Southeast Asian Bulletin of Mathematics (2023) 47: 707–715

Southeast Asian
Bulletin of
Mathematics

Convexity Conditions for 2×2 Quaternionic Numerical Range

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Received 3 April 2021
Accepted 9 October 2021

Communicated by Xiuyun Guo

AMS Mathematics Subject Classification(2020): 15A33, 15A60

Abstract. Quaternionic numerical range is not always convex, and so it is natural to characterize those matrices with convex quaternionic numerical range. In this paper, we present a necessary and sufficient condition in terms of matrix entries for the quaternionic numerical range of a 2×2 matrix to be convex. As a consequence, all 2×2 matrices with convex quaternionic numerical range are essentially Hermitian, skew-Hermitian, or real matrices up to real translation and unitary similarity.

Keywords: Quaternions; Quaternionic numerical range; Convexity.

1. Introduction

Let $\mathbf{H}$ be the skew algebra of quaternions generated by $\{1, i, j, k\}$ over the reals $\mathbf{R}$:

$$\mathbf{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbf{R}\}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Denote $q_0 = re(q)$ and $q_1 i + q_2 j + q_3 k = im(q)$. Define the conjugate of q as

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k,$$

and the length of q as

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Note that $q\overline{q} = \overline{q}q = |q|^2$. Let $\mathbf{H}^n$ be the collection of quaternionic vectors with n components. For basic properties of quaternionic vectors and matrices, see [9].

Definition 1.1. *Let A be an $n \times n$ quaternionic matrix. The quaternionic numerical range (QNR) of A is defined as*

$$W(A) = \{x^*Ax : x^*x = 1, x \in \boldsymbol{H}^n\} \subset \boldsymbol{H}$$

where * denotes the conjugate transposition of vector or matrix.

Some basic properties of QNR are immediate:
(i) $W(U^*AU) = W(A)$ for any unitary matrix U, i.e., $UU^* = U^*U = I_n$,
(ii) $W(\alpha I_n + \beta A) = \alpha + \beta W(A)$ for any $\alpha, \beta \in \mathbf{R}$.

The first person to study QNR was Kippenhahn [4], in particular, he considered the convexity question of QNR. Unfortunately, Kippenhahn made a false claim that QNR is convex. The details of his mistake and the early development of QNR were discussed by W. So in [7].

In general, $W(A)$ is not convex. The easiest counter example is to take A to be the 1×1 matrix with the only entry i. Hence $W(A) = \{q : re(q) = 0, |q| = 1\}$ is NOT convex because $i, -i \in W(A)$ but the mid-point $\frac{1}{2}(i + (-i)) = 0 \notin W(A)$. This example was first mentioned in literature by Au-Yeung in [2]. As a folklore, we have the characterization.

Theorem 1.2. *Let $A = [a]$ be an 1×1 matrix. Then $W(A)$ is convex iff $im(a) = 0$, i.e., $a \in \mathbf{R}$.*

It will be nice if we can have a similar characterization in terms of matrix entries for a general $n \times n$ matrix. Such problem seems very challenging. As a first step, we are able to obtain a necessary and sufficient condition in terms of matrix entries for the convexity of the QNR of a 2×2 matrix.

Theorem 1.3. *Let $A = \begin{bmatrix} a & 2b \\ 2d & c \end{bmatrix}$.*
(i) *If $b = d = 0$ then $W(A)$ is convex iff $re(a) = re(c)$ or $im(a) = im(c) = 0$.*
(ii) *If $b = 0$ and $d \neq 0$ or $b \neq 0$ and $d = 0$ then $W(A)$ is convex iff $im(a) = im(c) = 0$.*
(iii) *If $bd \neq 0$ and $b + \overline{d} = 0$ then $W(A)$ is convex iff $re(a) = re(c)$ or $im(a) = im(c) = 0$.*
(iv) *If $bd \neq 0$ and $b + \overline{d} \neq 0$ then $W(A)$ is convex iff*

$$(h - re(c))^2 \; im(a) + 4(h - re(c)) \; im(bd) + (b + \overline{d}) \; im(c) \; (\overline{b} + d) = 0$$

or equivalently

$$(\overline{b} + d) \; im(a) \; (b + \overline{d}) + 4(h - re(a)) \; im(db) + (h - re(a))^2 \; im(c) = 0$$

for both real roots of the equation $(h - re(a))(h - re(c)) = |b + \overline{d}|^2$.

In Section 2, we recall different convexity conditions of QNR from literature as a preparation for a proof of Theorem 1.3. Some are sufficient and some are both necessary and sufficient; some are implicit and some are explicit. Then we give a complete proof of Theorem 1.3 in Section 3 with many illustrative examples taken from literature. Finally we conclude that 2×2 matrices with convex QNR are essentially Hermitian, skew-Hermitian, or real matrices up to real translation and unitary similarity.

2. General Convexity Conditions

From now on, we assume $n \geq 2$. The problem of characterizing those matrices with convex QNR after discovering that QNR is not convex in general was proposed. $W(A)$ is convex if A is Hermitian, i.e., $A^* = A$, was proved. Later, Au-Yeung [1] proved that $W(A)$ is convex if A is skew-Hermitian, i.e., $A^* = -A$. Indeed, he gave a necessary and sufficient convexity condition for an $n \times n$ normal matrix, i.e., $AA^* = A^*A$, via its eigenvalues $h_t + s_t i$ where $h_1 \leq h_2 \leq \cdots \leq h_n$ and $s_t \geq 0$:

$$W(A) \text{ is convex if and only if } (h_1 - h_2)s_1 = 0 = (h_{n-1} - h_n)s_n.$$

This result was based on the following general (though implicit) convexity condition from the same paper.

Theorem 2.1. *$W(A)$ is convex iff $W(A) \cap \mathbf{R} = \{re(q) : q \in W(A)\}$.*

Later, So [6] was able to translate these implicit convexity condition into more explicit condition using the notion of quasi-diagonal elements of a matrix. Let A be an $n \times n$ matrix with $n \geq 2$, and $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$. Since H is Hermitian, we denote its real eigenvalues $h_1 \leq \cdots \leq h_n$ and corresponding orthonormal eigenvectors $u_1, \ldots, u_n$. Take $s_t = |u_t^* S u_t| \geq 0$ for $1 \leq t \leq n$. We call $h_t + is_t$ the quasi-diagonal elements of A because A is unitarily similar to a matrix with $h_t + is_t$ as its diagonal elements.

Theorem 2.2. *Let A be an $n \times n$ matrix with quasi-diagonal elements $h_t + is_t$ for $1 \leq t \leq n$. Then $W(A)$ is convex iff $(h_1 - h_2)s_1 = 0 = (h_{n-1} - h_n)s_n$.*

Note that the quasi-diagonal elements of a normal matrix are its eigenvalues. Hence Theorem 2.2 can be viewed as an extension of Au-Yeung's result on normal matrix to general matrix. An interesting sufficient convexity condition was observed by Carvalho, Diogo and Mendes [3]: real matrix always has convex QNR. We give a different proof via Theorem 2.2.

Corollary 2.3. *If A is a real matrix then $W(A)$ is convex.*

Proof. Let A be a real matrix and ^{T} denote the transpose of a matrix. Then $\frac{A+A^*}{2} = \frac{A+A^T}{2}$ is real symmetric with real eigenvalues $h_1, \cdots, h_n$ with corresponding real orthonormal eigenvectors $u_1, \ldots, u_n$. Hence $s_t = |u_t^T S u_t| = 0$ for all t because $S = \frac{A-A^T}{2}$ is real skew-symmetric. Consequently, A has quasi-diagonal elements $h_1 + 0i, \ldots, h_n + 0i$, i.e., real. By Theorem 2.2, $W(A)$ is convex. ∎

In section 3, we need the following specialization of Theorem 2.2 to the case $n = 2$.

Theorem 2.4. *Let A be a 2×2 matrix such that $H = \frac{1}{2}(A + A^*)$ has real eigenvalues $h_1 \le h_2$ with eigenvectors u_1 and u_2 respectively. Also let $S = \frac{1}{2}(A - A^*)$.*

Then $W(A)$ is convex iff either $h_1 = h_2$ or $u_t^* S u_t = 0$ for $t = 1, 2$.

3. 2 × 2 Convexity Results

In this section, we give the complete the proof of Theorem 1.3. We divide all 2×2 matrices into 3 types: (i) diagonal, (ii) (upper or lower) triangular, and (iii) generic; and then treat them separately in 3 theorems followed with some examples.

Theorem 3.1. *If $A = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ then*

$$W(A) \textit{ is convex iff } re(a) = re(c) \textit{ or } im(a) = im(c) = 0.$$

Proof. Then $H = \frac{1}{2}(A + A^*) = \begin{bmatrix} re(a) & 0 \\ 0 & re(c) \end{bmatrix}$ and $S = \frac{1}{2}(A - A^*) = \begin{bmatrix} im(a) & 0 \\ 0 & im(c) \end{bmatrix}$. Hence the eigenvalues of H are $re(a)$ and $re(c)$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. Hence, by Theorem 2.4, $W(A)$ is convex iff $re(a) = re(c)$ or $im(a) = im(c) = 0$. ∎

Example 3.2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then, by Theorem 3.1, $W(A)$ is convex because $im(1) = im(2) = 0$. Indeed, $W(A) = \{x \in \mathbf{R} : 1 \le x \le 2\}$.

Example 3.3. Let $A = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix}$. Then, by Theorem 3.1, $W(A)$ is convex because $re(i) = 0 = re(j)$. Indeed, $W(A) = \{q : re(q) = 0, |q| \le 1\}$.

Example 3.4. Let $A = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$. Then, by Theorem 3.1, $W(A)$ is NOT convex because $re(i) = 0 \neq 1 = re(1)$ and $im(i) = i \neq 0$. Indeed, $W(A) = \{\alpha + (1-\alpha)q : re(q) = 0, |q| = 1, 0 \leq \alpha \leq 1\}$, and so $i, -i \in W(A)$ but $0 \notin W(A)$.

Theorem 3.5. *If* $A = \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix}$ *or* $A = \begin{bmatrix} a & 0 \\ 2d & c \end{bmatrix}$ *with* $bd \neq 0$ *then*

$$W(A) \text{ is convex iff } im(a) = im(c) = 0.$$

Proof. Case 1: $A = \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix}$ with $b \neq 0$.

Then $H = \frac{1}{2}(A + A^*) = \begin{bmatrix} re(a) & b \\ \overline{b} & re(c) \end{bmatrix}$ and $S = \frac{1}{2}(A - A^*) = \begin{bmatrix} im(a) & b \\ -\overline{b} & im(c) \end{bmatrix}$. Let the real eigenvalue of H be h with an eigenvector $u = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$by = (h - re(a))x \quad \text{and} \quad \overline{b}x = (h - re(c))y$$

Since $b \neq 0$, we have $xy \neq 0$. Hence $|b|^2 = (h - re(a))(h - re(c))$, and so H has two distinct eigenvalues because $b \neq 0$:

$$h_{\pm} = \frac{1}{2}\left(re(a) + re(c) \pm \sqrt{(re(a) - re(c))^2 + 4|b|^2}\right).$$

Now

$$\begin{aligned} u^*Su &= \overline{x}im(a)x - \overline{y}\overline{b}x + \overline{x}by + \overline{y}im(c)y \\ &= \overline{x}im(a)x - \frac{\overline{x}b\overline{b}x}{h - re(c)} + \frac{\overline{x}b\overline{b}x}{h - re(c)} + \frac{\overline{x}bim(c)\overline{b}x}{(h - re(c))^2} \\ &= \overline{x}\left(\frac{(h - re(c))^2 im(a) + bim(c)\overline{b}}{(h - re(c))^2}\right)x \end{aligned}$$

and so if u is the eigenvector of H corresponds to h then $u^*Su = 0$ iff $(h - re(c))^2 im(a) + bim(c)\overline{b} = 0$. Finally, by Theorem 2.4, $W(A)$ is convex iff $u^*Su = 0$ for both eigenvectors of H corresponding to $h_{\pm}$ iff $(h_{-} - re(c))^2 im(a) + b\ im(c)\overline{b} = 0 = (h_{+} - re(c))^2 im(a) + b\ im(c)\overline{b}$ iff $im(a) = im(c) = 0$ because $h_{-} \neq h_{+}$.

Case 2: $A = \begin{bmatrix} a & 0 \\ 2d & c \end{bmatrix}$ with $d \neq 0$.

Let $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then U is unitary and $B = U^*AU = \begin{bmatrix} c & 2d \\ 0 & a \end{bmatrix}$. Hence $W(A) = W(B)$ is convex iff $im(c) = im(a) = 0$ by Case 1. ■

Example 3.6. Let $A = \begin{bmatrix} 1 & i + j + k \\ 0 & 2 \end{bmatrix}$. Then, by Theorem 3.5, $W(A)$ is convex because $im(1) = im(2) = 0$.

Example 3.7. [5] Let $A = \begin{bmatrix} -1+i & 3-4k \\ 0 & 1+i \end{bmatrix}$, $\begin{bmatrix} 3+4i & 16j \\ 0 & 20+i \end{bmatrix}$, or $\begin{bmatrix} 3+4i & 1-j \\ 0 & -2+5i \end{bmatrix}$. Then, by Theorem 3.5, $W(A)$ is not convex because $im(a) \neq 0$ for all three matrices.

Theorem 3.8. *Let* $A = \begin{bmatrix} a & 2b \\ 2d & c \end{bmatrix}$ *with* $bd \neq 0$.

(i) *If* $q = b + \overline{d} = 0$ *then*

$$W(A) \text{ is convex iff } re(a) = re(c) \text{ or } im(a) = im(c) = 0.$$

(ii) *If* $q = b + \overline{d} \neq 0$ *then*
$W(A)$ *is convex*
iff $(h - re(c))^2\ im(a) + 4(h - re(c))\ im(bd) + q\ im(c)\ \overline{q} = 0$
for both $h = \frac{1}{2}\left[(re(a) + re(c)) \pm \sqrt{(re(a) - re(c))^2 + 4|q|^2}\right]$
iff $\overline{q}\ im(a)\ q + 4(h - re(a))\ im(db) + (h - re(a))^2\ im(c) = 0$
for both $h = \frac{1}{2}\left[(re(a) + re(c)) \pm \sqrt{(re(a) - re(c))^2 + 4|q|^2}\right]$.

Proof. (i) Note that $H = \frac{1}{2}(A + A^*) = \begin{bmatrix} re(a) & q \\ \overline{q} & re(c) \end{bmatrix} = \begin{bmatrix} re(a) & 0 \\ 0 & re(c) \end{bmatrix}$ and $S = \frac{1}{2}(A - A^*) = \begin{bmatrix} im(a) & q \\ -\overline{q} & im(c) \end{bmatrix} = \begin{bmatrix} im(a) & 0 \\ 0 & im(c) \end{bmatrix}$. Hence the eigenvalues of H are $re(a)$ and $re(c)$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. Hence, by Theorem 2.4, $W(A)$ is convex iff $re(a) = re(c)$ or $im(a) = im(c) = 0$.

(ii) Note that $H = \frac{1}{2}(A + A^*) = \begin{bmatrix} re(a) & q \\ \overline{q} & re(c) \end{bmatrix}$ and $S = \frac{1}{2}(A - A^*) = \begin{bmatrix} im(a) & p \\ -\overline{p} & im(c) \end{bmatrix}$ where $p = b - \overline{d}$. Let the real eigenvalue of H be h with an eigenvector $u = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$qy = (h - re(a))x \quad \text{and} \quad \overline{q}x = (h - re(c))y.$$

Since $q \neq 0$, we have $xy \neq 0$. Hence $|q|^2 = (h - re(a))(h - re(c))$, and so H has two distinct eigenvalues because $q \neq 0$:

$$h_{\pm} = \frac{1}{2}\left(re(a) + re(c) \pm \sqrt{(re(a) - re(c))^2 + 4|q|^2}\right).$$

Now

$$\begin{aligned} u^*Ku &= \overline{x}im(a)x - \overline{y}\,\overline{p}x + \overline{x}py + \overline{y}im(c)y \\ &= \overline{x}im(a)x - \frac{\overline{x}q\overline{p}x}{h-re(c)} + \frac{\overline{x}p\overline{q}x}{h-re(c)} + \frac{\overline{x}qim(c)\overline{q}x}{(h-re(c))^2} \\ &= \overline{x}\left(\frac{(h-re(c))^2im(a) + 2(h-re(c))im(p\overline{q}) + qim(c)\overline{q}}{(h-re(c))^2}\right)x \\ &= \overline{x}\left(\frac{(h-re(c))^2im(a) + 4(h-re(c))im(bd) + qim(c)\overline{q}}{(h-re(c))^2}\right)x \end{aligned}$$

and so if u is the eigenvector of H corresponds to h then $u^*Su = 0$ iff $(h - re(c))^2im(a) + 4(h - re(c))im(bd) + qim(c)\overline{q} = 0$. Hence, by Theorem 2.4, $W(A)$ is convex iff $u^*Su = 0$ for both eigenvectors of H corresponding to $h_\pm$ iff $(h - re(c))^2im(a) + 4(h - re(c))im(bd) + qim(c)\overline{q} = 0$ for both h_+ and h_-. Finally, we observe that

$$(h-re(c))^2im(a) + 4(h-re(c))im(bd) + qim(c)\overline{q} = 0$$

iff

$$\overline{q}\;im(a)\;q + 4(h-re(a))\;im(db) + (h-re(a))^2\;im(c) = 0$$

because $|q|^2 = (h - re(a))(h - re(c))$ and $im(bd)(b + \overline{d}) = (b + \overline{d})im(db)$. ∎

Example 3.9. [8] Let $A = \begin{bmatrix} k_1 i & \gamma j \\ \gamma j & 1 + k_2 i \end{bmatrix}$ where k_1, k_2, γ are positive real numbers. Then $W(A)$ is not convex.

Proof. Note that $b = \frac{\gamma i}{2} \neq 0$ and $d = \frac{\gamma i}{2} \neq 0$, thus $q = b + \overline{d} = 0$. Moreover, $re(k_1 i) = 0 \neq 1 = re(1 + k_2 i)$, and $im(k_1 i) = k_1 i \neq 0$. Hence, by Theorem 3.8 (i), $W(A)$ is not convex. ∎

Example 3.10. Let $A = \begin{bmatrix} 12 - 8i & 12 + 6i \\ 6i & 3 + 8i \end{bmatrix}$. Then $W(A)$ is convex.

Proof. Note that $a = 12 - 8i$, $b = 6 + 3i$, $d = 3i$ and $c = 3 + 8i$. Then $q = b + \overline{d} = 6 + 3i - 3i = 6 \neq 0$, and so $|q| = 6$. Moreover, $re(a) = 12$ and $re(c) = 3$, hence

$$\begin{aligned} h &= \frac{1}{2}\left[(re(a) + re(c)) \pm \sqrt{(re(a) - re(c))^2 + 4|q|^2}\right] \\ &= \frac{1}{2}[(12+3) \pm \sqrt{(12-3)^2 + 4 \cdot 6^2}] \\ &= 15 \text{ or } 0. \end{aligned}$$

Note that $im(a) = -8i$, $im(bd) = 18i$, $im(c) = 8i$. Consequently,

$$\begin{aligned} & (h - re(c))^2 \; im(a) + 4(h - re(c)) \; im(bd) + q \; im(c) \; \overline{q} \\ = \; & (15-3)^2(-8i) + 4(15-3)(18i) + 6(8i)6 \\ = \; & -1152i + 864i + 288i \\ = \; & 0. \end{aligned}$$

and

$$\begin{aligned} & (h - re(c))^2 \; im(a) + 4(h - re(c)) \; im(bd) + q \; im(c) \; \overline{q} \\ = \; & (0-3)^2(-8i) + 4(0-3)(18i) + 6(8i)6 \\ = \; & -72i - 216i + 288i \\ = \; & 0. \end{aligned}$$

Hence, by Theorem 3.8 (ii), $W(A)$ is convex. ■

When $n \geq 2$, from Section 2, we know that $W(A)$ is convex if A is Hermitian, skew-Hermitian or real. It turns out that these are essentially all 2×2 matrices with convex QNR.

Theorem 3.11. *Let A be a 2×2 matrix with convex QNR. Then A is Hermitian, skew-Hermitian with a real translation, or unitarily similar to a real matrix.*

Proof. Let U be a unitary matrix such that

$$U^*AU = \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix}.$$

Then $W\left(\begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix}\right) = W(U^*AU) = W(A)$ is convex.

Case 1: $b = 0$.

By Theorem 3.1, $re(a) = re(c)$ or $im(a) = im(c) = 0$. Hence U^*AU is skew-Hermitian with a real translation or Hermitian, and so A is skew-Hermitian with a real translation or Hermitian.

Case 2: $b \neq 0$.

By Theorem 3.5, $im(a) = im(c) = 0$. Take $q = \frac{b}{|b|}$ and $D = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$. Then D is unitary and $D^*U^*AUD = \begin{bmatrix} re(a) & 2|b| \\ 0 & re(c) \end{bmatrix}$ is real. Hence A is unitarily similar to a real matrix. ■

The following example shows that 3×3 matrices with convex QNR have more varieties than those mentioned in Theorem 3.11.

Example 3.12. Let $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is NOT Hermitian, NOT skew-Hermitian with a real translation, and NOT unitarily similar to a real matrix. However, by Theorem 2.2, $W(A)$ is convex because A has quasi-diagonal elements $h_t + is_t$ with $h_1 = h_2 = 0$, $h_3 = 1$; and $s_1 = s_2 = 1$, $s_3 = 0$.

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