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# Fekete-Szegö Results for Certain Class of Non-Bazilevič Functions Involving Linear Operator

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**Abstract.** In this paper, by using the principal of subordination, we obtain sharp bounds for certain class of non-Bazilevič functions involving linear operator.

**Keywords:** Non-Bazilevič functions; Schwarz function; Fekete-Szegö results; Linear operator; Sharp bounds.

## 1. Introduction

Denote by $\mathbb{A}$ the class of univalent analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}). \tag{1}$$

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$ $(f(z) \prec g(z))$ in $\mathbb{U}$, if there exists function $\omega(z)$, analytic in $\mathbb{U}$ with

$\omega(0)=0$ and $|\omega(z)|<1$, $f(z)=g(\omega(z))$ $(z\in\mathbb{U})$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [3], [5], [12] and also [18]):

$$f(z)\prec g(z)\Longleftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

The Hadamard product of $f(z)$ and $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$ , is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k=(g*f)(z).$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0)=1$, $\phi^{'}(0)>0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. For $b\in\mathbb{C}^*=\mathbb{C}\backslash\{\mathbf{0}\}$, Ravichandran et al. [16] defined the classes $\mathcal{S}_b^*(\phi)$ and $\mathcal{C}_b(\phi)$ as follow:

$$\mathcal{S}_b^*(\phi)=\left\{f\in\mathbb{A}:1+\frac{1}{b}\left(\frac{zf^{'}(z)}{f(z)}-1\right)\prec\phi(z)\quad(z\in\mathbb{U})\right\}$$

and

$$\mathcal{C}_b(\phi)=\left\{f\in\mathbb{A}:1+\frac{1}{b}\frac{zf^{''}(z)}{f^{'}(z)}\prec\phi(z)\quad(z\in\mathbb{U})\right\}.$$

We note that:

(i) $\mathcal{S}_b^*(\frac{1+z}{1-z})=\mathcal{S}^*(b)$ (see [14]).

(ii) $\mathcal{C}_b(\frac{1+z}{1-z})=\mathcal{C}(b)$ (see [23] and also [13]).

(iii) $\mathcal{S}_b^*(\frac{1+(1-2\alpha)z}{1-z})=\mathcal{S}_\alpha^*(b)$ (see [9]) $(0\le\alpha<1)$.

(iv) $\mathcal{C}_b(\frac{1+(1-2\alpha)z}{1-z})=\mathcal{C}_\alpha(b)$ (see [9]) $(0\le\alpha<1)$.

(v) $\mathcal{S}_1^*(\phi)=\mathcal{S}^*(\phi)$ and $\mathcal{C}_1(\phi)=\mathcal{C}(\phi)$ (see [11]).

For $-1\le B<A\le 1, 0<\alpha<1, \gamma\in\mathbb{C}$, Wang et al. [22] introduced and studied the class $N(\gamma,\alpha;A,B)$ of $f(z)\in\mathbb{A}$ satisfying

$$(1+\gamma)\left(\frac{z}{f(z)}\right)^{\alpha}-\gamma\frac{zf^{'}(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{1+Az}{1+Bz}.$$

For $l\ge 0,\lambda>0,\ m\in\mathbb{N}_0=\{0,1,2,...\}$, Catas et al. [6] defined the operator $D_{\lambda,l}^m:\mathbb{A}\to\mathbb{A}$ as follow:

$$D_{\lambda,l}^m f(z)=z+\sum_{k=2}^{\infty}\left(\frac{l+1+\lambda(k-1)}{l+1}\right)^m a_k z^k.$$

The operator $D_{\lambda,l}^m$ generalizes many others see ( [1], [7], [8], [10], [17] and [21]).

By making use of the operator $D_{\lambda,l}^m$, $b\in\mathbb{C}^*$ and the principle of subordination between analytic functions, we now introduce the following class of non-Bazilevič functions.

**Definition 1.1.** *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$, $\phi'(0) > 0, \gamma \in \mathbb{C}$ and $0 < \alpha < 1$. A function $f(z) \in \mathbb{A}$ is said to be in the class $R^{\alpha,\gamma}_{m,\lambda,l}(b,\phi)$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D^m_{\lambda,l}f(z)}\right)^{\alpha} - \gamma\frac{z\left(D^m_{\lambda,l}f(z)\right)'}{D^m_{\lambda,l}f(z)}\left(\frac{z}{D^m_{\lambda,l}f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z).$$

We note that:

(i) $R^{\alpha,\gamma}_{0,\lambda,l}(1,\phi) = R^{\alpha,\gamma}(\phi)$ (see [19]).

(ii) $R^{\alpha,\gamma}_{0,1,0}\left(1, \frac{1+Az}{1+Bz}\right) = R^{\alpha,\gamma}(A,B)$ $(-1 \le B < A \le 1)$ (see [22]).

(iii) $R^{\alpha,-1}_{0,1,0}\left(1, \frac{1+(1-2\rho)z)}{1-z}\right) = R^{\alpha}(\rho)$ $(0 \le \rho < 1)$ (see [20]).

(iv) $R^{\alpha,-1}_{0,1,0}\left(1, \frac{1+z}{1-z}\right) = R^{\alpha}$ (see [15]).

Also, we obtain new subclasses for different values of $\lambda$ and $l$ :

(i) $R^{\alpha,\gamma}_{m,1,l}(b,\phi) = R^{\alpha,\gamma}_{m,l}(b,\phi) =$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{I^m_l f(z)}\right)^{\alpha} - \gamma\frac{z\left(I^m_l f(z)\right)'}{I^m_l f(z)}\left(\frac{z}{I^m_l f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z),$$

where $I^m_l f(z)$ was introduced by Cho and Srivastava [8] and Cho and Kim [7].

(ii) $R^{\alpha,\gamma}_{m,1,0}(b,\phi) = R^{\alpha,\gamma}_{m}(b,\phi) =$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D^m f(z)}\right)^{\alpha} - \gamma\frac{z\left(D^m f(z)\right)'}{D^m f(z)}\left(\frac{z}{D^m f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z),$$

where $D^m f(z)$ was introduced by Sălăgean [17] (see also [4]).

(iii) $R^{\alpha,\gamma}_{m,\lambda,0}(b,\phi) = R^{\alpha,\gamma}_{m,\lambda}(b,\phi) =$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D^m_{\lambda} f(z)}\right)^{\alpha} - \gamma\frac{z\left(D^m_{\lambda} f(z)\right)'}{D^m_{\lambda} f(z)}\left(\frac{z}{D^m_{\lambda} f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z),$$

where $D^m_{\lambda} f(z)$ was introduced by Al-Oboudi [1].

(iv) $R^{\alpha,\gamma}_{m,1,1}(b,\phi) = R^{\alpha,\gamma}_{m}(b,\phi) =$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{I^m f(z)}\right)^{\alpha} - \gamma\frac{z\left(I^m f(z)\right)'}{I^m f(z)}\left(\frac{z}{I^m f(z)}\right)^{\alpha} - 1\right\} \prec \phi(z),$$

where $I^m f(z)$ was introduced by Uralegaddi and Somanatha [21].

(v) For $b=(1-\beta)\cos\theta\ e^{-i\theta}\left(|\theta|<\frac{\pi}{2},0\leq\beta<1\right)$, the class $R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$ reduces to the class

$$R_{m,\lambda,l}^{\alpha,\gamma}(\beta,\theta,\phi)=\left\{f:\frac{e^{-i\theta}\Psi f(z)-\beta\cos\theta-i\sin\theta}{(1-\beta)\cos\theta}\prec\phi(z)\right\},\qquad(2)$$

where

$$\Psi f(z)=(1+\gamma)\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-\gamma\frac{z\left(D_{\lambda,l}^{m}f(z)\right)'}{D_{\lambda,l}^{m}f(z)}\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}$$

and also, putting $\gamma=-1$ in (2), we have the following subclass:

$$R_{m,\lambda,l}^{\alpha,-1}(\beta,\theta,\phi)=\left\{f:\frac{e^{-i\theta}\frac{z\left(D_{\lambda,l}^{m}(z)\right)'}{D_{\lambda,l}^{m}f(z)}\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-\beta\cos\theta-i\sin\theta}{(1-\beta)\cos\theta}\prec\phi(z)\right\}.$$

## 2. Main Results

In order to prove our results, we need the following lemmas.

**Lemma 2.1.** [11] *If $p(z)=1+c_1z+c_2z^2+\dots$ is a function with positive real part in $\mathbb{U}$ and $\mu$ is a complex number, then*

$$\left|c_2-\mu c_1^2\right|\leq 2\max\{1;|2\mu-1|\}.$$

*The result is sharp for the functions*

$$p(z)=\frac{1+z^2}{1-z^2}\ \text{and}\ p(z)=\frac{1+z}{1-z}.$$

**Lemma 2.2.** [11] *If $p(z)=1+c_1z+c_2z^2+\dots$ is an analytic function with a positive real part in $\mathbb{U}$, then*

$$\left|c_2-vc_1^2\right|\leq\begin{cases}-4v+2 & \text{if}\quad v\leq 0,\\ 2 & \text{if}\quad 0\leq v\leq 1,\\ 4v-2 & \text{if}\quad v\geq 1.\end{cases}$$

*When $v<0$ or $v>1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$, the equality holds if and only if*

$$p(z)=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}\quad(0\leq\lambda\leq 1),$$

*or one of its rotations. If* $v = 1$*, the equality holds if and only if* $p$ *is the reciprocal of one of the functions such that equality holds in the case of* $v = 0$*. Also the above upper bound is sharp, and it can be improved as follows when* $0 < v < 1$*:*

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \le 2 \quad \left(0 \le v \le \frac{1}{2}\right)$$

*and*

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \le 2 \quad \left(\frac{1}{2} \le v \le 1\right).$$

*The technique used by (see [2]) is applied here.*

Unless otherwise mentioned, we assume that $l \ge 0, \lambda > 0$, $m \in \mathbb{N}_0$, $\phi(0) = 1, \phi^{'}(0) > 0, \gamma \in \mathbb{C}, b \in \mathbb{C}^*$ and $0 < \alpha < 1$.

**Theorem 2.3.** *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *with* $B_1 > 0$*. If* $f(z)$ *given by* (1) *belongs to the class* $R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$ *with* $\alpha + \gamma \neq 0$ *and* $\alpha + 2\gamma \neq 0$*, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|B_1}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m} \max\left\{1; \left|\frac{B_2}{B_1} - \frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2} + \frac{\mu b(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}\right|\right\}.$$

*Proof.* If $f(z) \in R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$, then there is a function $\omega$, analytic with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D_{\lambda,l}^m f(z)}\right)^\alpha - \gamma\frac{z\left(D_{\lambda,l}^m f(z)\right)^{'}}{D_{\lambda,l}^m f(z)}\left(\frac{z}{D_{\lambda,l}^m f(z)}\right)^\alpha - 1\right\} = \phi(\omega(z)). \quad (3)$$

Define the function $p(z)$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + .... \quad (4)$$

Since $\omega(z)$ is a Schwarz function, we see that $\Re\{p(z)\} > 0$ and $p(0) = 1$. Therefore,

$$\begin{aligned} \phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\ &= \phi\left\{\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + ...\right]\right\} \\ &= 1 + \frac{1}{2}c_1 B_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}c_1^2 B_2\right]z^2 + ... \quad . \end{aligned} \quad (5)$$

Now by subsituting (5) into (3), we have

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-\gamma\frac{z\left(D_{\lambda,l}^{m}(z)\right)'}{D_{\lambda,l}^{m}f(z)}\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-1\right\}$$
$$=1+\frac{1}{2}c_1B_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}c_1^2B_2\right]z^2+\ldots \quad .$$

So, we obtain

$$-(\alpha+\gamma)\left(\frac{l+\lambda+1}{l+1}\right)^{m}a_2=\frac{1}{2}bc_1B_1,$$

$$-(\alpha+2\gamma)\left[\left(\frac{l+2\lambda+1}{l+1}\right)^{m}a_3-\frac{1}{2}(\alpha+1)\left(\frac{l+\lambda+1}{l+1}\right)^{2m}a_2^2\right]$$
$$=\frac{1}{2}bB_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}bB_2c_1^2,$$

or, equivalenty,

$$a_2=\frac{-bc_1B_1}{2(\alpha+\gamma)\left(\frac{l+\lambda+1}{l+1}\right)^{m}},$$
$$a_3=\frac{-bB_1}{2(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^{m}}\left\{c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2}\right]c_1^2\right\}.$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{-bB_1}{2(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^{m}}\left[c_2-vc_1^2\right], \tag{6}$$

where

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2}-\frac{\mu b(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^{m}B_1}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}\right].$$

Our result now follows by using Lemma 2.1. The result is sharp for the functions

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-\gamma\frac{z\left(D_{\lambda,l}^{m}f(z)\right)'}{D_{\lambda,l}^{m}f(z)}\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-1\right\}=\phi(z^2)$$

and

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-\gamma\frac{z\left(D_{\lambda,l}^{m}f(z)\right)'}{D_{\lambda,l}^{m}f(z)}\left(\frac{z}{D_{\lambda,l}^{m}f(z)}\right)^{\alpha}-1\right\}=\phi(z).$$

This completes the proof. ■

Putting $m = 0$ in Theorem 2.3, we obtain the following result.

**Corollary 2.4.** *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. If $f(z)$ given by* (1) *belongs to the class $R^{\alpha,\gamma}(b, \phi)$ with $\alpha + \gamma \neq 0$ and $\alpha + 2\gamma \neq 0$ , then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|B_1}{|\alpha+2\gamma|} \max\left\{1; \left|\frac{B_2}{B_1} - \frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2} + \frac{\mu b(\alpha+2\gamma)B_1}{(\alpha+\gamma)^2}\right|\right\}.$$

Putting $\gamma = -1$ and $\alpha = 0$ in Corollary 2.4, we obtain the following result which modifies the result obtained by Ravichandran et al. [16].

**Corollary 2.5.** *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. If $f(z)$ given by* (1) *belongs to the class $\mathcal{S}_b^*(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|B_1}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - 2\mu)\, bB_1\right|\right\}.$$

Putting $b = (1 - \beta)\cos\theta\, e^{-i\theta}\left(|\theta| < \frac{\pi}{2}, 0 \leq \beta < 1\right)$ in Theorem 2.3, we obtain the following result.

**Corollary 2.6.** *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. If $f(z)$ given by* (1) *belongs to the class $R_{m,\lambda,l}^{\alpha,\gamma}(\theta, \beta, \phi)$ with $\alpha + \gamma \neq 0$ and $\alpha + 2\gamma \neq 0$, then*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \leq\ & \frac{(1-\beta)\cos\theta B_1}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m} \\ & \times \max\left\{1; \left|\begin{array}{l}\frac{B_2}{B_1}e^{i\theta} - \frac{(\alpha+1)(\alpha+2\gamma)(1-\beta)\cos\theta B_1}{2(\alpha+\gamma)^2} \\ + \frac{\mu(1-\beta)\cos\theta(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}\end{array}\right|\right\}.\end{aligned}$$

*Remark 2.7.* For $\gamma = -1, b = 1$ and $\phi(z) = \frac{1+(1-2\rho)z}{1-z}$ $(0 \leq \rho < 1)$ in Corollary 2.4, we obtain the result of Tuneski and Darus [20].

**Theorem 2.8.** *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. Let*

$$\sigma_1 = \frac{\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}{2\left(\frac{l+2\lambda+1}{l+1}\right)^m}\left[(\alpha + 1) - \frac{2(\alpha + \gamma)^2(B_2 - B_1)}{b(\alpha + 2\gamma)B_1^2}\right],$$

$$\sigma_2 = \frac{\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}{2\left(\frac{l+2\lambda+1}{l+1}\right)^m}\left[(\alpha + 1) - \frac{2(\alpha + \gamma)^2(B_2 + B_1)}{b(\alpha + 2\gamma)B_1^2}\right],$$

*and*

$$\sigma_3 = \frac{\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}{2\left(\frac{l+2\lambda+1}{l+1}\right)^{m}}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2 B_2}{b(\alpha+2\gamma)B_1^2}\right].$$

*If $f(z)$ given by (1) belongs to the class $R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$ with $\alpha+2\gamma\neq 0$, then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|b|B_2}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m} - \frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2\left(\frac{l+2\lambda+1}{l+1}\right)^m} + \frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}} & \mu\geq\sigma_1, \\ \frac{|b|B_1}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m} & \sigma_2\leq\mu\leq\sigma_1, \\ \frac{-|b|B_2}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m} + \frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2\left(\frac{l+2\lambda+1}{l+1}\right)^m} - \frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}} & \mu\leq\sigma_2. \end{cases}$$

*Further, if $\sigma_3\leq\mu\leq\sigma_1$, then*

$$\left|a_3-\mu a_2^2\right| + \frac{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}{|b|\,|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1^2} \times\left[B_1 - B_2 + \frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2} - \frac{\mu b(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1^2}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}\right]|a_2|^2 \leq \frac{|b|B_1}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m}.$$

*If $\sigma_2\leq\mu\leq\sigma_3$, then*

$$\left|a_3-\mu a_2^2\right| + \frac{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}{|b|\,|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1^2} \times\left[B_1 + B_2 - \frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2} + \frac{\mu b(\alpha+2\gamma)\left(\frac{l+2\lambda+1}{l+1}\right)^m B_1^2}{(\alpha+\gamma)^2\left(\frac{l+\lambda+1}{l+1}\right)^{2m}}\right]|a_2|^2 \leq \frac{|b|B_1}{|\alpha+2\gamma|\left(\frac{l+2\lambda+1}{l+1}\right)^m}.$$

*Proof.* The results of Theorem 2.8 follows by applying Lemma 2.2 to (6). To show that the bounds are sharp, we define the functions $\chi_{\phi n}\,(n=2,3,4,...)$, $F_\lambda$ and $\xi_\lambda$ $(0\leq\lambda\leq 1)$ by

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D_{\lambda,l}^m\chi_{\phi n}(z)}\right)^{\alpha} - \gamma\frac{z\left(D_{\lambda,l}^m\chi_{\phi n}(z)\right)'}{D_{\lambda,l}^m\chi_{\phi n}(z)}\left(\frac{z}{D_{\lambda,l}^m\chi_{\phi n}(z)}\right)^{\alpha} - 1\right\} = \phi\left(z^{n-1}\right),$$

$$\chi_{\phi n}(0) = 0 = \chi'_{\phi n}(0) - 1,$$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D^m_{\lambda,l}F_\lambda(z)}\right)^\alpha - \gamma\frac{z\left(D^m_{\lambda,l}F_\lambda(z)\right)'}{D^m_{\lambda,l}F_\lambda(z)}\left(\frac{z}{D^m_{\lambda,l}F_\lambda(z)}\right)^\alpha - 1\right\}$$
$$= \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$F_\lambda(0) = 0 = F'_\lambda(0) - 1,$$

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{D^m_{\lambda,l}\xi_\lambda(z)}\right)^\alpha - \gamma\frac{z\left(D^m_{\lambda,l}\xi_\lambda(z)\right)'}{D^m_{\lambda,l}\xi_\lambda(z)}\left(\frac{z}{D^m_{\lambda,l}\xi_\lambda(z)}\right)^\alpha - 1\right\}$$
$$= \phi\left(-\frac{1+\lambda z}{z(z+\lambda)}\right),$$

$$\xi_\lambda(0) = 0 = \xi'_\lambda(0) - 1.$$

Clearly, the functions $\chi_{\phi n}, F_\lambda$ and $\xi_\lambda \in R^{\alpha,\gamma}_{m,\lambda,l}(b,\phi)$. If $\mu > \sigma_1$ or $\mu < \sigma_2$, then the equality holds if and only if $f(z)$ is $\chi_{\phi 2}$, or one of its rotations. When $\sigma_2 < \mu < \sigma_1$, the equality holds if and only if $f(z)$ is $\chi_{\phi 3}$, or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is $F_\lambda$, or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is $\xi_\lambda$, or one of its rotations. ■

Taking $m = 0$ in Theorem 2.8, we obtain the following result for the function belonging to the class $R^{\alpha,\gamma}(b,\phi)$

**Corollary 2.9.** *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *with* $B_1 > 0$. *Let*

$$\sigma_4 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2(B_2 - B_1)}{b(\alpha+2\gamma)B_1^2}\right],$$

$$\sigma_5 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2(B_2 + B_1)}{b(\alpha+2\gamma)B_1^2}\right],$$

*and*

$$\sigma_6 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2 B_2}{b(\alpha+2\gamma)B_1^2}\right].$$

*If $f(z)$ given by (1) belongs to the class $R^{\alpha,\gamma}(b,\phi)$ with $\alpha+2\gamma\neq 0$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{|b|B_2}{|\alpha+2\gamma|}-\frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2}+\frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2} & \text{if } \mu\geq\sigma_4,\\ \frac{|b|B_1}{|\alpha+2\gamma|} & \text{if } \sigma_5\leq\mu\leq\sigma_4,\\ \frac{-|b|B_2}{|\alpha+2\gamma|}+\frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2}-\frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2} & \text{if } \mu\leq\sigma_5.\end{cases}$$

*Further, if $\sigma_6\leq\mu\leq\sigma_4$, then*

$$\left|a_3-\mu a_2^2\right|+\frac{(\alpha+\gamma)^2}{|b|\,|\alpha+2\gamma|\,B_1^2}\times\left[B_1-B_2+\frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2}-\frac{\mu b(\alpha+2\gamma)B_1^2}{(\alpha+\gamma)^2}\right]|a_2|^2\leq\frac{|b|B_1}{|\alpha+2\gamma|}.$$

*If $\sigma_5\leq\mu\leq\sigma_6$, then*

$$\left|a_3-\mu a_2^2\right|+\frac{(\alpha+\gamma)^2}{|b|\,|\alpha+2\gamma|\,B_1^2}\times\left[B_1+B_2-\frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2}+\frac{\mu b(\alpha+2\gamma)B_1^2}{(\alpha+\gamma)^2}\right]|a_2|^2\leq\frac{|b|B_1}{|\alpha+2\gamma|}.$$

*Remark 2.10.* For $b=1$ in Corollary 2.9, we obtain the result of Shanmugam et al. [19].

*Remark 2.11.* Specializing the parameters $\lambda, l$ and $m$ in the above results, we obtain corresponding results for the subclasses (i)–(iv) involving the different operators mentioned in the introduction.

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