# A Compact Difference Scheme on Graded Meshes for the Fourth-order Fractional Integro-differential Equation with Initial Singularity* 

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#### Abstract

In this paper, a compact finite difference scheme is constructed and investigated for the fourth-order time-fractional integro-differential equation with singular kernels. In temporal direction, the Caputo derivative is treated by $L 1$ discrete formula and the Riemann-Liouville fractional integral is discretized by trapezoidal PI rule respectively. In spatial direction, the fourth order derivative is approximating by high-order accuracy compact difference method. The detailed analysis shows that the proposed scheme is unconditionally stable and convergent with the convergence order $O\left(N^{-\min \{r \sigma, 2-\alpha\}}+M^{-4}\right)$. $N, M$ denote the numbers of grids in temporal direction and in spatial direction, $\alpha \in(0,1)$ is the fractional order of the Caputo derivative and $\sigma$ is a regularity parameter. At last, some numerical results are also given to confirm our theoretical statement.


Keywords: Integro-differential equation; Initial singularity; Graded meshes; Finite difference scheme; Stability and convergence.

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## 1. Introduction

Numerical methods will be studied for the fourth-order time-fractional integrodifferential equation as follows:

$$
\begin{align*}
& { }_{0}^{C} D_{t}^{\alpha} u(x, t)+\mu u_{x x x x}(x, t)+\int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(x, s) d s=f(x, t),  \tag{1}\\
& \quad 0<x<L, \quad 0<t \leq T,
\end{align*}
$$

subject to the initial condition

$$
\begin{equation*}
u(x, 0)=0, \quad 0 \leq x \leq L \tag{2}
\end{equation*}
$$

and the boundary conditions

$$
\begin{align*}
& u(0, t)=\phi_{1}(t), \quad u(L, t)=\phi_{2}(t), \quad \frac{\partial^{2} u(0, t)}{\partial x^{2}}=\varphi_{1}(t)  \tag{3}\\
& \frac{\partial^{2} u(L, t)}{\partial x^{2}}=\varphi_{2}(t), \quad 0<t \leq T
\end{align*}
$$

where $\mu$ is a positive constant and $f(x, t)$ is a given function. The integral term in (1) is known as Riemann-Liouville fractional integral [19, 16, 18] with $0<\beta<1$, and the symbol ${ }_{0}^{C} D_{t}^{\alpha} u(x, t)$ means Caputo fractional derivative of order $\alpha$, i.e.

$$
{ }_{0}^{C} D_{t}^{\alpha} u(x, t)= \begin{cases}\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\partial u(x, s)}{\partial s} \frac{d s}{(t-s)^{\alpha}} & \text { if } 0<\alpha<1, \\ \partial_{t} u(x, t) & \text { if } \alpha=1\end{cases}
$$

In recent decades, fractional differential equations have received much attention from more and more scholars and been widely used in thermal systems, mechanical systems and other application fields. Specific applications can be found (see $[13,23,15,1,17,8,14]$ ). Currently, there are various numerical methods for the fourth-order time-fractional equations as follows.

$$
{ }_{0}^{C} D_{t}^{\alpha} u(x, t)+\mu u_{x x x x}(x, t)=f(x, t) .
$$

Liu [13] introduced an auxiliary variable, then the fourth-order equation can be splited into the coupled system of two second-order equations. Guo [6] proposed fully discrete local discontinuous Galerkin method for some time-fractional fourth-order differential equations. For Equation (1), due to the influence of Riemann-Liouville fractional integral, the theoretical results are hard to obtain in some way. Very recently, Qiao [19] construct a finite difference scheme for the fractional integro-differential equation (4) under good regularity assumption on uniform meshes and the theoretical analysis are investigated as well.

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} u(x, t)-\mu \Delta u(x, t)=\int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Delta u(x, s) d s+f(x, t) . \tag{4}
\end{equation*}
$$

In addition, Qiao [20] formulate and analyze an ADI-OSC numerical scheme for the multi-term time fractional integro-differential equation. Xu [28] analyze a compact finite difference scheme, the stability and convergence are proved by the discrete energy method, the Cholesky decomposition and the reduced-order method.

Referring to the literatures [11], our work is based on more factual initial singularity assumptions:

$$
\begin{equation*}
\left|\partial^{l} u(x, t) / \partial t^{l}\right| \leq C\left(1+t^{\sigma-l}\right), \quad 0<\sigma<1, \quad l=0,1,2 \tag{5}
\end{equation*}
$$

where $(x, t) \in[0, L] \times(0, T], \sigma$ is a regularity parameter, which depends on the orders of Caputo fractional derivative $\alpha$ and Riemann-Liouville integral $\beta$. To overcome the challenge caused by initial singularity, nonuniform mesh technic will be implemented in this work. In fact, the nonuniform mesh methods have been used for solving integro-differential equations with singular kernel for many years [10, 22, 29]. Tang [26] studied the numerical solutions of weakly singular Volterra integral equations by collocation method on graded meshes. Ma [4] investigated weakly singular Volterra integral equations by the graded mesh methods as well. Cen [2] studied the numerical method for time-fractional KdVBurgers' equation with initial singularity. Zhang [30] discussed the implicit finite difference scheme with nonuniform time steps for the time fractional diffusion equations.

We construct a compact finite difference scheme on graded meshes and deduce the stability and convergence results of the proposed numerical scheme. The L1 discrete formula is used to deal with the Caputo fractional derivative and the compact difference approximation is employed for spatial directional derivatives. Based on some cruical skills, the unconditional stability and convergence with $O\left(N^{-\min \{r \sigma, 2-\alpha\}}+M^{-4}\right)$ are obtained.

The rest of the paper is organized as follows. Some preliminary knowledge and the discrete scheme are introduced in Section 2. In Section 3, the stability is presented for the discrete scheme and the convergence result is provided for the discrete scheme. In Section 4, the results of numerical experiments are carried out. This paper ends with a brief conclusion.

## 2. Preliminaries

Firstly, we divide time interval and space interval as follows. In space interval, for a positive integer $M$, let $h=\frac{L}{M}, x_{i}=i h(0 \leq i \leq M)$. In time interval, we adopt graded mesh, for a positive interval $N$, let $t_{k}=T\left(\frac{k}{N}\right)^{r}(0 \leq k \leq N, r \geq 1), \tau_{k}=$ $t_{k}-t_{k-1}(1 \leq k \leq N)$. Let $V_{h}=\left\{u \mid u=\left(u_{0}, u_{1}, \ldots, u_{M}\right), u_{0}=u_{M}=0\right\}$. For any $u \in V_{h}$, denote

$$
\delta_{x} u_{i}=\frac{1}{h}\left(u_{i}-u_{i-1}\right), \quad \delta_{x}^{2} u_{i}=\frac{1}{h^{2}}\left(u_{i-1}-2 u_{i}+u_{i+1}\right), \quad \delta_{x}^{4} u_{i}=\delta_{x}^{2}\left(\delta_{x}^{2} u_{i}\right)
$$

For grid functions $u, v$, the notations of discrete inner products and norms are defined as follow:

$$
\|u\|^{2}=\langle u, u\rangle, \quad\langle u, v\rangle=h \sum_{i=1}^{M-1} u_{i} v_{i}, \quad\left\|\delta_{x}^{2} u\right\|^{2}=h \sum_{i=1}^{M-1}\left(\delta_{x}^{2} u_{i}\right)^{2}
$$

We introduce the compact operator

$$
\mathcal{H} u_{i}= \begin{cases}\frac{1}{6}\left(u_{i-1}+4 u_{i}+u_{i+1}\right)=\left(1+\frac{h^{2}}{6} \delta_{x}^{2}\right) u_{i} & \text { if } 1 \leq i \leq M-1 \\ u_{i} & \text { if } i=0, M\end{cases}
$$

According to [11, 9], we discretize the Caputo fractional derivative ${ }_{0}^{C} D_{t}^{\alpha} u(t)$ by the nonuniform $L 1$ formula. For ${ }_{0}^{C} D_{t}^{\alpha} u(t)$ at $t_{k}(1 \leq k \leq N)$, one has

$$
\begin{align*}
{ }_{0}^{C} D_{t}^{\alpha} u\left(x, t_{k}\right) & =\sum_{l=1}^{k} \int_{t_{l-1}}^{t_{l}} \frac{\left(t_{k}-s\right)^{-\alpha}}{\Gamma(1-\alpha)} \frac{u\left(t_{l}\right)-u\left(t_{l-1}\right)}{\tau_{l}} d s+\left(R_{t}^{\alpha}\right)^{k} \\
& =\sum_{l=1}^{k} a_{k-l}^{(k)}\left[u\left(t_{l}\right)-u\left(t_{l-1}\right)\right]+\left(R_{t}^{\alpha}\right)^{k} \tag{6}
\end{align*}
$$

where $a_{k-l}^{(k)}=\int_{t_{l-1}}^{t_{l}} \frac{\left(t_{k}-s\right)^{-\alpha}}{\Gamma(1-\alpha) \tau_{l}} d s$.
Denote

$$
\begin{equation*}
\partial_{t}^{\alpha} u\left(t_{k}\right)=\sum_{l=1}^{k} a_{k-l}^{(k)}\left[u\left(t_{l}\right)-u\left(t_{l-1}\right)\right], \quad 1 \leq k \leq N \tag{7}
\end{equation*}
$$

and

$$
P_{n-k}^{(n)}=\frac{1}{a_{0}^{(k)}} \begin{cases}1 & \text { if } k=n  \tag{8}\\ \sum_{j=k+1}^{n}\left(a_{j-k-1}^{(j)}-a_{j-k}^{(j)}\right) P_{n-j}^{(n)} & \text { if } 1 \leq k \leq n-1\end{cases}
$$

Lemma 2.1. [11] Let $\alpha \in(0,1)$. Under the assumptions in (5), one has

$$
\sum_{j=1}^{n} P_{n-j}^{(n)}\left|\left(R_{t}^{\alpha}\right)^{j}\right| \leq C N^{-\min \{r \sigma, 2-\alpha\}}, \quad n \geq 1
$$

Then, in order to approximate the Riemann-Liouville fractional ${ }_{0} I_{t}^{\beta} u(t)$, we introduce the following trapezoidal PI rule in [5, 12]:

$$
\begin{equation*}
{ }_{0} I_{t}^{\beta} u\left(t_{k}\right)=\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)}\left[w_{k} u\left(t_{0}\right)+\sum_{l=1}^{k} b_{k, l} u\left(t_{l}\right)\right]+\left(R_{t}^{\beta}\right)^{k}, \quad 1 \leq k \leq N \tag{9}
\end{equation*}
$$

where $w_{k}=\left(k^{r}-1\right)^{\beta+1}-k^{r \beta}\left(k^{r}-\beta-1\right), b_{k, l}=\phi_{k, l}(\beta, r)-\phi_{k, l+1}(\beta, r), b_{k, k}=$ $\left(k^{r}-(k-1)^{r}\right)^{\beta}$, with $\phi_{k, l}(\beta, r)=\frac{\left(k^{r}-(l-1)^{r}\right)^{\beta+1}-\left(k^{r}-l^{r}\right)^{\beta+1}}{l^{r}-(l-1)^{r}}$.
Denote

$$
\begin{equation*}
Q_{t}^{\beta} u\left(t_{k}\right)=\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)}\left[w_{k} u\left(t_{0}\right)+\sum_{l=1}^{k} b_{k, l} u\left(t_{l}\right)\right], \quad 1 \leq k \leq N \tag{10}
\end{equation*}
$$

Error estimate results are given as follow.

Lemma 2.2. [27] Let $\beta \in(0,1)$. Under the assumption (5), then

$$
\sum_{j=1}^{n} P_{n-j}^{(n)}\left|\left(R_{t}^{\beta}\right)^{j}\right| \leq C N^{-\min \{r \sigma, 2\}}, \quad n \geq 1
$$

Lemma 2.3. [21] If $g(x) \in C^{8}\left[x_{i-1}, x_{i+1}\right], 1 \leq i \leq M-1$, then it holds that

$$
\begin{aligned}
& \frac{1}{6}\left[g^{(4)}\left(x_{i-1}\right)+4 g^{(4)}\left(x_{i}\right)+g^{(4)}\left(x_{i+1}\right)\right] \\
= & \frac{\delta_{x}^{2} g\left(x_{i-1}\right)-2 \delta_{x}^{2} g\left(x_{i}\right)+\delta_{x}^{2} g\left(x_{i+1}\right)}{h^{2}}+\left(R_{x}\right)_{i}
\end{aligned}
$$

where $\left|\left(R_{x}\right)_{i}\right| \leq C M^{-4}$.

The difference scheme we will consider for (1)-(3) is as follows:

$$
\begin{align*}
& \mathcal{H} \partial_{t}^{\alpha} u_{i}^{k}+\mu \delta_{x}^{4} u_{i}^{k}+\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} w_{k} \mathcal{H} u_{i}^{0}+\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{k} b_{k, l} \mathcal{H} u_{i}^{l}=\mathcal{H} f_{i}^{k},  \tag{11}\\
& 1<i<M-1 \\
& u_{i}^{0}=0, \quad 0 \leq i \leq M  \tag{12}\\
& u_{0}^{k}=\phi_{1}(t), u_{L}^{k}=\phi_{2}(t), \delta_{x}^{2} u_{0}^{k}=\varphi_{1}(t), \delta_{x}^{2} u_{L}^{k}=\varphi_{2}(t), \quad 1 \leq k \leq N \tag{13}
\end{align*}
$$

It is easy to check that at each time level, the finite difference scheme (11)(13) is a linear tridiagonal system with strictly diagonally dominant coefficient matrix, thus the difference scheme has a unique solution.

## 3. Convergence and Stability of the Compact Scheme

For the analysis of the stability and convergence, we give some notations and lemmas as follows. By the Cholesky decomposition (square root method), there exists a real positive definite $\mathcal{B}$, which satisfies

$$
\mathcal{H}=\mathcal{B}^{\mathcal{T}} \mathcal{B}
$$

Lemma 3.1. [21] For any grid function $u, v \in V_{h}$, and $\delta_{x}^{2} u_{0}=\delta_{x}^{2} u_{M}=0$. It holds that

$$
\left\langle\delta_{x}^{4} u, v\right\rangle=\left\langle\delta_{x}^{2} u, \delta_{x}^{2} v\right\rangle
$$

Lemma 3.2. For any grid function $u, v \in V_{h}$, we have

$$
\langle\mathcal{H} u, v\rangle=\langle\mathcal{B} u, \mathcal{B} v\rangle .
$$

Proof. Using the definition of the inner product $\langle.,$.$\rangle and combining \mathcal{H}^{T}=\mathcal{H}=$ $\mathcal{B}^{\mathcal{T}} \mathcal{B}$, it follows that

$$
\langle\mathcal{H} u, v\rangle=h(\mathcal{H} u)^{T} v=h u^{T}\left(\mathcal{H}^{T} v\right)=h u^{T}\left(\mathcal{B}^{T} \mathcal{B} v\right)=h(\mathcal{B} u)^{T}(\mathcal{B} v)=\langle\mathcal{B} u, \mathcal{B} v\rangle
$$

The proof is finished.

Lemma 3.3. [27] For any grid function $\left\{v^{k} \mid \partial v^{k}=0\right\}, 1 \leq k \leq N$, it holds that

$$
\left\langle\partial_{t}^{\alpha} \mathcal{H} v^{k}, v^{k}\right\rangle \geq \frac{1}{2} \partial_{t}^{\alpha}\left\|\mathcal{B} v^{k}\right\|^{2}
$$

Lemma 3.4. [21] For any grid function $u \in V_{h}$, it holds that

$$
\frac{1}{3}\langle u, u\rangle \leq\langle\mathcal{B} u, \mathcal{B} u\rangle \leq\langle u, u\rangle
$$

Lemma 3.5. [27] Let $w_{k},\left\{b_{k, l}\right\}$ be defined as (9). Then we have
(i) $w_{k} \leq \beta$,
(ii) $\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{k-1} b_{k, l}<\frac{T^{\beta}}{\Gamma(\beta+1)}$.

Lemma 3.6. [9] For any finite time $T$ and nonnegative sequences $\left(\lambda_{l}^{(k)}\right)_{l=0}^{k-1}$, assume that there exists a constant $\lambda$, independent of time steps, such that $\lambda \geq$ $\max _{1 \leq k \leq n} \sum_{l=0}^{k-1} \lambda_{l}^{(k)}$. Suppose that the grid function $\left\{v^{k} \mid k \geq 0\right\}$ satisfies

$$
\partial_{t}^{\alpha}\left(v^{k}\right)^{2} \leq \sum_{l=1}^{k} \lambda_{k-l}^{(k)}\left(v^{l}\right)^{2}+v^{k}\left(\xi^{k}+\eta^{k}\right), \quad 1 \leq k \leq n
$$

where $\xi^{k}, \eta^{k}$ are nonnegative sequences. When $\tau_{n} \leq \sqrt[\alpha]{\frac{1}{2 \Gamma(2-\alpha) \lambda}}$, it holds that

$$
v^{k} \leq 2 E_{\alpha}\left(2 \lambda t_{k}^{\alpha}\right)\left(v^{0}+\max _{1 \leq j \leq k} \sum_{l=1}^{j} P_{j-l}^{(j)} \xi^{l}+\omega_{1+\alpha}\left(t_{k}\right) \max _{1 \leq j \leq k} \eta^{j}\right), \quad 1 \leq k \leq n
$$

where $E_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(1+k \alpha)}$ is the well-known Mittag-Leffler function.
We now consider the convergence of our numerical scheme.

Theorem 3.7. Assume that $u_{i}^{n} \in[0, L] \times(0, T]$ is the solution of problem (1)-(3). Let $\left\{U_{i}^{n} \mid 0 \leq i \leq M, 0 \leq n \leq N\right\}$ be the solution of (11)-(13), denote

$$
e_{i}^{n}=u_{i}^{n}-U_{i}^{n}, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N
$$

When assumptions (5) hold and $\tau_{N} \leq \sqrt[\alpha]{\frac{1}{2 \Gamma(2-\alpha) \lambda}}$ with $\lambda=\frac{2 T^{\beta}}{\Gamma(\beta+1)}$, then there exist positive constant $C$ such that

$$
\left\|e^{n}\right\| \leq C\left(N^{-\min \{r \sigma, 2-\alpha\}}+M^{-4}\right), \quad 1 \leq n \leq N
$$

Proof. We can get the following error equation

$$
\begin{align*}
& \partial_{t}^{\alpha} \mathcal{H} e_{i}^{n}+\mu \delta_{x}^{4} e_{i}^{n}+\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} w_{n} \mathcal{H} e_{i}^{0}+\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{n} b_{n, l} \mathcal{H} e_{i}^{l}=R_{i}^{n},  \tag{14}\\
& 1<i<M-1,1 \leq n \leq N \\
& e_{i}^{0}=0, \quad 0 \leq i \leq M \\
& e_{0}^{n}=0, \quad e_{M}^{n}=0, \delta_{x}^{2} e_{0}^{n}=0, \delta_{x}^{2} e_{M}^{n}=0,1 \leq n \leq N,
\end{align*}
$$

where $R_{i}^{n}=-\mathcal{H}\left(R_{t}^{\alpha}\right)^{n}-\mathcal{H}\left(R_{t}^{\beta}\right)^{n}-\mu\left(R_{x}\right)^{n}$.
Making the inner product of Equation (14) with $e^{n}$, we obtain

$$
\begin{align*}
& \left\langle\partial_{t}^{\alpha} \mathcal{H} e^{n}, e^{n}\right\rangle+\mu\left\langle\delta_{x}^{4} e^{n}, e^{n}\right\rangle+\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} w_{n}\left\langle\mathcal{H} e^{0}, e^{n}\right\rangle \\
= & -\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{n} b_{n, l}\left\langle\mathcal{H} e^{l}, e^{n}\right\rangle+\left\langle R^{n}, e^{n}\right\rangle \tag{15}
\end{align*}
$$

Lemmas 3.1, 3.3 and 3.4 give that

$$
\begin{align*}
\left\langle\partial_{t}^{\alpha} \mathcal{H} e^{n}, e^{n}\right\rangle & \geq \frac{1}{2} \partial_{t}^{\alpha}\left\|\mathcal{B} e^{n}\right\|^{2}  \tag{16}\\
\frac{\tau_{1}^{\beta} w_{n}}{\Gamma(\beta+2)}\left\langle\mathcal{H} e^{0}, e^{n}\right\rangle & =0, \frac{\tau_{1}^{\beta} b_{n, n}}{\Gamma(\beta+2)}\left\langle\mathcal{H} e^{n}, e^{n}\right\rangle \geq 0 \\
\left\langle\delta_{x}^{4} e^{n}, e^{n}\right\rangle & =\left\|\delta_{x}^{2} e^{n}\right\|^{2} \geq 0,\left\langle R^{n}, e^{n}\right\rangle \leq \sqrt{3}\left\|R^{n}\right\|\left\|\mathcal{B} e^{n}\right\|
\end{align*}
$$

To the term $\sum_{l=1}^{n-1} b_{n, l}\left\langle\mathcal{H} e^{l}, e^{n}\right\rangle$, it holds that

$$
\begin{align*}
\sum_{l=1}^{n-1} b_{n, l}\left\langle\mathcal{H} e^{l}, e^{n}\right\rangle & =\sum_{l=1}^{n-1} b_{n, l}\left\langle\mathcal{B} e^{l}, \mathcal{B} e^{n}\right\rangle  \tag{17}\\
& \geq-\frac{1}{2} \sum_{l=1}^{n-1} b_{n, l}\left\|\mathcal{B} e^{l}\right\|^{2}-\frac{1}{2} \sum_{l=1}^{n-1} b_{n, l}\left\|\mathcal{B} e^{n}\right\|^{2}
\end{align*}
$$

Denote

$$
\lambda_{n-l}^{(n)}= \begin{cases}\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} b_{n, l} & \text { if } 1 \leq l \leq n-1 \\ \frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{n-1} b_{n, l} & \text { if } l=n\end{cases}
$$

Substituting (16), (17) into (15), we have

$$
\begin{align*}
\partial_{t}^{\alpha}\left\|\mathcal{B} e^{n}\right\|^{2} & \leq \sum_{l=1}^{n} \lambda_{n-l}^{(n)}\left\|\mathcal{B} e^{l}\right\|^{2}+\left\|\mathcal{B} e^{n}\right\|\left\|2 \sqrt{3} R^{n}\right\| \\
& \leq \sum_{l=1}^{n} \lambda_{n-l}^{(n)}\left\|\mathcal{B} e^{l}\right\|^{2}+\left\|\mathcal{B} e^{n}\right\|\left\|\left(R_{t}\right)^{n}\right\|+\left\|\mathcal{B} e^{n}\right\|\left\|\left(R_{s}\right)^{n}\right\|, \tag{18}
\end{align*}
$$

where $\left\|\left(R_{t}\right)^{n}\right\|=2 \sqrt{3}\left(\left\|\mathcal{H}\left(R_{t}^{\alpha}\right)^{n}\right\|+\left\|\mathcal{H}\left(R_{t}^{\beta}\right)^{n}\right\|\right), \quad\left\|\left(R_{s}\right)^{n}\right\|=2 \sqrt{3}\left\|\mu\left(R_{x}\right)^{n}\right\|$.
By Lemma 3.5, there exists a constant $\lambda=\frac{2 T^{\beta}}{\Gamma(\beta+1)}$, such that $\lambda \geq$ $\max _{1 \leq k \leq n} \sum_{l=0}^{k-1} \lambda_{l}^{(k)}$. When $\tau_{n} \leq \sqrt[\alpha]{\frac{1}{2 \Gamma(2-\alpha) \lambda}}$, Lemma 3.6 implies that
$\left\|\mathcal{B} e^{n}\right\| \leq 2 E_{\alpha}\left(2 \lambda t_{n}^{\alpha}\right)\left(\left\|\mathcal{H} e^{0}\right\|+\max _{1 \leq j \leq n} \sum_{l=1}^{j} P_{j-l}^{(j)}\left\|\left(R_{t}\right)^{l}\right\|+\omega_{1+\alpha}\left(t_{n}\right) \max _{1 \leq j \leq n}\left\|\left(R_{s}\right)^{j}\right\|\right)$.
Based on Lemmas 2.1, 2.2, 2.3 and 3.4, one has

$$
\left\|e^{n}\right\| \leq C\left(N^{-\min \{r \sigma, 2-\alpha\}}+M^{-4}\right)
$$

The proof is completed.

Theorem 3.8. Following the idea of the proof for Theorem 3.7, stability statement is obtained for the proposed compact scheme (11)-(13). Assume that $U_{i}^{k}, V_{i}^{k}$ are the solutions of the scheme. Denote

$$
\varepsilon_{i}^{k}=U_{i}^{k}-V_{i}^{k}, 0 \leq i \leq M, 0 \leq k \leq N
$$

We have stability equations as follows

$$
\begin{align*}
& \partial_{t}^{\alpha} \mathcal{H} \varepsilon_{i}^{k}+Q_{t}^{\beta} \mathcal{H} \varepsilon_{i}^{k}+\mu \delta_{x}^{4} \varepsilon_{i}^{k}=g_{i}^{k}, 1 \leq i \leq M-1,1 \leq k \leq N  \tag{19}\\
& \varepsilon_{i}^{0}=\psi_{i}, 0 \leq i \leq M \\
& \varepsilon_{0}^{k}=0, \varepsilon_{M}^{k}=0, \delta_{x}^{2} \varepsilon_{0}^{k}=0, \delta_{x}^{2} \varepsilon_{M}^{k}=0,1 \leq k \leq N
\end{align*}
$$

Taking the inner product of (19) with $\varepsilon^{k}$, we get

$$
\begin{aligned}
& \left\langle\partial_{t}^{\alpha} \mathcal{H} \varepsilon^{k}, \varepsilon^{k}\right\rangle+\frac{\tau_{1}^{\beta} w_{k}}{\Gamma(\beta+2)}\left\langle\mathcal{H} \varepsilon^{0}, \varepsilon^{k}\right\rangle+\mu\left\langle\delta_{x}^{4} \varepsilon^{k}, \varepsilon^{k}\right\rangle \\
= & -\frac{\tau_{1}^{\beta}}{\Gamma(\beta+2)} \sum_{l=1}^{k} b_{k, l}\left\langle\mathcal{H} \varepsilon^{l}, \varepsilon^{k}\right\rangle+\left\langle g^{k}, \varepsilon^{k}\right\rangle
\end{aligned}
$$

From Theorem 3.7, there exists a constant $C$ such that

$$
\left\|\varepsilon^{k}\right\| \leq C\left(\|\psi\|+\max _{1 \leq j \leq N}\left\|g^{j}\right\|\right), 1 \leq k \leq N .
$$

## 4. Numerical Experiments

We present numerical results which support the analyses of preceding sections. We suppose $L=\pi, T=\mu=1$. Let $U^{n}$ be the numerical solutions.

Denote

$$
\begin{aligned}
E_{2}(M, N) & =\max _{0 \leq n \leq N}\left\|u^{n}-U^{n}\right\|, \\
\text { rate } 1 & =\log _{2}\left(\frac{E_{2}\left(M, \frac{N}{2}\right)}{E_{2}(M, N)}\right), \quad \text { rate } 2=\log _{2}\left(\frac{E_{2}\left(\frac{M}{2}, N\right)}{E_{2}(M, N)}\right) .
\end{aligned}
$$

Example 4.1. The following problem is considered:

$$
\begin{aligned}
& { }_{0}^{C} D_{t}^{\alpha} u(x, t)+\mu u_{x x x x}(x, t)+\int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(x, s) d s=f(x, t) \\
& \quad 0<x<L, \quad 0<t \leq T \\
& u(x, 0)=0, \quad 0 \leq x \leq L \\
& u(0, t)=0, u(L, t)=0, \frac{\partial^{2} u(0, t)}{\partial x^{2}}=0, \frac{\partial^{2} u(L, t)}{\partial x^{2}}=0, \quad 0<t \leq T
\end{aligned}
$$

where

$$
f(x, t)=\sin (x)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)} t^{(\alpha+\beta)}+t^{\alpha}\right) .
$$

The exact solution for this problem is $u(x, t)=\sin (x) t^{\alpha}$.

Table 1: Numerical convergence orders in temporal direction with $M=100, r=$ $\frac{2-\alpha}{\alpha}$.

| $N$ | $\alpha=0.5, \beta=0.3$ |  |  | $\alpha=0.4, \beta=0.7$ |  |  | $\alpha=0.6, \beta=0.6$ |  |
| :--- | :---: | :---: | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $E_{2}(M, N)$ | Rate 1 |  | $E_{2}(M, N)$ | Rate 1 |  | $E_{2}(M, N)$ | Rate1 |
| 64 | $8.8761 \mathrm{e}-04$ | $*$ |  | $6.4774 \mathrm{e}-04$ | $*$ |  | $1.2000 \mathrm{e}-03$ | $*$ |
| 128 | $3.2928 \mathrm{e}-04$ | 1.4306 |  | $2.2414 \mathrm{e}-04$ | 1.2747 |  | $4.8490 \mathrm{e}-04$ | 1.3213 |
| 256 | $1.2014 \mathrm{e}-04$ | 1.4546 |  | $7.6738 \mathrm{e}-05$ | 1.3228 |  | $1.9052 \mathrm{e}-04$ | 1.3478 |
| 512 | $4.3413 \mathrm{e}-05$ | 1.4686 |  | $2.6091 \mathrm{e}-05$ | 1.5565 |  | $7.3974 \mathrm{e}-05$ | 1.3648 |
| 1024 | $1.5595 \mathrm{e}-05$ | 1.4771 |  | $8.8231 \mathrm{e}-06$ | 1.5642 |  | $2.8498 \mathrm{e}-05$ | 1.3761 |

Table 2: Numerical convergence orders in temporal direction with $M=100, r=$ 1.

| $N$ | $\alpha=0.5, \beta=0.3$ |  |  | $\alpha=0.4, \beta=0.7$ |  |  | $\alpha=0.6, \beta=0.6$ |  |
| :--- | :---: | :---: | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $E_{2}(M, N)$ | Rate 1 |  | $E_{2}(M, N)$ | Rate 1 |  | $E_{2}(M, N)$ | Rate1 |
| 64 | $2.3200 \mathrm{e}-02$ | $*$ |  | $3.3100 \mathrm{e}-02$ | $*$ |  | $1.5800 \mathrm{e}-02$ | $*$ |
| 128 | $1.7200 \mathrm{e}-02$ | 0.4344 |  | $2.6200 \mathrm{e}-02$ | 0.3378 |  | $1.0700 \mathrm{e}-02$ | 0.5603 |
| 256 | $1.2500 \mathrm{e}-02$ | 0.4554 |  | $2.0500 \mathrm{e}-02$ | 0.3547 |  | $7.2000 \mathrm{e}-03$ | 0.5747 |
| 512 | $9.1000 \mathrm{e}-03$ | 0.4697 |  | $1.5900 \mathrm{e}-02$ | 0.3663 |  | $4.8000 \mathrm{e}-03$ | 0.5836 |
| 1024 | $6.5000 \mathrm{e}-03$ | 0.4794 |  | $1.2300 \mathrm{e}-02$ | 0.3747 |  | $3.2000 \mathrm{e}-03$ | 0.5894 |

Table 3: Numerical convergence orders in spatial direction with $N=100000, r=$ $\frac{2-\alpha}{\alpha}$.

| $M$ | $\alpha=0.5, \beta=0.5$ |  |
| :--- | :---: | :---: |
|  | $E_{2}(M, N)$ | Rate 2 |
| 5 | $8.1419 \mathrm{e}-05$ | $*$ |
| 10 | $4.9889 \mathrm{e}-06$ | 4.0286 |
| 20 | $3.0295 \mathrm{e}-07$ | 4.0416 |
| 40 | $1.6899 \mathrm{e}-08$ | 4.1641 |

We choose different graded mesh coefficients $r$ in the experiments, such as $r=\frac{2-\alpha}{\alpha}$ in Table 1 and Table 3, $r=1$ in Table 2. The convergence orders in temporal direction with $M=100$ is reported in Table 1, Table 2. The temporal rate with graded mesh is $O\left(N^{-\min \{r \sigma, 2-\alpha\}}\right)$ with $\sigma=\alpha$. The convergence rate in spatial direction is $O\left(M^{-4}\right)$, which is listed in Table 3 with $N=100000$. The numerical results match that of the theoretical ones.

## 5. Conclusions

In this article, we construct a compact finite difference scheme on graded meshes for the fourth-order time-fractional integro-differential equation with initial singularity. L1 formula and trapezoidal PI rule with nonuniform mesh are adopted to approximate the Caputo derivative and the Riemann-Liouville integral. The compact difference scheme is stable and convergent with the convergence order $O\left(N^{-\min \{r \sigma, 2-\alpha\}}+M^{-4}\right)$. The theoretical results have been verified by some numerical experiments. In the future, we plan to investigate the possibility to construct higher order schemes.

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