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Uniqueness of an Entire Function with its Derivatives Sharing Two Polynomials

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Abstract. The uniqueness problems of entire function that share a non-zero finite value have been studied and many results on this topic have been obtained. In this paper we prove a uniqueness theorem for an entire function, which shares polynomials with its higher order derivatives. In particular, the result of the paper is an improvement of the corresponding results of H. Zhong [8] and I. Lahiri and G.K. Ghosh [5].

Keywords: Uniqueness; Entire function; Polynomial; Sharing; Derivatives.

1. Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$ and $a = a(z)$ be a polynomial. We denote by $E(a; f)$ the set of zeros of $f - a$, counted with multiplicities and by $\overline{E}(a; f)$ the set of distinct zeros of $f - a$.

If for two meromorphic functions f and g, $E(a; f) = E(a; g)$ then we say that f and g share a CM and if $\overline{E}(a; f) = \overline{E}(a; g)$ then we say that f and g share a IM.

For standard definitions and notations of the value distribution theory we refer the reader to [3] and [6].

There are some results related to value sharing. In the begining, G. Jank, E. Mues and L. Volkmann [4] considered the case when an entire function shared a single value with its first two derivatives and proved the following theorem.

Theorem 1.1. [4] *Let f be a non-constant entire function and a be a non-zero* *finite value. If* $\overline{E}(a;f) = \overline{E}(a;f^{(1)}) \subset \overline{E}(a;f^{(2)})$, *then* $f \equiv f^{(1)}$.

In 2002, J. Chang and M. Fang [1] extended Theorem 1.1 in the following way.

Theorem 1.2. [1] *Let* f *be a non-constant entire function and* a, b *be two non-zero finite constants. If* $\overline{E}(a;f) \subset \overline{E}(a;f^{(1)}) \subset \overline{E}(b;f^{(2)})$, *then either* $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ *or* $f = \lambda e^{\frac{bz}{a}} + a$, *where* $\lambda(\neq 0)$ *is a constant.*

Following example shows that in Theorem 1.1 the second derivative cannot be replaced by any higher order derivatives.

Example 1.3. [8] Let $k(\geq 3)$ be an integer and $\omega(\neq 1)$ be a $(k-1)^{th}$ root of unity. We put $f = e^{\omega z} + \omega - 1$. Then f, $f^{(1)}$ and $f^{(k)}$ share the value ω CM, but $f \not\equiv f^{(1)}$.

On the basis of this example, H. Zhong [8] improved Theorem 1.1 by considering higher order derivatives in the following way.

Theorem 1.4. [8] *Let* f *be a non-constant entire function and* a *be a non-zero finite complex constant. If* $E(a;f) = E(a;f^{(1)})$ *and* $\overline{E}(a;f) \subset \overline{E}(a;f^{(n)}) \cap \overline{E}(a;f^{(n+1)})$ *for* $n(\geq 1)$, *then* $f \equiv f^{(n)}$.

For further discussion we need the following notation.

Let f be a non-constant meromorphic function. For $A \subset \mathbb{C}$, we define $N_A(r,a;f)$ as follows

$$N_A(r,a;f) = \int_0^r \frac{n_A(t,a;f) - n_A(0,a;f)}{t} dt + n_A(0,a;f)\log r,$$

where $n_A(t,a;f)$ is the number of zeros of $f - a$, counted with multiplicities, which lie in $\{z : |z| \leq r\} \cap A$. For $A \subset \mathbb{C} \cup \{\infty\}$, the counting function (reduced counting function) of those a-points of f which belong to A is denote by $N_A(r,a;f)(\overline{N}_A(r,a;f))$. Let $T(r,f)$ be the characteristic function of f. We denote by $S(r,f)$ is any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ defined in $\mathbb{C}$ is called a small function of f if $T(r,a) = S(r,f)$.

For two subsets A and B of $\mathbb{C}$, we denote by $A \triangle B$ the symmetric difference of A and B i.e., $A \triangle B = (A - B) \cup (B - A)$.

In 2011, I. Lahiri and G.K. Ghosh [5] improved Theorem 1.4 in the following manner.

Theorem 1.5. [5] *Let* f *be a non-constant entire function and* a, b *be two non-zero finite constants. Suppose further that* $A = \overline{E}(a;f) \setminus \overline{E}(a;f^{(1)})$ *and* $B =$

$\overline{E}(a; f^{(1)}) \setminus \overline{E}(a; f^{(n)}) \cap \overline{E}(b; f^{(n+1)})$ *for* $n(\geq 1)$.

If

(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$,

(ii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then either $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ *or* $f = \lambda e^{\frac{bz}{a}} + a$, *where* $\lambda(\neq 0)$ *is a constant.*

In Theorem 1.5, I. Lahiri and G.K. Ghosh considered an entire function which shares constants with its derivatives. In this paper we improve Theorem 1.5 by considering an entire function which shares polynomials. The main result of the paper is the following theorem.

Theorem 1.6. *Let* f *be a non-constant entire function and* $a(\not\equiv 0)$, $b(\not\equiv 0)$ *be two polynomials of degree* $p(\geq 1)$ *and* $q(\geq 1)$ *respectively. Also suppose that* $n(\geq \max\{p, q\})$ *be a positive integer. Further suppose that* $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$ *and* $B = \overline{E}(a; f^{(1)})\setminus\{\overline{E}(a; f^{(n)}) \cap \overline{E}(b; f^{(n+1)}) \cap \overline{E}(a; f^{(n+2)})\}$.

If

(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$,

(ii) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$, $E_{1)}(a; f)$ *are the simple zeros of* $f - a$,

(iii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then the following statements hold:

(i) *for* $n = 1$, $f = \lambda e^z$, *where* $\lambda(\neq 0)$ *is a constant,*

(ii) *for* $n > 1$, *either* $f = \lambda e^z$ *or* $a \equiv b$ *and* $f = \lambda e^z + a$,

where $\lambda(\neq 0)$ *is a constant.*

Putting $A = B = \Phi$, we get the following corollary.

Corollary 1.7. *Let* f *be a non-constant entire function and* $a(\not\equiv 0)$, $b(\not\equiv 0)$ *be two polynomials of degree* $p(\geq 1)$ *and* $q(\geq 1)$ *respectively. Also suppose that* $n(\geq \max\{p, q\})$ *be a positive integer. If* $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; f^{(n)}) \cap \overline{E}(b; f^{(n+1)}) \cap \overline{E}(a; f^{(n+2)})\}$, *then the conclusion of Theorem 1.6 holds.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. [3, pp. 47] *Let* f *be a non-constant meromorphic function and* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\nu) = S(r, f)$ *for* $\nu = 1, 2, 3$. *Then*

$$T(r, f) \leq \sum_{\nu=1}^{3} \overline{N}(r, a_\nu; f) + S(r, f).$$

Lemma 2.2. [7] *Let g be a transcendental meromorphic function and $\phi(\not\equiv 0)$ be a meromorphic function satisfying $T(r,\phi) = S(r,g)$. Then*

$$T(r,g) \leq C_n\{N(r,0;g) + \overline{N}(r,0;g^{(n)} - \phi)\} + S(r,g),$$

where C_n is a constant depending only on $n(\geq 1)$.

Following lemma is an easy consequence of Lemma 2.2.

Lemma 2.3. *Let f be a transcendental meromorphic function. Also let a and b be two meromorphic functions satisfying $b - a^{(n)} \not\equiv 0$ and $T(r,a) + T(r,b) = S(r,f)$. Then*

$$T(r,f) \leq C_n\{N(r,a;f) + \overline{N}(r,b;f^{(n)})\} + S(r,f),$$

where C_n is a constant depending only on $n(\geq 1)$.

Proof. Putting $g = f - a$ and $\phi = b - a^{(n)}$ in Lemma 2.2, we obtain Lemma 2.3. ∎

Lemma 2.4. [3, pp. 57] *Suppose that g be a non-constant meromorphic function and $\psi = \sum_{\nu=0}^{l} a_\nu g^{(\nu)}$, where $a_\nu{}'s$ are meromorphic functions satisfying $T(r,a_\nu) = S(r,g)$ for $\nu = 1,2,\ldots,l$. If ψ is non-constant, then*

$$T(r,g) \leq \overline{N}(r,\infty;g) + N(r,0;g) + \overline{N}(r,1;\psi) + S(r,g).$$

The above lemma motivates us to prove the following:

Lemma 2.5. *Let f be a transcendental meromorphic function and a be a polynomial. Then for any positive integer n,*

$$T(r,f) \leq \overline{N}(r,\infty;f) + N(r,a;f) + \overline{N}(r,a;f^{(n)}) + S(r,g).$$

Proof. Putting $g = f - a$ and $\psi = \frac{g^{(n)}}{a - a^{(n)}}$ in Lemma 2.4, we obtain Lemma 2.5. ∎

Lemma 2.6. [3, pp. 69] *Let f be a non-constant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial generated by f and of degree at most $n-1$.

If $N(r,\infty;f) + N(r,0;g) = S(r,f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

Let us note the special case, where $P_{n-1}(f) = a_0(z)f^{n-1}+$ terms of degree $n-2$ at most. Then $h^{n-1}(z)a(z) = a_0(z)h^{n-1}(z)$ and so $a(z) = a_0(z)$. Hence

$$g(z) = \left(f(z) + \frac{a_0(z)}{n}\right)^n .$$

Lemma 2.7. [6, pp. 92] *Suppose that $f_1, f_2, \ldots, f_n (n \geq 3)$ are meromorphic functions which are not constants except for f_n. Furthermore, let $\sum_{j=1}^{n} f_j \equiv 1$.*

If $f_n \not\equiv 0$ and $\sum_{j=1}^{n} N(r, 0; f_j) + (n-1)\sum_{j=1}^{n} \overline{N}(r, \infty; f_j) < \{\lambda + o(1)\}T(r, f_k)$, where $r \in I$, $k = 1, 2, \ldots, n-1$ and $\lambda < 1$, then $f_n \equiv 1$.

Lemma 2.8. [2] *Let f be a non-constant meromorphic function and n be a positive integer. If there exist meromorphic functions $a_0(\not\equiv 0)$, a_1, $a_2, \ldots, a_n$ such that*

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r, f) \leq nT(r, a_0) + \sum_{j=1}^{n} m(r, a_j) + (n-1)\log 2.$$

Lemma 2.9. *Let f be a meromorphic function. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q} \quad (a_0 b_0 \not\equiv 0),$$

where $a_0, a_1, a_2, \ldots, a_p$, $b_0, b_1, b_2, \ldots, b_q$ are meromorphic functions, then

$$T(r, R(f)) \leq O(T(r, f) + \sum_{i=1}^{p} T(r, a_i) + \sum_{j=1}^{q} T(r, b_j)).$$

Proof. The Lemma follows from the properties of the characteristic function and the First Fundamental Theorem. ∎

3. Proof of the Main Theorem

First we verify that f is not a polynomial. If f is a polynomial then $T(r, f) = O(\log r)$ and so $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$ implies that $A = B = \Phi$. Therefore by the hypothesis

$$\overline{E}(a; f)\Delta\overline{E}(a; f^{(1)}) = \{\overline{E}(a; f) - \overline{E}(a; f^{(1)})\} \cup \{\overline{E}(a; f^{(1)}) - \overline{E}(a; f)\} = \Phi.$$

This implies

$$\overline{E}(a;f)=\overline{E}(a;f^{(1)})\subset\{\overline{E}(a;f^{(n)})\cap\overline{E}(b;f^{(n+1)})\cap\overline{E}(a;f^{(n+2)})\}. \tag{1}$$

Let $deg(f)=u$. If $u\geq p+1$, then $deg(f-a)=u$, $deg(f^{(1)}-a)\leq u-1$. From (1) and each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, we arrive at a contradiction.

If $u\leq p-1$, then $deg(f-a)=p$ and $deg(f^{(1)}-a)=p$. By (1) and each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, we can write $f^{(1)}-a\equiv c(f-a)$, where $c(\neq 0)$ is a constant.

If $c\neq 1$, then $cf-f^{(1)}\equiv(c-1)a$, which is impossible as $deg((c-1)a)=p>u=deg(cf-f^{(1)})$.

If $c=1$ then $f=f^{(1)}$, which is again a contradiction.

Finally if $u=p$, then from (1), $c_1f\equiv a\equiv c_2f^{(1)}$, for some nonzero constants c_1,c_2. This is again a contradiction.

Therefore f is a transcendental entire function and $T(r,a)=S(r,f)$.

Since $a-a^{(1)}=(f^{(1)}-a^{(1)})-(f^{(1)}-a)$, a common zero of $f-a$ and $f^{(1)}-a$ of multiplicity $v(\geq 2)$ is a zero of $a-a^{(1)}$ with multiplicity $v-1(\geq 1)$. Therefore

$$\begin{aligned} N_{(2}(r,a;f) &\leq 2N(r,0;a-a^{(1)})+N_A(r,a;f)\\ &= S(r,f). \end{aligned} \tag{2}$$

To prove our result, we first consider the following function

$$F=f-a.$$

Then from

$$\omega=\frac{f^{(1)}-a}{f-a}, \tag{3}$$

we obtain

$$\begin{aligned} F^{(1)} &= f^{(1)}-a^{(1)}\\ &= f^{(1)}-a+(a-a^{(1)})\\ &= \omega F+(a-a^{(1)})\\ &= \alpha_1F+\beta_1, \end{aligned} \tag{4}$$

where $\alpha_1=\omega$ and $\beta_1=a-a^{(1)}=r$ (say).

Differentiating both sides of (4) and then using (4), we have

$$\begin{aligned} F^{(2)} &= \alpha_1F^{(1)}+\alpha_1^{(1)}F+\beta_1^{(1)}\\ &= \alpha_1(\alpha_1F+\beta_1)+\alpha_1^{(1)}F+\beta_1^{(1)}\\ &= (\alpha_1\alpha_1+\alpha_1^{(1)})F+\alpha_1\beta_1+\beta_1^{(1)}\\ &= \alpha_2F+\beta_2, \end{aligned} \tag{5}$$

where $\alpha_2 = \alpha_1\alpha_1 + \alpha_1^{(1)}$ and $\beta_2 = \alpha_1\beta_1 + \beta_1^{(1)}$.

Similarly,

$$F^{(k)} = \alpha_k F + \beta_k, \tag{6}$$

where $\alpha_{k+1} = \alpha_1\alpha_k + \alpha_k^{(1)}$ and $\beta_{k+1} = \beta_1\alpha_k + \beta_k^{(1)}$, for $k = 1, 2, \ldots$.

Now we shall prove that

$$T(r,\omega) = S(r,f). \tag{7}$$

If ω is a constant, then we get $T(r,\omega) = S(r,f)$.

So we suppose that ω is non-constant. Clearly from the hypothesis, we obtain

$$\begin{aligned} N(r,0;\omega) + N(r,\infty;\omega) &\le N_A(r,a;f) + N_A(r,a;f^{(1)}) \\ &= S(r,f). \end{aligned} \tag{8}$$

Now putting $k = 1$ in $\alpha_{k+1} = \alpha_1\alpha_k + \alpha_k^{(1)}$, we have

$$\begin{aligned} \alpha_2 &= \alpha_1\alpha_1 + \alpha_1^{(1)} \\ &= \omega^2 + \omega^{(1)} \\ &= \omega^2 + \omega h_1, \end{aligned}$$

where $h_1 = \frac{\omega^{(1)}}{\omega}$.

Again putting $k = 2$ in $\alpha_{k+1} = \alpha_1\alpha_k + \alpha_k^{(1)}$, we get

$$\begin{aligned} \alpha_3 &= \alpha_1\alpha_2 + \alpha_2^{(1)} \\ &= \omega(\omega^2 + \omega h_1) + (\omega^2 + \omega h_1)^{(1)} \\ &= \omega^3 + \omega^2 h_1 + 2\omega\omega^{(1)} + \omega h_1^{(1)} + \omega^{(1)} h_1 \\ &= \omega^3 + \omega^2 h_1 + 2\omega^2 h_1 + \omega h_1^{(1)} + \omega h_1^2 \\ &= \omega^3 + 3h_1\omega^2 + h_2\omega, \end{aligned}$$

where $h_2 = h_1^{(1)} + h_1^2$.

Similarly,

$$\omega_4 = \omega^4 + 6h_1\omega^3 + (h_2 + 6h_1^2 + 3h_1^{(1)})\omega^2 + (h_2^{(1)} + h_1 h_2)\omega.$$

Therefore in general, we get for $k \ge 2$

$$\alpha_k = \omega^k + \sum_{j=1}^{k-1} \gamma_j \omega^j, \tag{9}$$

where

$$\begin{aligned} T(r,\gamma_j) &= O(\overline{N}(r,0;\omega) + \overline{N}(r,\infty;\omega)) + S(r,\omega) \\ &= S(r,f), \end{aligned}$$

for $j = 1, 2, \ldots, k-1$.

Now putting $k = 1$ in $\beta_{k+1} = \beta_1 \alpha_k + \beta_k^{(1)}$, we get

$$\begin{aligned}\beta_2 &= \beta_1 \alpha_1 + \beta_1^{(1)} \\ &= \omega r + r^{(1)}.\end{aligned}$$

Also putting $k = 2$ in $\beta_{k+1} = \beta_1 \alpha_k + \beta_k^{(1)}$, we have

$$\begin{aligned}\beta_3 &= \beta_1 \alpha_2 + \beta_2^{(1)} \\ &= r(\omega^2 + \omega h_1) + (\omega r + r^{(1)})^{(1)} \\ &= r\omega^2 + r h_1 \omega + \omega r^{(1)} + \omega^{(1)} r + r^{(2)} \\ &= r\omega^2 + (r^{(1)} + 2 r h_1)\omega + r^{(2)}.\end{aligned}$$

Similarly,

$$\beta_4 = r\omega^3 + (5h_1\omega + r^{(1)})\omega^2 + (3r^{(1)}h_1 + 4rh_1^{(1)} + r^2 + h_2 r)\omega + r^{(3)}.$$

Therefore in general, we get for $k \geq 2$

$$\beta_k = \sum_{j=1}^{k-1} \delta_j \omega^j + r^{(k-1)}, \tag{10}$$

where

$$\begin{aligned}T(r, \delta_j) &= O(\overline{N}(r, 0; \omega) + \overline{N}(r, \infty; \omega)) + S(r, \omega) \\ &= S(r, f),\end{aligned}$$

for $j = 1, 2, \ldots, k-1$.

Before going to prove (7), let us divide the proof into the following two cases.

Case 1. In this case we suppose that $p = n = q = 1$. Here we have to consider following subcases.

Subcase 1.1. Let $f^{(1)} \not\equiv f^{(2)}$. Then we have two possibilities either $bf^{(1)} \equiv af^{(2)}$ or $bf^{(1)} \not\equiv af^{(2)}$.

Subcase 1.1.1. First we suppose that $bf^{(1)} \equiv af^{(2)}$. If $\overline{E}(a; f^{(1)}) \cap \overline{E}(b; f^{(2)}) \cap \overline{E}(a; f^{(3)}) = \Phi$, then $N(r, a; f^{(1)}) = N_B(r, a; f^{(1)}) = S(r, f)$.

Now from hypothesis and (2), we have

$$\begin{aligned}N(r, a; f) &\leq N_A(r, a; f) + N(r, a; f | f^{(1)} = a) \\ &\leq N_{1)}(r, a; f | f^{(1)} = a) + N_{(2}(r, a; f | f^{(1)} = a) + S(r, f) \\ &\leq \overline{N}(r, a; f | f^{(1)} = a) + N_{(2}(r, a; f^{(1)}) + S(r, f) \\ &\leq N(r, a; f^{(1)}) + S(r, f) \\ &= S(r, f),\end{aligned}$$

where $N_{1)}(r, a; f | f^{(1)} = a)$ denotes the simple a-points of f which are also a-points of $f^{(1)}$.

Using Lemma 2.3, we have $T(r, f) = S(r, f)$, which is a contradiction. Hence $\overline{E}(a; f^{(1)}) \cap \overline{E}(b; f^{(2)}) \cap \overline{E}(b; f^{(3)}) \neq \Phi$. Now differentiating both sides of $bf^{(1)} \equiv af^{(2)}$, we obtain

$$bf^{(2)} + b^{(1)}f^{(1)} \equiv af^{(3)} + a^{(1)}f^{(2)}.$$

This implies

$$\begin{aligned} af^{(3)} &\equiv (b - a^{(1)})f^{(2)} + b^{(1)}f^{(1)} \\ &\equiv \left(\frac{b^2}{a} - \frac{ba^{(1)}}{a} + b^{(1)}\right) f^{(1)}. \end{aligned}$$

If z_1 is a zero of $f^{(1)} - a$ which is also zero of $f^{(2)} - b$ and $f^{(3)} - a$, then from the above identity, we get z_1 is a zero of $a^2 - b^2 - ab^{(1)} + a^{(1)}b$.

If $a^2 - b^2 - ab^{(1)} + a^{(1)}b \not\equiv 0$, then using (2),

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(r, a; f^{(1)}|f^{(2)} = b, f^{(3)} = a) \\ &\leq O(\log r) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Again

$$\begin{aligned} \overline{N}(r, a; f^{(1)}) &\leq N_B(r, a; f^{(1)}) + \overline{N}(r, a; f^{(1)}|f^{(2)} = b, f^{(3)} = a) \\ &\leq O(\log r) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Applying Lemma 2.3, we get $T(r, f) = S(r, f)$, which is a contradiction.

Hence

$$a^2 - b^2 - ab^{(1)} + a^{(1)}b \equiv 0.$$

This implies

$$\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^{(1)} \equiv 1.$$

Therefore

$$\frac{a}{b} \equiv \frac{e^{2z} - c_1}{e^{2z} + c_1},$$

where c_1 is a constant.

Since a and b are polynomials, so $\frac{a}{b}$ is a rational function, we get $c_1 = 0$. Therefore from the above equality, we have $a \equiv b$. Hence $bf^{(1)} \equiv af^{(2)}$ implies that $f^{(1)} \equiv f^{(2)}$, a contradiction.

Subcase 1.1.2. Next we suppose that $bf^{(1)} \not\equiv af^{(2)}$. Then by the hypothesis of theorem, we have

$$\begin{aligned}
\overline{N}(r,a;f^{(1)}) &\le N\left(r,\frac{b-a^{(2)}}{a-a^{(1)}};\frac{f^{(2)}-a^{(2)}}{f^{(1)}-a^{(1)}}\right)+N_B(r,a;f^{(1)})\\
&\le T\left(r,\frac{f^{(2)}-a^{(2)}}{f^{(1)}-a^{(1)}}\right)+S(r,f)\\
&= m\left(r,\frac{f^{(2)}-a^{(2)}}{f^{(1)}-a^{(1)}}\right)+N\left(r,\frac{f^{(2)}-a^{(2)}}{f^{(1)}-a^{(1)}}\right)+S(r,f)\\
&\le N(r,a^{(1)};f^{(1)})+S(r,f). \qquad (11)
\end{aligned}$$

Again

$$\begin{aligned}
m(r,a;f) &= m\left(r,\frac{f^{(1)}-a^{(1)}}{f-a}\cdot\frac{1}{f^{(1)}-a^{(1)}}\right)\\
&\le m\left(r,\frac{f^{(1)}-a^{(1)}}{f-a}\right)+m\left(r,\frac{1}{f^{(1)}-a^{(1)}}\right)\\
&= m(r,a^{(1)};f^{(1)})+S(r,f)\\
&= T(r,f^{(1)})-N(r,a^{(1)};f^{(1)})+S(r,f)\\
&\le T(r,f)-N(r,a^{(1)};f^{(1)})+S(r,f).
\end{aligned}$$

This implies

$$N(r,a^{(1)};f^{(1)}) \le N(r,a;f)+S(r,f). \qquad (12)$$

Combining (11) and (12), we get

$$\overline{N}(r,a;f^{(1)}) \le N(r,a;f)+S(r,f).$$

Applying Lemma 2.5 and using above equality, we obtain

$$T(r,f) \le 2N(r,a;f)+S(r,f). \qquad (13)$$

Let

$$\Phi = \frac{(a-a^{(1)})f^{(2)}-b(f^{(1)}-a^{(1)})}{f-a}. \qquad (14)$$

Then by the Lemma of logarithmic derivative, we get $m(r,\Phi)=S(r,f)$.

Now by the hypothesis of our result and using (2), we have

$$\begin{aligned}
N(r,\Phi) &\le N_A(r,a;f)+N_B(r,a;f^{(1)})+N_{(2}(r,a;f)+S(r,f)\\
&= S(r,f).
\end{aligned}$$

Therefore $T(r,\Phi)=S(r,f)$.

Since in this case $p = 1$ i.e., a is a linear polynomial, so we must have $a^{(2)} = 0$. Now from (14), we obtain

$$\Phi F = rF^{(2)} - bF^{(1)}.$$

Substituting the values of $F^{(1)}$ and $F^{(2)}$ in the above equation, we get

$$\Phi F = r((\omega^2 + \omega h_1)F + \omega r + r^{(1)}) - b(\omega F + r).$$

Which implies

$$\left[r\omega^2 + (rh_1 - b)\omega - \Phi\right] F + r^2\omega - (b - a^{(1)})r = 0. \tag{15}$$

If $r\omega^2 + (rh_1 - b)\omega - \Phi \not\equiv 0$, then from (15) we have

$$F = -\frac{r^2\omega - (b - a^{(1)})r}{r\omega^2 + (rh_1 - b)\omega - \Phi}. \tag{16}$$

Applying Lemma 2.9 to the above equation, we get

$$T(r, F) = O(T(r, \omega)) + S(r, f).$$

Hence

$$\begin{aligned} T(r, f) &= T(r, F + a) \\ &\leq T(r, F) + T(r, a) + \log 2 \\ &= T(r, F) + S(r, f). \end{aligned}$$

Again

$$\begin{aligned} T(r, F) &= T(r, f - a) \\ &\leq T(r, f) + T(r, a) + \log 2 \\ &= T(r, f) + S(r, f). \end{aligned}$$

Therefore

$$\begin{aligned} T(r, f) &= T(r, F) + S(r, f) \\ &= O(T(r, \omega)) + S(r, f), \end{aligned}$$

which implies that $S(r, f)$ is replaced by $S(r, \omega)$.

From (16) we see that F is a rational function in ω, which can be made irreducible. We set

$$F = \frac{A_\phi(\omega)}{B_{\phi+1}(\omega)}, \tag{17}$$

where $A_\phi(\omega)$ and $B_{\phi+1}(\omega)$ are relatively prime polynomials in ω of respective degrees ϕ and $\phi+1$ ($\phi = 0, 1$). The coefficients of both the polynomials are small functions of ω. Without loss of generality we assume that $B_{\phi+1}(\omega)$ is a monic polynomial. Also we note that the counting function of the common zeros of $A_{\phi}(\omega)$ and $B_{\phi+1}(\omega)$ is $S(r,\omega)$, because $A_{\phi}(\omega)$ and $B_{\phi+1}(\omega)$ are relatively prime and the coefficients are small functions of ω.

Again since $N(r,\infty;F) = S(r,f) = S(r,\omega)$, then from (17), we get

$$N(r,0;B_{\phi+1}(\omega)) = S(r,\omega).$$

From (8), we can easily see that

$$N(r,\infty;\omega) = S(r,f) = S(r,\omega).$$

Applying Lemma 2.6, we obtain

$$B_{\phi+1}(\omega) = \left(\omega + \frac{Q}{\phi+1}\right)^{\phi+1}, \tag{18}$$

where Q is the coefficient in ω^{ϕ} in $B_{\phi+1}(\omega)$.

If $Q \not\equiv 0$, then using Lemma 2.1 we have

$$\begin{aligned} T(r,\omega) &\le \overline{N}(r,0;\omega) + \overline{N}(r,\infty;\omega) + \overline{N}\left(r, -\frac{Q}{\phi+1};\omega\right) + S(r,\omega) \\ &= \overline{N}(r,0;B_{\phi+1}(\omega)) + S(r,\omega) \\ &= S(r,\omega), \end{aligned}$$

which is a contradiction. Hence $Q \equiv 0$ and from (17) and (18), we have

$$F = \frac{A_{\phi}(\omega)}{\omega^{\phi+1}}. \tag{19}$$

Differentiating both sides of (19), we get

$$F^{(1)} = h_1 \frac{\omega A_{\phi}^{(1)}(\omega) - (\phi+1)A_{\phi}(\omega)}{\omega^{\phi+1}}, \tag{20}$$

where $h_1 = \frac{\omega^{(1)}}{\omega}$.

We note that

$$\begin{aligned} T(r,h_1) &= O(\overline{N}(r,0;\omega) + \overline{N}(r,\infty;\omega)) + m(r,h_1) \\ &= S(r,f) + S(r,\omega) \\ &= S(r,\omega). \end{aligned} \tag{21}$$

From (20), (21) and the properties of characteristic function, we obtain

$$T(r,F^{(1)}) \le (\phi+1)T(r,\omega) + S(r,\omega). \tag{22}$$

Again from (4) and (19), we get

$$\begin{aligned} F^{(1)} &= \omega F + r \\ &= \omega\left(\frac{A_\phi(\omega)}{\omega^{\phi+1}}\right) + r \\ &= \frac{A_\phi(\omega)}{\omega^{\phi}} + r, \end{aligned}$$

where $r = a - a^{(1)} \not\equiv 0$.

Therefore

$$T(r, F^{(1)}) \leq \phi T(r,\omega) + S(r,\omega). \tag{23}$$

Combining (22) and (23), we have

$$T(r,\omega) = S(r,\omega),$$

which is again a contradiction.

Now if $r\omega^2 + (rh_1 - b)\omega - \Phi \equiv 0$, then using Lemma 2.8 we conclude (7).

Again from (15) we have

$$r^2\left(\frac{b - a^{(1)}}{a - a^{(1)}} - \omega\right) = 0.$$

Since $r^2 \not\equiv 0$, we get

$$\omega = \frac{b - a^{(1)}}{a - a^{(1)}}. \tag{24}$$

From (3) and (24), we get

$$\frac{f^{(1)} - a}{f - a} = \frac{b - a^{(1)}}{a - a^{(1)}}$$

or

$$f^{(1)}(a - a^{(1)}) - f(b - a^{(1)}) - a(a - b) = 0. \tag{25}$$

Differentiating (25) twice, we get

$$f^{(3)}(a - a^{(1)}) + f^{(2)}(3a^{(1)} - b) - f^{(1)}2b^{(1)}) - 2a^{(1)}(a^{(1)} - b^{(1)}) = 0. \tag{26}$$

Now for a zero of $f - a$ which is common zero of $f^{(1)} - a$, $f^{(2)} - b$ and $f^{(3)} - a$, we have

$$(a^2 - b^2) + (3a^{(1)}b - 2ab^{(1)} - aa^{(1)}) - 2a^{(1)}(a^{(1)} - b^{(1)}) = 0. \tag{27}$$

If $a \not\equiv b$, then the left hand side of (27) is not identically equal to zero.

Then we have

$$\begin{aligned} N(r,a;f) \le\ & N_A(r,a;f) + N_B(r,a;f^{(1)}) + N(r,a;f|f^{(1)}=a, f^{(2)}=b, f^{(3)}=a) \\ =\ & O(\log r) + S(r,f) \\ =\ & S(r,f). \end{aligned}$$

From (13) and above equality, we get

$$T(r,f) = S(r,f),$$

which is a contradiction.

If $a \equiv b$ then from (25), $f^{(1)} = f$ and so $f^{(1)} = f^{(2)}$, which is again a contradiction.

Subcase 1.2 Next we suppose that $f^{(1)} \equiv f^{(2)}$. Then on integration, we get

$$f = \lambda e^z + \eta, \tag{28}$$

where $\lambda(\neq 0)$, η are constants.

Then obviously from (28) we have

$$f = f^{(1)} + \eta. \tag{29}$$

If $f-a$ and $f^{(1)}-a$ have no common zero then $N(r,a;f) = S(r,f)$ and from (13), $T(r,f) = S(r,f)$, a contradiction.

So $f-a$ and $f^{(1)}-a$ have some common zeros and from (29), $\eta = 0$.

Therefore $f = \lambda e^z$, $\lambda(\neq 0)$ is a constant.

Case 2. In this case, we suppose that $n > 1$. We now consider the following subcases.

Subcase 2.1. Let $f^{(n)} \not\equiv f^{(n+1)}$. Then we have two possibilities either $af^{(n+1)} \equiv bf^{(n)}$ or $af^{(n+1)} \not\equiv bf^{(n)}$.

Subcase 2.1.1. Let $af^{(n+1)} \equiv bf^{(n)}$. Then following the similar arguments of Subcase 1.1.1, we can easily prove that $a \equiv b$ and then $af^{(n+1)} \equiv bf^{(n)}$ implies that $f^{(n+1)} \equiv f^{(n)}$, which contradicts our assumption $f^{(n+1)} \not\equiv f^{(n)}$.

Subcase 2.1.2. Let $af^{(n+1)} \not\equiv bf^{(n)}$. Then following the similar arguments of Subcase 1.1.2 and applying Lemma 2.5, we can prove that

$$T(r,f) \le 2N(r,a;f) + S(r,f). \tag{30}$$

Now we suppose that

$$\Psi = \frac{(a-a^{(n)})f^{(n+1)} - b(f^{(n)} - a^{(n)})}{f-a}. \tag{31}$$

Then by (2) and the hypothesis of Theorem 1.6, we have

$$\begin{aligned} N(r,\Psi) \le\ & N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f) + S(r,f) \\ =\ & S(r,f). \end{aligned}$$

Clearly, $m(r, \Psi) = S(r, f)$. Hence $T(r, \Psi) = S(r, f)$.
Now (31) can be rewritten as

$$\Psi F - (a - a^{(n)})F^{(n+1)} + bF^{(n)} \equiv 0,$$

where $F = f - a$.

Now proceeding as in Subcase 1.1.2, we have $T(r, \omega) = S(r, f)$, where ω is given in (3).

Therefore $T(r, \alpha_k) + T(r, \beta_k) = S(r, f)$ for $k = 1, 2, \ldots$, where α_k and β_k are defined respectively in (9) and (10).

Let z_3 be a zero of $F = f - a$ such that $z_3 \notin A \cup B$. For $k = n + 1$, we get from (6)

$$F^{(n+1)} = \alpha_{n+1}F + \beta_{n+1}$$

or

$$f^{(n+1)} = \alpha_{n+1}(f - a) + \beta_{n+1}. \tag{32}$$

Since $z_3 \notin A \cup B$, then z_3 must be a zero of $f - a$, $f^{(1)} - a$, $f^{(n)} - a$, $f^{(n+1)} - b$, $f^{(n+2)} - a$.

Therefore $f(z_3) = a(z_3)$ and $f^{(n+1)}(z_3) = b(z_3)$.

From (32), we have

$$b(z_3) = \beta_{n+1}(z_3).$$

If $\beta_{n+1}(z) \not\equiv b(z)$, then we obtain

$$\begin{aligned} N(r, a; f) &\leq N_A(r, 0; f - a) + N(r, 0; b - \beta_n) + S(r, f) \\ &= S(r, f). \end{aligned}$$

From (30) we get $T(r, f) = S(r, f)$, a contradiction.

Hence

$$\beta_{n+1}(z) \equiv b(z).$$

Differentiating both sides of (32), we have

$$f^{(n+2)} = \alpha_{n+1}(f^{(1)} - a^{(1)}) + \alpha_{n+1}^{(1)}(f - a) + \beta_{n+1}^{(1)}.$$

At the point z_3, we get

$$a(z_3) = \alpha_{n+1}(z_3)(a(z_3) - a^{(1)}(z_3)) + \beta_{n+1}^{(1)}(z_3).$$

Again if

$$\alpha_{n+1}(z)(a(z) - a^{(1)}(z)) + \beta_{n+1}^{(1)}(z) \not\equiv a(z),$$

then we have

$$\begin{aligned} N(r,a;f) &\le N_A(r,0;f-a) + N(r,0;a-\alpha_{n+1}(a-a^{(1)}) - \beta^{(1)}_{n+1}) + S(r,f) \\ &= S(r,f). \end{aligned}$$

Again from (30), we get $T(r,f) = S(r,f)$, which is a contradiction.

Hence

$$\alpha_{n+1}(z)(a(z) - a^{(1)}(z)) + \beta^{(1)}_{n+1}(z) \equiv a(z)$$

or

$$\alpha_{n+1}(z)(a(z) - a^{(1)}(z)) + b^{(1)}(z) = a(z).$$

This implies

$$\alpha_{n+1} = \frac{a - b^{(1)}}{a - a^{(1)}}.$$

Putting the values of α_{n+1} and β_{n+1} in (32), we have

$$f^{(n+1)} = \frac{a - b^{(1)}}{a - a^{(1)}}(f - a) + b. \tag{33}$$

Rewriting (33), we get

$$\frac{1}{f-a} = \frac{1}{b}\left(\frac{f^{(n+1)}}{f-a} - \frac{a-b^{(1)}}{a-a^{(1)}}\right).$$

Hence

$$\begin{aligned} m(r,a;f) &\le O(\log r) + S(r,f) \\ &= S(r,f). \end{aligned}$$

Therefore

$$T(r,f) = N(r,a;f) + S(r,f). \tag{34}$$

Now if possible let $a \not\equiv b$, then from (33), we can see that the number of common zeros of $f - a$ and $f^{(n)} - a$ at most finite.

Hence by hypothesis, we have

$$\begin{aligned} N(r,a;f) &\le N_A(r,a;f) + N(r,a;f|f^{(n)} = a) \\ &= O(\log r) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{35}$$

Combining (34) and (35), we get $T(r,f) = S(r,f)$, which is a contradiction. Therefore $a \equiv b$. Now from (33), we get

$$f^{(n+1)} \equiv f. \tag{36}$$

Solving (36), we obtain

$$f = m_1 e^{\mu_1 z} + m_2 e^{\mu_2 z} + \cdots + m_s e^{\mu_s z},$$

where $\mu_1, \mu_2, \ldots, \mu_s$ are distinct roots of $z^{n+1} - 1 = 0$ and $m_1, m_2, \ldots, m_s$ are constants or polynomials.

Differentiating both sides of the above equation, we have

$$f^{(1)} = (m_1\mu_1 + m_1^{(1)})e^{\mu_1 z} + (m_2\mu_2 + m_2^{(1)})e^{\mu_2 z} + \cdots + (m_s\mu_s + m_s^{(1)})e^{\mu_s z}$$

From (3), we get

$$\omega f - f^{(1)} = a(\omega - 1).$$

Now from above three equations, we obtain

$$\sum_{j=1}^{s}(\omega m_j - m_j\mu_j - m_j^{(1)})e^{\mu_j z} = a(\omega - 1).$$

If $\omega \not\equiv 1$, then from above equation, we have

$$\sum_{j=1}^{s} \frac{(\omega m_j - m_j\mu_j - m_j^{(1)})}{a(\omega - 1)} e^{\mu_j z} \equiv 1. \tag{37}$$

Also we see that $T(r, f) = O(T(r, e^{\mu_j z}))$ for $j = 1, 2, \ldots, s$.

First we suppose that the left hand side of (37) contains more than two terms. Then using Lemma 2.7 we have

$$\frac{(\omega m_j - m_j\mu_j - m_j^{(1)})}{a(\omega - 1)} e^{\mu_j z} \equiv 1,$$

for one value of $j \in \{1, 2, \ldots, s\}$.

From the above equality, we can easily see that

$$T(r, e^{\mu_j z}) = S(r, f) = S(r, e^{\mu_j z}),$$

which is a contradiction.

Next we suppose that the left hand side of (37) contains exactly two terms. Then

$$\frac{(\omega m_t - m_t\mu_t - m_t^{(1)})}{a(\omega - 1)} e^{\mu_t z} + \frac{(\omega m_l - m_l\mu_l - m_l^{(1)})}{a(\omega - 1)})e^{\mu_l z} \equiv 1,$$

where $1 \le t, l \le s$.

Applying Lemma 2.1, we get

$$\begin{aligned}
T(r, e^{\mu_t z}) &\le \overline{N}(r, 0; e^{\mu_t z}) + \overline{N}(r, \infty; e^{\mu_t z}) + \overline{N}(r, \frac{a(\omega - 1)}{(\omega m_t - m_t\mu_t - m_t^{(1)})}; e^{\mu_t z}) \\
&\quad + S(r, e^{\mu_t z}) \\
&= \overline{N}(r, 0; e^{\mu_l z}) + S(r, e^{\mu_t z}) \\
&= S(r, e^{\mu_t z}),
\end{aligned}$$

a contradiction.

Finally we suppose that the left hand side of (37) contains only one term. That is,

$$\frac{(\omega m_t - m_t\mu_t - m_t^{(1)})}{a(\omega-1)}e^{\mu_t z} \equiv 1.$$

Which implies

$$T(r, e^{\mu_t z}) = S(r,f) = S(r, e^{\mu_t z}),$$

which is again a contradiction.

Hence $\omega \equiv 1$. Therefore $f^{(1)} \equiv f$. This implies $f^{(n+1)} \equiv f^{(n)}$, which is again a contradiction.

Subcase 2.2. Let $f^{(n+1)} \equiv f^{(n)}$. Since f is transcendental, we get $f^{(n)} \not\equiv 0$. Then on integration, we have

$$f^{(n)} = \lambda e^z,$$

where $\lambda(\neq 0)$ is a constant. On further integration, we get

$$f = \lambda e^z + P(z) = f^{(n)} + P(z),$$

where $P(z)$ is a polynomial of degree $K(< n)$. This subcase can be divided into two subcases.

Subcase 2.2.1. First we suppose that $P \equiv a$. Then $f = \lambda e^z + a$. Also $f^{(n+1)} = \lambda e^z = f^{(n+2)}$. Let z_4 be a zero of $f^{(n+1)} - b$, which is also a zero of $f^{(n+2)} - a$. Then z_4 is a zero of $a - b$. If $a - b \not\equiv 0$, then by Lemma 2.1, we have

$$\begin{aligned} T(r, f^{(n+1)}) &\leq \overline{N}(r, 0; f^{(n+1)}) + \overline{N}(r, \infty; f^{(n+1)}) + \overline{N}(r, 0; f^{(n+1)} - b) + S(r,f) \\ &= \overline{N}(r, 0; a-b) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{38}$$

Again

$$\begin{aligned} T(r,f) &= T(r, \lambda e^z + a) \\ &= T(r, f^{(n+1)} + a) \\ &\leq T(r, f^{(n+1)}) + T(r,a) + \log 2 \\ &= T(r, f^{(n+1)}) + S(r,f). \end{aligned} \tag{39}$$

Combining (38) and (39), we get

$$T(r,f) = S(r,f),$$

which is a contradiction. Hence $a \equiv b$.

Subcase 2.2.2. Next we suppose that $P \not\equiv a$ and P is non-constant. Now by Lemma 2.1, we have

$$\begin{aligned} T(r, \lambda e^z) &\le \overline{N}(r, 0; \lambda e^z) + \overline{N}(r, \infty; \lambda e^z) + \overline{N}(r, a - P; \lambda e^z) + S(r, \lambda e^z) \\ &= \overline{N}(r, a; f) + S(r, \lambda e^z). \end{aligned}$$

Now let z_5 is a zero of $f - a$ such that $z_5 \notin A \cup B$, then from $f(z) = f^{(n)}(z) + P(z)$, we get $P(z_5) = 0$. Hence

$$\begin{aligned} \overline{N}(r, a; f) &\le N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(r, 0; P) \\ &= S(r, f). \end{aligned}$$

Combining above two identity, we get

$$T(r, \lambda e^z) = S(r, \lambda e^z),$$

which is a contradiction. Therefore $P(z)$ is a constant, say, C. Hence

$$f = \lambda e^z + C = f^{(n)} + C.$$

Since f dose not assume the values C and ∞, using Lemma 2.1, we have $\overline{N}(r, a; f) \neq S(r, f)$. Also since $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, we get $\overline{E}(a; f) \cap \overline{E}(b, f^{(n+1)}) \neq \Phi$. Hence $C = 0$. Therefore $f = \lambda e^z$. This proves the theorem.

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