -\frac{\frac{\lambda p(1-p^2)}{1-p^4\lambda^2}z^2\omega(z)}{(1-zp)(1-\frac{z}{p})}$.

Straightforward computations give

$$\begin{aligned} h(z) &= \frac{z(1-\frac{zp(1-p^2\lambda^2)}{1-p^4\lambda^2})}{(1-zp)(1-\frac{z}{p})} \\ &= z\left(\frac{1}{1-\lambda^2p^4}\frac{1}{1-\frac{z}{p}} - \frac{\lambda^2p^4}{1-\lambda^2p^4}\frac{1}{1-zp}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{p^n(1-\lambda^2p^4)} - \frac{\lambda^2p^4p^n}{1-\lambda^2p^4}\right)z^{n+1} \\ &= \sum_{n=1}^{\infty}\frac{1-\lambda^2p^{2n+2}}{p^{n-1}(1-\lambda^2p^4)}z^n, |z|<p, \end{aligned} \tag{50}$$

and

$$\begin{aligned} g(z) &= -\frac{\frac{\lambda p(1-p^2)}{1-p^4\lambda^2}z^2\omega(z)}{(1-zp)(1-\frac{z}{p})} \\ &= -\omega(z)\left(\frac{\lambda p}{1-\lambda^2p^4}\frac{1}{1-\frac{z}{p}} - \frac{\lambda p^3}{1-\lambda^2p^4}\frac{1}{1-zp}\right)z^2 \end{aligned}$$

$$= -\omega(z)\sum_{n=0}^{\infty}\left(\frac{\lambda p}{p^n(1-\lambda^2p^4)} - \frac{\lambda p^3 p^n}{1-\lambda^2 p^4}\right) z^{n+2}$$

$$= -\omega(z)\sum_{n=0}^{\infty}\frac{\lambda p(1-p^{2n+2})}{p^n(1-\lambda^2p^4)} z^{n+2}, |z| < p. \tag{51}$$

Hence, it remains to show the nth Taylor coefficient of $\omega(z)$. Let

$$\omega(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{52}$$

Then (51) becomes

$$g(z) = \sum_{n=2}^{\infty} b_n(\omega) z^n, |z| < p. \tag{53}$$

By (51), (52) and (53), we have

$$b_n(\omega) = -\sum_{k=0}^{n-2} c_k \frac{\lambda p^2}{p^{n-k-1}} \frac{1-p^{2(n-k)-2}}{1-\lambda^2p^4}. \tag{54}$$

For convenience, we set $m = n-2$. Then (54) is written as

$$b_m(\omega) = -\sum_{k=0}^{m} c_k \frac{1-p^{2(m-k)+2}}{p^{m-k+1}} \frac{\lambda p^2}{1-\lambda^2p^4}.$$

By Lemma 4.5, for ω as (52) and any $m \geq 0$, we have

$$|b_m(\omega)| = \left|\sum_{k=0}^{m} c_k \frac{1-p^{2(m-k)+2}}{p^{m-k}} \frac{\lambda p^2}{p(1-\lambda^2p^4)}\right| \leq \frac{1-p^{2m+2}}{p^m} \frac{\lambda p^2}{p(1-\lambda^2p^4)}. \tag{55}$$

It's easy to check that (55) is equivalent to

$$|b_n(\omega)| \leq \frac{\lambda p^2(1-p^{2n-2})}{p^{n-1}(1-\lambda^2p^4)}. \tag{56}$$

Combining (49), (50), (51) and (56), we obtain (48). ■

Remark 4.7. When $\alpha = 0$, Theorems 2.2 and 4.6 correspond to Lemmas 1.1 and 1.2 respectively.

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