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Estimates of Prime Factors of the Generalized Fermat Number

Wei Xie

School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi, 710000, China
Email: xiewei1729@163.com

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Abstract. Let $F_n = 2^{2^n} + 1$ and $F_{a,n} = a^{2^n} + 1$ where a is a positive even integer. For any integer $k > 1$, let $P(k)$ be the largest prime factor of k. In this paper, we obtain an improvement of the lower bound of $P(F_{a,n})$. We also obtain the lower bounds of $P(F_n)$ and $P(F_{a,n})$ when F_n and $F_{a,n}$ are square-free.

Keywords: Generalized Fermat number; Greatest prime factor; Square-free number; The lower bound.

1. Introduction

Let $F_n = 2^{2^n} + 1$ and $F_{a,n} = a^{2^n} + 1$ $(2|a)$. For any integer $k > 1$, let $P(k)$ be the largest prime factor of k. In [4], Le proved that $P(F_n) > 2^{n-4}n$ when $n \geqslant 2^{18}$. Grytczuk, Wójtowicz and Luca showed that $P(F_n) > 2^{n+2}(4n+9)+1$ when $n \geqslant 4$ and $P(F_{a,n}) > 2^{n-4}n$ when $n \geqslant a^{18}$(see [3]). After that, Grytczuk and Mędryk proved that $P(F_{a,n}) > 2^{n+2}(n+1)\log_a 2 + 1$ in [2]. Chen showed that $P(F_n) > 2^{n+2}(4n + 16 + 4\log_2(n+7)) + 1$ when $n \geqslant 4$ (see [1]). In this paper, we obtain an improvement of the lower bound of $P(F_{a,n})$.

2. Main Results

Theorem 2.1. *If $n \geqslant 3$, then*

$$P(F_{a,n}) \geqslant 2^{n+2}[(n+1)\log_a 2 + \log_a \log_a 2^{2(n+1)}] + 1.$$

Proof. From [2], $F_{a,n} = \prod_{i=1}^{s}(2^{n+1}h_i + 1), 1 \leqslant h_1 \leqslant h_2 \leqslant \cdots \leqslant h_s$. It is easy to know that $2^{n+1} | \sum h_i$ and $F_{a,n} > \prod_i (2^{n+1}h_i)$. Then

$$\sum_i ((n+1)\log_a 2 + \log_a h_i) \leqslant 2^n \leqslant \frac{1}{2}\sum_i h_i.$$

So there exists at least one i with $2(n+1)\log_a 2 + 2\log_a h_i \leqslant h_i$. We have

$$\begin{aligned} h_i &\geqslant 2(n+1)\log_a 2 + 2\log_a(2(n+1)\log_a 2 + 2\log_a h_i) \\ &\geqslant 2(n+1)\log_a 2 + 2\log_a \log_a 2^{2(n+1)}. \end{aligned}$$

This proves Theorem 2.1. ∎

Theorem 2.2. *Let $b = \log_a 2^{2n+1} + \log_a \log_a 2^{3n+1}$. If $F_{a,n}$ is square-free, then*

$$P(F_{a,n}) > 2^{n+1}\left(\frac{2^n}{\max(b,1)} - 0.5\right) + 1.$$

Proof. Let $\sum_{i=1}^{s} h_i = 2^{n+1}\Delta$. Since $F_{a,n}$ has no square factors, we have

$$s \leqslant \sqrt{s(s+1)} \leqslant \sqrt{2\sum h_i} = 2^{\frac{n}{2}+1}\sqrt{\Delta}. \tag{1}$$

Note that $2^n < s\log_a\left(1 + \frac{1}{2^{n+1}}\right) + (n+1)\, s\log_a 2 + \log_a \prod_{i=1}^{s} h_i$. So

$$\begin{aligned} \frac{2^n}{s} &< \log_a\left(1 + \frac{1}{2^{n+1}}\right) + (n+1)\log_a 2 + \log_a \frac{\sum_{i=1}^{s} h_i}{s} \\ &< (n+2)\log_a 2 + \log_a \frac{2^{n+1}\Delta}{s} \\ &< (n+2)\log_a 2 + \log_a\left[2\Delta\left((n+2)\log_a 2 + \log_a 2^{n+1}\Delta\right)\right] \\ &= \log_a(2^{n+3}\Delta) + \log_a\log_a(2^{2n+3}\Delta). \end{aligned} \tag{2}$$

By (1) and (2), it follows that

$$2^{\frac{n-2}{2}} < (\log_a(2^{n+3}\Delta) + \log_a\log_a(2^{2n+3}\Delta))\sqrt{\Delta}. \tag{3}$$

From (3), $\sqrt{\Delta} > \frac{2^{\frac{n-2}{2}}}{b}$ when $\sqrt{\Delta} < 2^{\frac{n-2}{2}}$. Hence,

$$\sqrt{\Delta} \geqslant \frac{2^{\frac{n-2}{2}}}{\max(b,1)}. \tag{4}$$

Note that

$$h_s > \sqrt{2\sum_{i=0}^{s-1}(h_s - i) - 0.5} \geqslant \sqrt{2\sum_{i=1}^{s} h_i - 0.5} = 2^{\frac{n+2}{2}}\sqrt{\Delta} - 0.5. \tag{5}$$

(4) and (5) complete the proof. ■

From [2], all divisors of F_n for $n > 1$ are of the form $2^{n+2}k + 1$ ($k \in \mathbb{N}$). Similar to the discussion of $F_{a,n}$, we have the following theorem.

Theorem 2.3. *If F_n is square-free, then*

$$P(F_n) > 2^{n+2}[\frac{2^n}{2n + 1 + \log_2(3n + 1)} - 0.5] + 1.$$

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