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On Characterization of Quadratic Exponential Invertible Graphs

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Abstract. In this paper, we investigate the notion of quadratic exponential invertible graphs whose vertex set is reduced residue system $mod\ n$ and there will be an edge between x and y such that $x^{2^{\alpha}} \equiv y^{2^{\alpha}}\,(mod\,n)$ for some positive integer α. The proposed graph is completely characterized for each positive integer n and also, we find the class of integers in which quadratic exponential invertible graphs are isomorphic to each other. Moreover, the class of those integers is investigated in which the proposed graph is a complete graph.

Keywords: Invertible elements; Quadratic exponential invertible graphs; Complete graphs.

1. Introduction

Graph theory plays a dynamic role in various fields such as informatics, chemistry, physics, biology, etc. Many applications in biology have been proposed to be means of graph theory [10]. The various applications in the era of chemistry, physics, social, and information systems have been incorporated by using the notion of the graph in [6]-[8]. In today's digital world, cryptography is one of the main fields where cyber security is a major concern. In cryptography, the strong code is generated by means of a larger prime. In number theory, such large prime numbers can easily be generated to secure most of the encrypted messages. Maurer [13] derived an efficient algorithm to generate such numbers with the help of number theory. The congruence relations play an essential role in cryptography [18].

Finding the new family of graphs has a vital role in the theory of graphs. New families of graphs based on totient, super totient, and hyper totient numbers are discussed in [5, 3, 11] and [2], [12]. In [17], the family of graphs was introduced which is based on the mapping $x^2 \equiv y^2 (mod\ n)$. Furthermore, some new families of graphs are discussed in [4, 14, 16, 15], which are based on algebraic structure.

Here we discuss some preliminary concepts, notations, and results on graph theory. For more details about these topics, we refer the reader to the manuscripts [9]. A *graph* $G = (V, E)$ is an ordered pair of set of *vertices* V and set of *edges* E. If each vertices are adjacent to each other then the graph will be a *connected graph*. Two graphs $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ are *isomorphic* if there exist a bijection $f : V \to V^{'}$ such that $uv \in E$ if and only if $f(u)f(v) \in E^{'}$. A finite graph is called *complete* if all its vertices are pairwise adjacent. The complete graph of order n is denoted K_n. There are a few results of [1], given below will be used in the sequel.

Theorem 1.1. [1] *Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a positive integer. If t_i represent the distinct solutions of $f(x) \equiv 0(mod\ p_i^{\alpha_i})$, then there are $t_1 t_2 \cdots t_k$ number of solutions of $f(x) \equiv 0(mod\ m)$.*

Theorem 1.2. [1] *If a is an odd integer, then the following statements hold:*

(i) $x^2 \equiv a(mod\ 2)$ *is always solvable and has exactly one solution.*
(ii) $x^2 \equiv a(mod\ 4)$ *is solvable if and only if $a \equiv 1(mod\ 4)$, in which case there are precisely two solutions.*
(iii) $x^2 \equiv a(mod\ 2^k)$, $k \geq 3$ *is solvable if and only if $a \equiv 1(mod\ 8)$, in which case there are exactly four solutions.*

2. Quadratic Exponential Invertible Graphs

In this section, we propose the notion of quadratic exponential invertible graphs and then we characterize it complectly for each positive integer n.

Definition 2.1.A graph $\mathcal{G}(2^\alpha, n) = (V, E)$ is called quadratic exponential invertible if vertex and edge set is defined as,

$$V = \{u_i \mid (u_i, n) = 1, u_i \in \mathbb{Z}_n\}, \quad E = \{e_i = u_i u_j \mid u_i^{2^\alpha} \equiv u_j^{2^\alpha} (mod\ n), i \neq j\}.$$

Where, $\mathbb{Z}_n = \{0, 1, 2 \cdots, n-1\}$ and α is any positive integer. The quadratic exponential invertible graph for $n = 50$ and $\alpha = 3$ as shown in Figure 1.

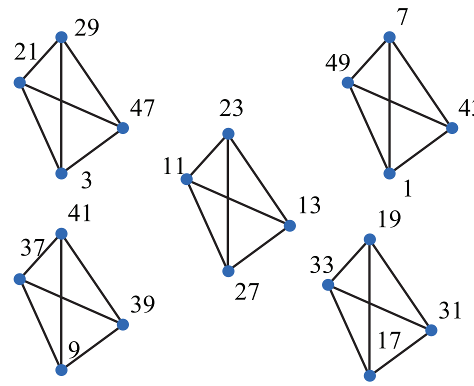


Figure 1: A quadratic exponential invertible graph $\mathcal{G}(2^3, 50)$.

The following result is characterize the quadratic exponential invertible graphs for $n = 2^\beta$ and each positive integer α, β.

Proposition 2.2. *Let α and β be positive integers. Then*

$$\mathcal{G}(2^\alpha, 2^\beta) = \begin{cases} K_{2^{\beta-1}} & \text{if } \beta \leq \alpha + 2, \\ \frac{\varphi(2^\beta)}{2^{\alpha+1}} K_{2^{\alpha+1}} & \text{if } \beta > \alpha + 2. \end{cases}$$

Proof. Let α and β be positive integers. The set of invertible elements of Z_{2^β} is $\{2k - 1 \mid k = 1, 2, \cdots, 2^{\beta-1}\}$. Without any loss, we take any inventible element (say u) of Z_{2^β}. Then for $\beta \leq \alpha + 2$, there is only when congruence which is

$$u^{2^\alpha} \equiv 1 (mod\ 2^\beta). \tag{1}$$

Since, u is an inventible element so the congruence (1), has exactly $\varphi(2^\beta) = 2^{\beta-1}$ number of solutions. When $\beta > \alpha + 2$, then there are $2^{\beta-\alpha-2}$ number of distinct congruence.

$$u^{2^\alpha} \equiv (2k - 1)^{2^\alpha} (mod\ 2^\beta), \quad k = 1, 2, \cdots, 2^{\beta-\alpha-2}. \tag{2}$$

Proof is done if we show that, each congruence in (2) has $2^{\alpha+1}$ number of solutions. Since $(2k - 1)^{2^\alpha}$ is an odd number in modulo 2^β (say a) for each positive integer k. That is, our claim is that $u^{2^\alpha} \equiv a(mod\ 2^\beta)$ has $2^{\alpha+1}$ number of solutions. By using mathematical method, this has required number of solutions. For basis step assume $\alpha = 1$, then $u^2 \equiv a(mod\ 2^\beta)$ has four distinct solutions namely, $1, 2^\beta - 1, 2^{\beta-1} - 1, 2^{\beta-1} + 1$, because $\beta > 3$ by using Theorem 1.2. For inductive step, we assume that the congruence

$$u^{2^k} \equiv a(mod\ 2^\beta),$$

has 2^{k+1} number of solutions. Since, the congruence $u^{2^k} \equiv a(mod\ 2^\beta)$ and $u^{2^k} \equiv 1(mod\ 2^\beta)$, behave a same. Therefore for $\alpha = k+1$, we have

$$u^{2^{k+1}} = (u^{2^k})^2 \equiv 1(mod\ 2^\beta),$$
$$\Rightarrow u^{2^k} \equiv \pm 1(mod\ 2^\beta),$$

so, by inductive hypothesis the congruence $u^{2^k} \equiv 1(mod\ 2^\beta)$ and $u^{2^k} \equiv -1(mod\ 2^\beta)$ have 2^{k+1} number of solutions each. Thus, for $\alpha = k+1$ the number of solutions is 2^{k+2}. ■

The following result is characterize the quadratic exponential invertible graphs for $n = p^\beta$ (p is an odd prime) and each positive integer α, β.

Proposition 2.3. *Let p be an odd prime. Then*

$$\mathcal{G}(2^\alpha, p^\beta) = \begin{cases} \frac{\varphi(p^\beta)}{2} K_2 & \text{if } \alpha = 1, \\ \frac{\varphi(p^\beta)}{2^2} K_{2^2} & \text{if } \alpha > 1 \text{ and } p \equiv 1(mod\ 4), \\ \frac{\varphi(p^\beta)}{2} K_2 & \text{if } \alpha > 1 \text{ and } p \equiv 3(mod\ 4). \end{cases}$$

Proof. When $\alpha = 1$, then proof is done from Theorem 2.1 of [17]. For $\alpha > 1$, we choose an invertible element u such that $(u, p^\alpha) = 1$

$$u^{2^\alpha} \equiv 1(mod\ p^\beta), \tag{3}$$

the congruence (3) has 2 and 4 solutions when $p \equiv 3(mod\ 4)$ and $p \equiv 1(mod\ 4)$ respectively because $\gcd(2^\alpha, p) = 2$, and $\gcd(2^\alpha, p) = 4$, whenever $p \equiv 3(mod\ 4)$ and $p \equiv 1(mod\ 4)$ respectively. Thus there are $\frac{\varphi(p^\beta)}{2}$ and $\frac{\varphi(p^\beta)}{2^2}$ copies of complete graphs K_2 and K_4 respectively according to prime $p \equiv 3(mod\ 4)$ and $p \equiv 1(mod\ 4)$. ■

If $n = \prod_{i=1}^t p_i^{\gamma_i}$ is a positive integer with odd primes $p_i^{'}s$ then by Proposition 2.3 and Theorem 1.1, we have the following proposition.

Proposition 2.4. *Let $n = \prod_{i=1}^t p_i^{\gamma_i}$ be a positive integer with odd primes $p_i^{'}s$. Then*

$$\mathcal{G}(2^\alpha, n) = \begin{cases} \frac{\varphi(n)}{2^t} K_{2^t} & \text{if } \alpha = 1, \\ \frac{\varphi(n)}{2^{2t}} K_{2^{2t}} & \text{if } \alpha > 1\ p_i \equiv 1(mod\ 4), \\ \frac{\varphi(n)}{2^t} K_{2^t} & \text{if } \alpha > 1\ p_i \equiv 3(mod\ 4). \end{cases}$$

Proposition 2.5. *Let* $n = \prod_{i=1}^{s} p_i^{\gamma_i} \cdot \prod_{i=1}^{t} q_i^{\gamma_i}$ *be a positive integer with odd primes* p_i*'s and* q_i*'s such that* $p_i \equiv 1(mod\ 4)$, $q_i \equiv 3(mod\ 4)$. *Then*

$$\mathcal{G}(2^\alpha, n) = \begin{cases} \frac{\varphi(n)}{2^{s+t}} K_{2^{s+t}} & if \ \ \alpha = 1, \\ \frac{\varphi(n)}{2^{2s+t}} K_{2^{2s+t}} & if \ \ \alpha > 1. \end{cases}$$

Proof. Since n is the product of the primes of the form $p_i \equiv 1(mod\ 4)$, $q_i \equiv 3(mod\ 4)$. When $\alpha = 1$, then the proof is done from Theorem 2.2 of [17]. If $\alpha > 1$, then we have by using Proposition 2.4 and Theorem 1.1, we have Proposition 2.5. ■

Proposition 2.6. *If* $n = 2^\beta \cdot \prod_{i=1}^{s} p_i^{\gamma_i} \cdot \prod_{j=1}^{t} q_i^{\delta_i}$ *with* $p_i \equiv 1(mod\ 4)$ *and* $q_i \equiv 3(mod\ 4)$. *Then*

$$\mathcal{G}(2^\alpha, n) = \begin{cases} \frac{\varphi(n)}{2^{s+t}} K_{2^{s+t}} & if \ \ \alpha = 1, \ \beta \in \{0, 1\}, \\ \frac{\varphi(n)}{2^{s+t+1}} K_{2^{s+t+1}} & if \ \ \alpha = 1, \ \beta = 2, \\ \frac{\varphi(n)}{2^{s+t+2}} K_{2^{s+t+2}} & if \ \ \alpha = 1, \ \beta \geq 3, \\ \frac{\varphi(n)}{2^{2s+t}} K_{2^{2s+t}} & if \ \ \alpha > 1, \ \beta = 0, \\ \frac{\varphi(n)}{2^{2s+t+\beta-1}} K_{2^{2s+t+\beta-1}} & if \ \ \alpha > 1, \ \beta \leq \alpha + 2, \\ \frac{\varphi(n)}{2^{2s+t+\alpha+1}} K_{2^{2s+t+\alpha+1}} & if \ \ \alpha > 1, \ \beta > \alpha + 2. \end{cases}$$

Proof. The proof is done in the case, when $\alpha = 1$ and β is any non-negative integer by using Theorem 2.3 of [17]. If $\alpha > 1$ and $\beta = 0$, then we have desired result by using Proposition 2.4. In case, when $\alpha > 1$ and $\beta \leq \alpha + 2$, then proof is done by using Theorem 1.1, Propositions 2.2, and 2.5. In last case, for $\alpha > 1$, $\beta > \alpha + 2$, by means of Theorem 1.1, and last cases of Propositions 2.2 and 2.5. ■

The quadratic exponential invertible graphs $\mathcal{G}(2, 27)$, $\mathcal{G}(2^2, 25)$, $\mathcal{G}(2^3, 34)$ are shown in Figure 2.

3. Isomorphism of Quadratic Exponential Invertible Graphs

In this section, we find the class of integers in which quadratic exponential invertible graphs are isomorphic to each other. Also, we find the condition on integers in which proposed graph is a complete graph.

Theorem 3.1. *Let* $m = 2^{\beta_1} \cdot \prod_{i=1}^{s_1} p_i^{\gamma_i} \cdot \prod_{j=1}^{t_1} q_i^{\delta_i}$ $n = 2^{\beta_2} \cdot \prod_{i=1}^{s_2} p_i^{\gamma_i} \cdot \prod_{j=1}^{t_2} q_i^{\delta_i}$ *be two positive integers with* $p_i \equiv 1(mod\ 4)$ *and* $q_i \equiv 3(mod\ 4)$. *Then quadratic exponential invertible graphs* $\mathcal{G}(2^{\alpha_1}, m)$ *and* $\mathcal{G}(2^{\alpha_2}, n)$ *are isomorphic if and only if* $\varphi(m) = \varphi(n)$ *and*

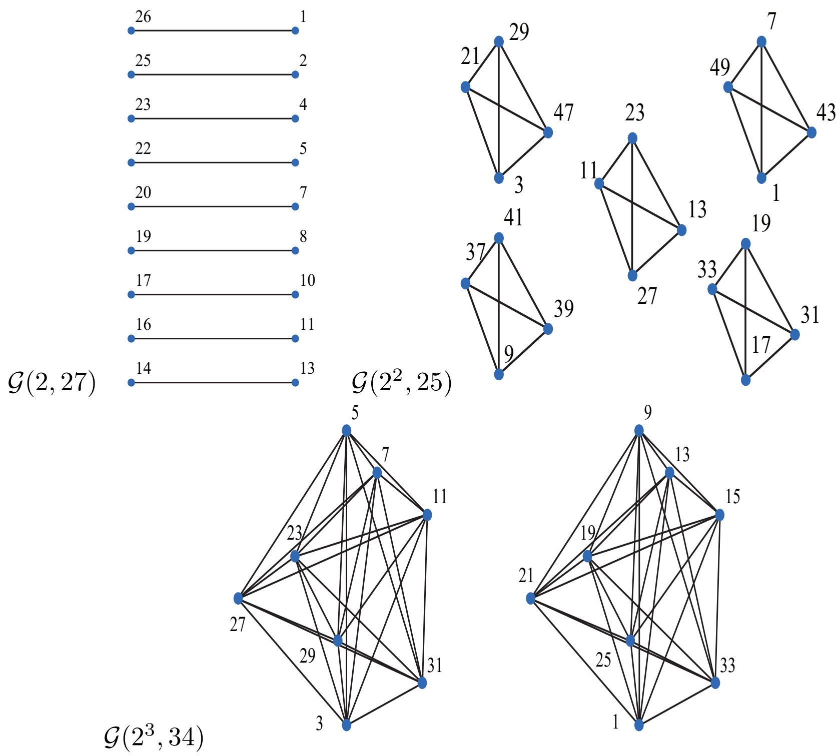


Figure 2: Quadratic exponential invertible graphs $\mathcal{G}(2,27)$, $\mathcal{G}(2^2,25)$, $\mathcal{G}(2^3,34)$.

(i) *if* $\alpha_1 = 1, \quad \beta_1 \in \{0,1\}$, *then*

$$\begin{cases} |(s_1+t_1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(s_1+t_1)-(s_2+t_2+1)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(s_1+t_1)-(s_2+t_2+2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(s_1+t_1)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(s_1+t_1)-(2s_2+t_2+\beta_2-1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 \leq \alpha_2+2, \\ |(s_1+t_1)-(2s_2+t_2+\alpha_2+1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 > \alpha_2+2. \end{cases}$$

(ii) *If* $\alpha_1 = 1, \quad \beta_1 = 2$, *then*

$$\begin{cases} |(s_1+t_1+1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(s_1+t_1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(s_1+t_1+1)-(s_2+t_2+2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(s_1+t_1+1)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(s_1+t_1+1)-(2s_2+t_2+\beta_2-1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 \leq \alpha_2+2, \\ |(s_1+t_1+1)-(2s_2+t_2+\alpha_2+1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 > \alpha_2+2. \end{cases}$$

(iii) *If* $\alpha_1 = 1, \quad \beta_1 \geq 3$, *then*

$$\begin{cases} |(s_1+t_1+2)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(s_1+t_1+2)-(s_2+t_2+1)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(s_1+t_1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(s_1+t_1+2)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(s_1+t_1+2)-(2s_2+t_2+\beta_2-1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 \leq \alpha_2+2, \\ |(s_1+t_1+2)-(2s_2+t_2+\alpha_2+1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 > \alpha_2+2. \end{cases}$$

(iv) *If* $\alpha_1 > 1, \quad \beta_1 = 0$, *then*

$$\begin{cases} |(2s_1+t_1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(2s_1+t_1)-(s_2+t_2+1)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(2s_1+t_1)-(s_2+t_2+2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(2s_1+t_1)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(2s_1+t_1)-(2s_2+t_2+\beta_2-1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 \leq \alpha_2+2, \\ |(2s_1+t_1)-(2s_2+t_2+\alpha_2+1)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 > \alpha_2+2. \end{cases}$$

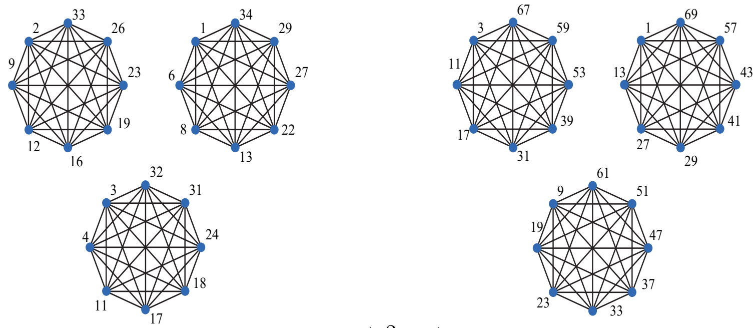
(v) *If* $\alpha_1 > 1, \quad \beta_1 \leq \alpha_1 + 2$, *then*

$$\begin{cases} |(2s_1+t_1+\beta_1-1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(2s_1+t_1+\beta_1-1)-(s_2+t_2+1)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(2s_1+t_1+\beta_1-1)-(s_2+t_2+2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(2s_1+t_1+\beta_1-1)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(2s_1+t_1+\beta_1)-(2s_2+t_2+\beta_2)| = 0 & \text{if } \alpha_2 > 1, \beta_2 \leq \alpha_2+2, \\ |(2s_1+t_1+\beta_1-1)-(2s_2+t_2+\alpha_2+1)| = 0 & \text{if } \alpha_2 > 1, \beta_2 > \alpha_2+2. \end{cases}$$

(vi) *If* $\alpha_1 > 1, \quad \beta_1 > \alpha_1 + 2$, *then*

$$\begin{cases} |(2s_1+t_1+\alpha_1+1)-(s_2+t_2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \in \{0,1\}, \\ |(2s_1+t_1+\alpha_1+1)-(s_2+t_2+1)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 = 2, \\ |(2s_1+t_1+\alpha_1+1)-(s_2+t_2+2)| = 0 & \text{if } \alpha_2 = 1,\ \beta_2 \geq 3, \\ |(2s_1+t_1+\alpha_1+1)-(2s_2+t_2)| = 0 & \text{if } \alpha_2 > 1,\ \beta_2 = 0, \\ |(2s_1+t_1+\alpha_1+1)-(2s_2+t_2+\beta_2-1)| = 0 & \text{if } \alpha_2 > 1, \beta_2 \leq \alpha_2+2, \\ |(2s_1+t_1+\alpha_1)-(2s_2+t_2+\alpha_2)| = 0 & \text{if } \alpha_2 > 1, \beta_2 > \alpha_2+2. \end{cases}$$

Proof. Clearly from Proposition 2.6, quadratic exponential invertible graphs $\mathcal{G}(2^{\alpha_1}, m)$ and $\mathcal{G}(2^{\alpha_2}, n)$ are isomorphic if and only if when $\varphi(m) = \varphi(n)$ and same number of copies of complete graphs. This is only possible when the exponent of 2 is same for both m and n, so we have the following 36 possibilities which is in Theorem 3.1. ■



$\mathcal{G}(2^2, 35)$ $\mathcal{G}(2^2, 70)$

Figure 3: Isomorphism of quadratic exponential invertible graphs $\mathcal{G}(2^2, 35) \cong \mathcal{G}(2^2, 70)$.

Two quadratic exponential invertible graphs for $n = 35$, $m = 70$ and $\alpha = 2$ are shown in Figure 3.

The quadratic exponential invertible graphs $\mathcal{G}(2^\alpha, n)$ is a complete graph if and only if there is only one copy graph for each positive integer α. From Proposition 2.6, we have the following result.

Proposition 3.2. *If* $n = 2^\beta \cdot \prod_{i=1}^{s} p_i^{\gamma_i} \cdot \prod_{j=1}^{t} q_i^{\delta_i}$ *with* $p_i \equiv 1(mod\ 4)$ *and* $q_i \equiv 3(mod\ 4)$*. Then* $\mathcal{G}(2^\alpha, n)$ *is a compete graph if and only if*

$$\varphi(n) = \begin{cases} 2^{s+t}, & if\ \alpha = 1,\ \beta \in \{0, 1\}, \\ 2^{s+t+1}, & if\ \alpha = 1,\ \beta = 2, \\ 2^{s+t+2}, & if\ \alpha = 1,\ \beta \geq 3, \\ 2^{2s+t}, & if\ \alpha > 1,\ \beta = 0, \\ 2^{2s+t+\beta-1}, & if\ \alpha > 1,\ \beta \leq \alpha + 2, \\ 2^{2s+t+\alpha+1}, & if\ \alpha > 1,\ \beta > \alpha + 2. \end{cases}$$

4. Conclusion

In this work, we investigate the notion of quadratic exponential invertible graphs whose vertices are inventible elements of the ring $\mathbb{Z}_n$ and there will be an edge between two invertible elements x and y ($x \neq y$) if it is satisfied the exponential mapping $x^{2^\alpha} \equiv y^{2^\alpha} (mod\ n)$, for some positive integer α. We characterized the proposed quadratic exponential invertible graphs for each positive integer n. Moreover, we note that each positive integer n quadratic exponential invertible graph is a complete graph or copies of complete graphs. Furthermore, we find the class of integers in which exponential invertible graphs are isomorphic and a complete graph. In further work, ones can be generalized this idea by means of exponential mapping $x^{n^\alpha} \equiv y^{n^\alpha} (mod\ n)$, for each positive integer α and n.

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