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# Neighborhood Pseudo Achromatic Number of a Complete $p$-Partite Graph

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**Abstract.** A pseudo coloring of the vertices of a graph $G$, is a coloring assigned to the vertices of $G$ such that adjacent vertices can receive the same colors. A pseudo complete coloring of a graph $G$ is a pseudo coloring of the vertices of $G$, such that for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors. The maximum number of colors used in a pseudo complete coloring of $G$ is called the pseudo achromatic number, denoted by $\psi_s(G)$. A neighborhood pseudo chromatic number of a graph $G$, denoted by $\psi_{nhd}(G)$, is the maximum number of colors used in a pseudo coloring of $G$, such that every vertex has at least two vertices in its closed neighborhood receiving the same color. Here, we defined a new coloring parameter called neighborhood pseudo achromatic number of a graph $G$ and obtained neighborhood pseudo achromatic number of a complete $p$-partite graph by analysing various possibilities.

**Keywords:** Neighborhood pseudo coloring; Achromatic number; Pseudoachromatic number; Neighborhood pseudo achromatic number; Complete $n$-partite graph.

## 1. Introduction

A coloring of $V(G)$ is called proper if no two adjacent vertices are assigned the same color and minimum number of colors used in such a coloring is called the chromatic number of $G$, denoted by $\chi(G)$. F. Harary and S.T. Hedetneimi [5] introduced the concept of proper complete coloring and obtained the bounds for achromatic number $\psi(G)$.

A pseudo coloring of the vertices of a graph $G$, is a coloring assigned to the vertices such that adjacent vertices can receive the same color. A pseudo complete coloring of a graph $G$ is a pseudo coloring of the vertices of $G$, such that for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors and the maximum number of colors required for such a coloring is called pseudo achromatic number of a graph $G$, denoted by $\psi_s(G)$. The pseudo achromatic number was first introduced by Gupta [4]. This problem has been extensively studied in graph theory and combinatorial optimization [8, 10, 12]. In 2000, V. Yegnarayanan [11] obtained the pseudo achromatic number of a complete $p$-partite graph partially and was obtained for the remaining case by M. Liu et al. [7] in 2011.

In 2014, B. Sooryanarayana and N. Narahari [9] introduced a neighborhood pseudo chromatic number of a graph $G$, denoted by $\psi_{nhd}(G)$, is the maximum number of colors used in a pseudo coloring of $G$ such that every vertex has at least two vertices in its closed neighborhood receiving the same color. Further, in the articles [3, 6], neighborhood pseudo chromatic polynomial was introduced and obtained the same for some graphs. We use the standard terminology, the terms not defined here, are found in [1, 2].

## 2. Neighborhood Pseudo Complete Coloring

Let $G(V, E)$ be a non trivial, simple, connected graph. For $v \in V(G)$, $N_G(v) = N(v) = \{x : vx \in E(G)\}$ and $N_G[v] = N[v] = N(v) \cup \{v\}$, are open and closed neighborhood of a vertex $v$ respectively.

**Definition 2.1.** *A pseudo complete coloring $c : V(G) \to \{1, 2, 3, \ldots, |V|\}$ in which for each vertex $v$, there exits $u, w$ in $N[v]$ satisfying $c(u) = c(w)$, is called a neighborhood pseudo complete coloring of graph $G$.*

More precisely, a coloring $c : V(G) \to \{1, 2, 3, \ldots, |V|\}$ satisfying,

(i) for each vertex $v$, there exits $u, w$ in $N[v]$ with $c(u) = c(w)$, the criteria of neighborhood pseudo coloring,

(ii) for any pair of distinct colors in range of $c$, there is at least one edge whose end vertices are colored with this pair of colors, the criteria of pseudo complete coloring,

is called a neighborhood pseudo complete coloring, shortly *npc* coloring of graph $G$.

The maximum number of colors used for neighborhood pseudo complete coloring of a graph $G$, is called the neighborhood pseudo achromatic number of $G$, denoted by $\psi_{S_{nhd}}(G)$. A surjective *npc* coloring $c : V(G) \to \{1, 2, \ldots, \psi_{S_{nhd}}(G)\}$, is called an optimal *npc* coloring of graph $G$ and $G$ is said to be neighborhood pseudo complete $\psi_{S_{nhd}}(G)$-colorable.

For any coloring $c$ defined on $V(G)$, we use the following conventions and notations:

(i) If the vertices of the graph are colored with $k$ colors, assume the range set to be $\{1, 2, ..., k\}$.
(ii) $c(V(G)) =$ Range of $c = \{c(v) : v \in V(G)\}$.
(iii) $c(S) = \{c(v) : v \in S\}$, for any set $S \subseteq V(G)$.
(iv) $c(N_G(x)) = c(N(x)) = \{c(w) : xw \in E(G)\}$, $c(N_G[x]) = c(N[x]) = c(N_G(x)) \cup \{c(x)\}$.
(v) $span\ of\ c = max\{c(w) : w \in V(G)\}$.

**Observation 2.2.** A coloring $c$ defined on $V(G)$, satisfies the criteria of neighborhood pseudo coloring if and only if $|c(N_G[v])| < |N_G[v]|$, for each vertex $v \in V(G)$ and hence, $|c(V(G)| < |V(G)|$.

**Lemma 2.3.** *For any graph $G$ on $n$ vertices, $1 \leq \psi_{S_{nhd}}(G) \leq \psi_{nhd}(G)$.*

*Proof.* The maximum $k$ colors admits for a graph $G$ to satisfy neighborhood pseudo complete $k$-coloring only if it is possible to neighborhood pseudo color the graph $G$ with at least $k$ colors. Hence the result. ■

## 3. *npc* Coloring of Complete *p*-Partite Graph

The join of disjoint graphs $\overline{K}_{m_i}$, $i = 1, 2, \ldots, p (p > 1)$ is called the complete $p$-partite graph, denoted by $K_{m_1,m_2,\ldots,m_p}$. That is, $K_{m_1,m_2,\ldots,m_p} = \overline{K}_{m_1} \vee \overline{K}_{m_2} \vee \ldots \vee \overline{K}_{m_p}$. Let $V(K_{m_1,m_2,\ldots,m_p}) = V_1 \cup V_2 \cup \ldots \cup V_p$, $|V_i| = m_i$, $|V| = \sum_{i=1}^{p} |V_i| = \sum_{i=1}^{p} m_i = n$ and $V_i = \{v_{i1}, v_{i2}, \ldots, v_{im_i}\}$ for every $i$ with $1 \leq i \leq p$. W.l.g, let $m_1 \leq m_2 \leq \ldots \leq m_p$.

V. Yegnanarayanan [11] and M. Liu et al. [7] obtained the following results on pseudo achromatic number of complete $p$-partite graph:

**Theorem 3.1.** [11] *Let $m_1, m_2, \ldots, m_p$ be $p$ positive integers with $p \geq 3$ and $n = \sum m_i$, where $m_1 \leq m_2 \leq \ldots \leq m_p$. If $m_p \leq (\frac{n-p+2}{2})$ then $\psi_S(K_{m_1,m_2,\ldots,m_p}) = \lfloor \frac{n+p}{2} \rfloor$.*

**Theorem 3.2.** [7] *Let $m_1, m_2, \ldots, m_p$ be $p$ positive integers with $p \geq 2$ and $n = \sum m_i$, where $m_1 \leq m_2 \leq \ldots \leq m_p$. If $m_p \geq \lceil \frac{n-p+2}{2} \rceil$ then $\psi_S(K_{m_1,m_2,\ldots,m_p}) = m_1 + m_2 + \ldots + m_{p-1} + 1$.*

In this article, we defined a new way of coloring to obtain the neighborhood pseudo achromatic number of complete $p$-partite graph in all the cases uniquely by studying few properties. Also, proved that neighborhood pseudo achromatic number coincide with pseudo achromatic number when $m_p \leq (\frac{n-p+2}{2})$ and are different otherwise.

**Theorem 3.3.** *For a complete p-partite graph,* $\psi_{S_{nhd}}(K_{m_1,m_2,\ldots,m_p}) = \sum_{i=1}^{p-1} m_i$ *if* $m_p \geq \sum_{i=1}^{p-1} m_i$.

*Proof.* Each vertex of $\bigcup_{i=1}^{p-1} V_i$ are colored differently using $\sum_{i=1}^{p-1} m_i$ colors and assign the same $\sum_{i=1}^{p-1} m_i$ colors to the vertices of $V_p$ choosing each color at least once. Then, such a coloring satisfies the criteria of *npc* coloring of $K_{m_1,m_2,\ldots,m_p}$, because for every pair of distinct colors, there exist an edge between the partitions $\bigcup_{i=1}^{p-1} V_i$ and $V_p$, which satisfies the criteria of pseudo complete coloring and each vertex $v \in \bigcup_{i=1}^{p-1} V_i$ is adjacent to a vertex in $V_p$ which receives the same color as that of $v$, satisfies the criteria of neighborhood pseudo coloring. Hence, $\psi_{S_{nhd}}(K_{m_1,m_2,\ldots,m_p}) \geq \sum_{i=1}^{p-1} m_i$. Also, $\psi_{S_{nhd}}(K_{m_1,m_2,\ldots,m_p}) \leq \sum_{i=1}^{p-1} m_i$ as every vertices of $\bigcup_{i=1}^{p-1} V_i$ are colored differently using $\sum_{i=1}^{p-1} m_i$ colors and no additional color can be assigned to any vertex $v$ of $V_p$, since $v$ does not satisfies the criteria of neighborhood pseudo coloring. Combining, $\psi_{S_{nhd}}(K_{m_1,m_2,\ldots,m_p}) = \sum_{i=1}^{p-1} m_i$. ∎

**Corollary 3.4.** *For a complete bipartite graph* $K_{m_1,m_2}$, $\psi_{S_{nhd}}(K_{m_1,m_2}) = m_1 = min\{m_1, m_2\}$.

**Lemma 3.5.** *A coloring c on* $V(K_{m_1,m_2,\ldots,m_p})$ *with* $p \geq 3$ *is a npc coloring if* $|c(V_k) \cap c(\bigcup_{i \neq k} V_i)| \geq |c(V_k)| - 1$ *for every k with* $1 \leq k \leq p$ *and* $c(V_l) \cap c(V_m) \neq \phi$ *for some* $l, m$ *with* $l \neq m$.

*Proof.* Let $V_k$ be any partition, $|c(V_k)| = l_k$ and $c$ be a coloring of $V(K_{m_1,m_2,\ldots,m_p})$ satisfies the condition as in statement. W.l.g, let $c(v_{kj}) = j$ for fixed $k$ and $1 \leq j \leq l_k$. Consider the following two cases.

*Case 1:* $|c(V_k) \cap c(\bigcup_{i \neq k} V_i)| = l_k - 1$.

Proof follows from the following steps:

(i) Let $c(V_k) \cap c(\bigcup_{i \neq k} V_i) = \{2, 3, \ldots, l_k\}$ which implies colors $\{2, 3, \ldots, l_k\}$ appear in two partite sets, $c(v_{k1}) = 1$. By the definition of complete $p$-partite graph, each vertex in $V_k$ is adjacent to every vertices of $\bigcup_{i \neq k} V_i$, so that there exists an edge between every pair of colors in $\{1, 2, 3, \ldots, l_k\}$, assigned to the vertices of $V_k$, which satisfies the criteria of pseudo complete coloring for the vertices of $V_k$.

(ii) $c(V_l) \cap c(V_m) \neq \phi$ and each vertex $v$ of $V_k$ is adjacent to every vertices of $V_l$ and $V_m$, which implies, $v$ is adjacent to at least two vertices in its closed neighborhood receiving the same color, satisfies the criteria of neighborhood pseudo coloring of vertices of $V_k$.

From (i) and (ii), coloring $c$ on $V_k$ satisfies *npc* coloring.

(iii) For any partition $V_i$, $i \neq k$ and for any vertex $v \in V_i$ which receives the new color say $c(v)$, then as $v$ is adjacent to each vertices of $V_k$, there exists an edge between every pair of colors in $\{1, 2, 3, \ldots, l_k, c(v)\}$.

As the vertex $v \in V_i$, the sets $V_i$, $V_k$ are arbitrary, the coloring $c$ defined on $V(K_{m_1,m_2,\ldots,m_p})$ is a *npc* coloring.

*Case 2:* $|c(V_k) \cap c(\bigcup_{i\neq k} V_i)| > l_k - 1$.

That is, $|\ c(V_k) \cap c(\bigcup_{i\neq k} V_i)\ | = l_k$, because $|c(V_k)| = l_k$. Let $c(V_k) \cap c(\bigcup_{i\neq k} V_i) = \{1, 2, 3, \ldots, l_k\}$ which implies colors $\{1, 2, 3, \ldots, l_k\}$ appear in two partite sets. Hence, there exists an edge between every pair of colors in $\{1, 2, 3, \ldots, l_k\}$, assigned to the vertices of $V_k$ and each vertex of $V_k$ has at least two vertices in its closed neighborhood receiving the same color, satisfies *npc* coloring of vertices of $V_k$. Also, for any partition $V_i$, $i \neq k$ and for any vertex $v$ of $V_i$ which receives the new color with the same explanation as that of Case 1, it follows that, the coloring $c$ defined on $V(K_{m_1,m_2,\ldots,m_p})$ is a *npc* coloring. ∎

**Lemma 3.6.** *A coloring $c$ on $V(K_{m_1,m_2,\ldots,m_p})$, $p \geq 3$ satisfying $|c(V_k) \cap c(\bigcup_{i\neq k} V_i)| < |c(V_k)| - 1$, for some partition $V_k$, is not an npc coloring.*

*Proof.* Let $c$ be a coloring on $V(K_{m_1,m_2,\ldots,m_p})$ and $V_k$ be the partition for which $|c(V_k) \cap c(\bigcup_{i\neq k} V_i)| < l_k - 1$, where $|c(V_k)| = l_k$. Then, there exist at least two vertices say $v_{k1}$, $v_{k2}$ in $V_k$, assigned with distinct colors $c(v_{k1})$, $c(v_{k2})$ such that $c(v_{k1}), c(v_{k2}) \notin c(V_k) \cap c(\bigcup_{i\neq k} V_i)$. But then, there exist no edge whose end vertices are colored with $c(v_{k1})$ and $c(v_{k2})$ and hence, $c$ on $V(K_{m_1,m_2,\ldots,m_p})$ is not an *npc* coloring. ∎

**Lemma 3.7.** *For a complete p-partite graph $K_{m_1,m_2,\cdots,m_p}$ with $p \geq 3$ and $m_p < m_1 + m_2 + \cdots + m_{p-1}$,*

$$\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) \geq \lceil \tfrac{m_1+m_2+\cdots+m_p}{2} \rceil.$$

*Proof.* Let $n = m_1 + m_2 + \cdots + m_p$. Define $c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, n\}$ by $c(v_{1j}) = j$ for $1 \leq j \leq m_1$, $c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$ for all $i$ with $1 < i \leq k_1 - 1$, $j$ varies with $1 \leq j \leq m_i$ and for $i = k_1$, $j$ varies with $1 \leq j \leq k_2$, for some $k_1, k_2$ satisfying $c(v_{k_1k_2}) = \lceil \frac{n}{2} \rceil$ and remaining $\lfloor \frac{n}{2} \rfloor$ vertices of $K_{m_1,m_2,\cdots,m_p}$ are colored differently from $\{1, 2, 3, \cdots, \lceil \frac{n}{2} \rceil\}$, such that each of $\lfloor \frac{n}{2} \rfloor$ colors should appear in exactly two partitions. Then, such a coloring satisfies the criteria of *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$, since, every color appears in two partitions (except one color when $n$ is odd). Hence, $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) \geq \lceil \frac{m_1+m_2+\cdots+m_p}{2} \rceil$. ∎

**Theorem 3.8.** *For a complete p-partite graph $K_{m_1,m_2,\cdots,m_p}$ with $p \geq 3$,*

$$\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \tfrac{m_1+m_2+\cdots+m_p}{2} \rceil \text{ if } \textstyle\sum_{i=1}^{p-1} m_i = m_p + 1.$$

*Proof.* Let $\sum_{i=1}^{p-1} m_i = m_p + 1$. Then $n = m_1 + m_2 + \cdots + m_p = \sum_{i=1}^{p-1} m_i + m_p = m_p + 1 + m_p = 2m_p + 1$ which implies, $n$ is odd and $\lceil \frac{n}{2} \rceil = |\bigcup_{i=1}^{p-1} V_i| = \sum_{i=1}^{p-1} m_i$.

Consider a surjective coloring $c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ defined by $c(v_{1j}) = j$ for $1 \leq j \leq m_1$, $c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$ for each $i$, $2 \leq i \leq p-1$, $j$ varies with $1 \leq j \leq m_i$ and $c(v_{pj}) = j + 1$ for $1 \leq j \leq m_p$. Then, from Lemma 3.7, such a coloring satisfies the criteria of *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$. The coloring $c$ is optimal, because each vertices of $\bigcup_{i=1}^{p-1} V_i$ are assigned with different colors and no new color can be assigned to any vertex $v \in V_p$, as $v$ does not satisfies the criteria of neighborhood pseudo coloring. Hence, $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \frac{n}{2} \rceil$ if $\sum_{i=1}^{p-1} m_i = m_p + 1$. ■

*Example 3.9.* Consider a complete 3-partite graph $K_{2,3,4}$. Here, $n = 2+3+4 = 9$, $m_p = |V_p| = 4$, $\sum_{i=1}^{p-1} m_i = 5$, so that $m_p < \sum_{i=1}^{p-1} m_i$ and $\sum_{i=1}^{p-1} m_i = m_p + 1$. From Theorem 3.8, $\psi_{S_{nhd}}(K_{2,3,4}) = \lceil \frac{n}{2} \rceil = \lceil \frac{9}{2} \rceil = 5$ and *npc* coloring of $K_{2,3,4}$ is as shown in Figure 1.

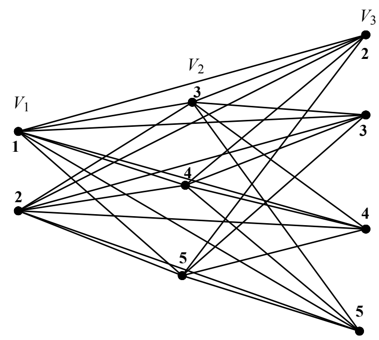


Figure 1: *npc* coloring of $K_{2,3,4}$ with $\psi_{S_{nhd}}(K_{2,3,4}) = 5$.

**Theorem 3.10.** *Let* $KI = K_{m_1,m_2,\cdots,m_p}$. *For a complete p-partite graph KI with* $p \geq 3$, $m_p < \sum_{i=1}^{p-1} m_i - 1$ *and k is the smallest positive integer such that* $\lfloor \frac{n}{2} \rfloor \leq \sum_{i=1}^{k} m_i$ *then,*

$$\psi_{S_{nhd}}(KI) = \begin{cases} \lfloor \frac{n}{2} \rfloor + (p-k) + \lfloor \frac{(2k-p)}{2} \rfloor & \text{if } k < p-1, \\ \sum\limits_{i=1}^{p-1} m_i & \text{if } k = p-1 \text{ and } \lfloor \frac{n}{2} \rfloor - m_p \leq \lfloor \frac{k}{2} \rfloor, \\ \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor & \text{if } k = p-1 \text{ and } \lfloor \frac{n}{2} \rfloor - m_p > \lfloor \frac{k}{2} \rfloor. \end{cases}$$

*Proof.* Let $n = \sum_{i=1}^{p} m_i$ and $k$ be the smallest positive integer such that $\lfloor \frac{n}{2} \rfloor \leq \sum_{i=1}^{k} m_i$. From Lemma 3.7, $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) \geq \lceil \frac{n}{2} \rceil$ as $m_p < \sum_{i=1}^{p-1} m_i$ and proof follows from the following cases.

*Case 1:* $k < p - 1$.

Consider a coloring $c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, n\}$ defined by $c(v_{1j}) = j$ for $1 \leq j \leq m_1$, $c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$ for all $i$ with $2 \leq i \leq k-1$, $j$ varies with $1 \leq j \leq m_i$ and for $i = k$, $j$ varies with $1 \leq j \leq k_1$, for some $k_1$ satisfying $c(v_{kk_1}) = \lfloor \frac{n}{2} \rfloor$ and remaining $\lceil \frac{n}{2} \rceil$ vertices are colored as follows. Now to satisfy the criteria of *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$ with maximum colors, from Lemma 3.5, colors of $((m_1-1)+(m_2-1)+\cdots+(m_{k-1}-1)+(k_1-1))$ vertices are assigned to the remaining $\lceil \frac{n}{2} \rceil$ vertices of $K_{m_1,m_2,\cdots,m_p}$. That is, assign the remaining $\lceil \frac{n}{2} \rceil$ vertices using already assigned $\lfloor \frac{n}{2} \rfloor$ colors except the $k$ colors say $c(v_{11}), c(v_{21}), \cdots, c(v_{k1})$ and the vertices in $\bigcup_{i=k}^{p} V_i$ are colored with maximum of $(p-k)+\lfloor \frac{2k-p}{2} \rfloor$ new colors as follows. Assign a new color to exactly one vertex say $v_{i1}$ of each $V_i$ by $c(v_{i1}) = \lfloor \frac{n}{2} \rfloor + i - k$ for every $i$ with $k+1 \leq i \leq p$. That is, total of $p-k$ new colors are assigned, which still preserve the criteria of *npc* coloring from Lemma 3.5. Further, to maximize the span of $c$, $k-(p-k) = 2k-p$ vertices in unassigned partition are colored using $\lfloor \frac{2k-p}{2} \rfloor$ new colors by coloring each $\lfloor \frac{2k-p}{2} \rfloor$ vertices in two different partitions say $V_{p-1}$, $V_p$. Assign $\lfloor \frac{2k-p}{2} \rfloor$ new colors as $c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lfloor \frac{n}{2} \rfloor + p - k + j$ with $1 \leq j \leq \lfloor \frac{2k-p}{2} \rfloor$. Also, unassigned $(\lceil \frac{n}{2} \rceil - k + k - ((p-k) + \lfloor \frac{2k-p}{2} \rfloor)) = (\lceil \frac{n}{2} \rceil + k - p - \lfloor \frac{2k-p}{2} \rfloor)$ vertices of $\bigcup_{i=k}^{p} V_i$ are colored using the colors from the set $\{\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\} - \{c(v_{11}), c(v_{21}), \cdots, c(v_{k1})\}\}$ choosing each color at least once, which satisfies the *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$ with maximum colors. Hence, $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lfloor \frac{n}{2} \rfloor + (p-k) + \lfloor \frac{(2k-p)}{2} \rfloor$.

*Case 2:* $k = p-1$.

Consider a coloring $c : V(K_{m_1,m_2,\cdots,m_p}) \to Z^+$ defined by $c(v_{1j}) = j$ for $1 \leq j \leq m_1$, $c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$ for all $i, j$ with $1 \leq j \leq m_i$ for $2 \leq i \leq p-2$ and $1 \leq j \leq k_1$ for $i = p-1$, for some $k_1$ satisfying $c(v_{(p-1)k_1}) = \lceil \frac{n}{2} \rceil$ and remaining $\lfloor \frac{n}{2} \rfloor$ vertices are colored as follows. Now to satisfy the criteria of *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$ with maximum colors, from Lemma 3.5, assign the remaining $\lfloor \frac{n}{2} \rfloor$ vertices of $V_{p-1} \cup V_p$ using already assigned $\lceil \frac{n}{2} \rceil$ colors except the $k$ colors say $c(v_{11}), c(v_{21}), \cdots, c(v_{k1})$ and hence, maximum of $\lfloor \frac{n}{2} \rfloor - m_p$ new colors when $(\lfloor \frac{n}{2} \rfloor - m_p) \leq \lfloor \frac{k}{2} \rfloor$ and $\lfloor \frac{k}{2} \rfloor$ new colors when $(\lfloor \frac{n}{2} \rfloor - m_p) > \lfloor \frac{k}{2} \rfloor$ are assigned as follows.

*Subcase 2.1:* $(\lfloor \frac{n}{2} \rfloor - m_p) \leq \lfloor \frac{k}{2} \rfloor$.

Assign a new color to exactly one vertex say $v_{p1}$ of $V_p$ by $c(v_{p1}) = \lceil \frac{n}{2} \rceil + 1$ and assign the remaining $\lfloor \frac{n}{2} \rfloor - m_p - 1$ new colors as $c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lceil \frac{n}{2} \rceil + 1 + j$ for every $j$ with $1 \leq j \leq \lfloor \frac{n}{2} \rfloor - m_p - 1$.

*Subcase 2.2:* $(\lfloor \frac{n}{2} \rfloor - m_p) > \lfloor \frac{k}{2} \rfloor$.

Assign a new color to exactly one vertex say $v_{p1}$ of $V_p$ by $c(v_{p1}) = \lceil \frac{n}{2} \rceil + 1$ and assign the remaining $\lfloor \frac{k}{2} \rfloor - 1$ new colors as $c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lceil \frac{n}{2} \rceil + 1 + j$ for every $j$ with $1 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1$.

Also, in both the sub cases, unassigned vertices of $V_{p-1} \cup V_p$ are colored using the colors from the set $\{\{1, 2, \cdots, \lceil \frac{n}{2} \rceil\} - \{c(v_{11}), c(v_{21}), \cdots, c(v_{k1})\}\}$ choosing each color at least once, which satisfies the *npc* coloring of $K_{m_1,m_2,\cdots,m_p}$ with maximum colors. Hence, if $k = p-1$ and $\lfloor \frac{n}{2} \rfloor - m_p \leq \lfloor \frac{k}{2} \rfloor$, then $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \frac{n}{2} \rceil + (\lfloor \frac{n}{2} \rfloor - m_p) = \sum_{i=1}^{p-1} m_i$; if $k = p-1$ and $\lfloor \frac{n}{2} \rfloor - m_p > \lfloor \frac{k}{2} \rfloor$, then $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor$. ■

*Example 3.11.* Consider a complete 5-partite graph $K_{2,3,4,4,5}$. Here, $n = 2 + 3 + 4 + 4 + 5 = 18$, $m_p = |V_p| = 5$, $\sum_{i=1}^{p-1} m_i - 1 = 13 - 1 = 12$, so that $m_p < \sum_{i=1}^{p-1} m_i - 1$ and $\lfloor \frac{n}{2} \rfloor = 9$. Thus, $k = 3$ and $k < p - 1$. Hence, from Theorem 3.10, $\psi_{S_{nhd}}(K_{2,3,4,4,5}) = \lfloor \frac{n}{2} \rfloor + (p - k) + \lfloor \frac{(2k-p)}{2} \rfloor = 9 + (5 - 3) + 0 = 11$ and an optimal *npc* coloring of $K_{2,3,4,4,5}$ is as shown in Figure 2.

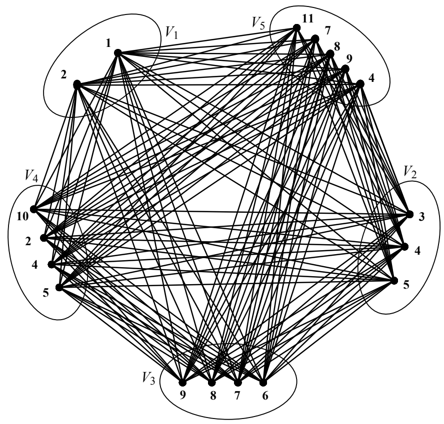


Figure 2: *npc* coloring of the graph $K_{2,3,4,4,5}$ with $\psi_{S_{nhd}}(K_{2,3,4,4,5}) = 11$.

*Example 3.12.* Consider a complete 6-partite graph $K_{5,5,5,6,6,6}$. Here, $n = 5 + 5 + 5 + 6 + 6 + 6 = 33$, $m_p = |V_p| = 6$, $\sum_{i=1}^{p-1} m_i - 1 = 27 - 1 = 26$, so that $m_p < \sum_{i=1}^{p-1} m_i - 1$ and $\lfloor \frac{n}{2} \rfloor = 16$. Thus, $k = 4$ and $k < p - 1$. Hence, from Theorem 3.10, $\psi_{S_{nhd}}(K_{5,5,5,6,6,6}) = \lfloor \frac{n}{2} \rfloor + (p-k) + \lfloor \frac{(2k-p)}{2} \rfloor = 16 + (6-4) + 1 = 19$ and an optimal *npc* coloring of vertices of $K_{5,5,5,6,6,6}$ (edges are not shown) is as shown in Figure 3.

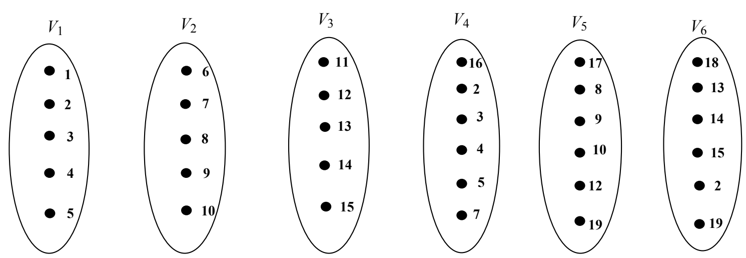


Figure 3: *npc* coloring of graph $K_{5,5,5,6,6,6}$ with $\psi_{S_{nhd}}(K_{5,5,5,6,6,6}) = 19$.

*Example 3.13.* Consider a complete 3-partite graph $K_{3,5,5}$. Here, $n = 3+5+5 = 13$, $m_p = |V_p| = 5$, $\sum_{i=1}^{p-1} m_i - 1 = 8 - 1 = 7$, so that $m_p < \sum_{i=1}^{p-1} m_i - 1$ and

$\lceil \frac{n}{2} \rceil = 7$. Thus, $k = 2$, $k = p - 1$ and $\lfloor \frac{n}{2} \rfloor - m_p \leq \lfloor \frac{k}{2} \rfloor$. Hence, from Theorem 3.10, $\psi_{S_{nhd}}(K_{3,5,5}) = \sum_{i=1}^{p-1} m_i = 8$ and an optimal *npc* coloring of $K_{3,5,5}$ is as shown in Figure 4.

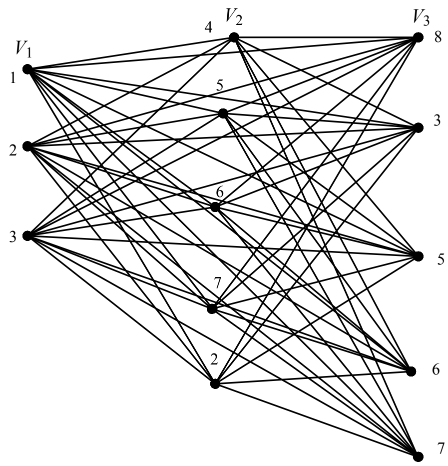


Figure 4: An optimal *npc* coloring of the graph $K_{3,5,5}$ with $\psi_{S_{nhd}}(K_{3,5,5}) = 8$.

*Example 3.14.* Consider a complete 4-partite graph $K_{2,3,4,10}$. Here, $n = 19$, $m_p = |V_p| = 10$, $\sum_{i=1}^{p-1} m_i = 9$, so that $m_p \geq \sum_{i=1}^{p-1} m_i$. From Theorem 3.3, $\psi_{S_{nhd}}(K_{2,3,4,9}) = \sum_{i=1}^{p-1} m_i = 9$.

**Observation 3.15.** By observing the pattern of coloring as defined in [11] and [7], there exists some classes of graphs for which $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \psi_S(K_{m_1,m_2,\cdots,m_p})$ and some classes of graphs $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) \neq \psi_S(K_{m_1,m_2,\cdots,m_p})$, are illustrated in the following examples.

*Example 3.16.* Consider the graph $K_{2,3,4,4,5}$.

The neighborhood pseudo achromatic number of $K_{2,3,4,4,5}$ from Example 3.11 is 11. Also, for the graph $K_{2,3,4,4,5}$, $n = 18$, $m_p = 5$, $p = 5$, so that, $\frac{n-p+2}{2} = \frac{18-5+2}{2} = \frac{15}{2}$ and hence, $m_p \leq (\frac{n-p+2}{2})$. From [11], $\psi_S(K_{m_1,m_2,\cdots,m_p}) = \lfloor \frac{n+p}{2} \rfloor$. Therefore, pseudo achromatic number of $K_{2,3,4,4,5}$ is $\lfloor \frac{18+5}{2} \rfloor = 11$. Hence, $\psi_S(K_{2,3,4,4,5}) = \psi_{S_{nhd}}(K_{2,3,4,4,5})$.

*Example 3.17.* Consider the graph $K_{2,3,4,10}$.

The neighborhood pseudo achromatic number of $K_{2,3,4,10}$ from Example 3.14 is 9. Also, for the graph $K_{2,3,4,10}$, $n = 19$, $m_p = 10$, $p = 4$, so that, $\frac{n-p+2}{2} = \frac{19-4+2}{2} = \frac{17}{2}$ and hence, $m_p \geq (\frac{n-p+2}{2})$. From [7], $\psi_S(K_{m_1,m_2,\cdots,m_p}) = m_1 + m_2 + \cdots + m_{p-1} + 1$. Therefore, pseudo achromatic number of $K_{2,3,4,10}$ is $2 +$

$3+4+1=10$. Hence, $\psi_S(K_{2,3,4,4,5}) \neq \psi_{S_{nhd}}(K_{2,3,4,4,5})$.

## 4. Algorithm to Find the Neighborhood Pseudo Achromatic Number of Complete $p$-Partite Graph

**Input:** The complete $p$-partite graph $K_{m_1,m_2,\cdots,m_p}$ with $V(K_{m_1,m_2,\cdots,m_p}) = V_1 \cup V_2 \cup \cdots V_p$, $|V_i| = m_i$, $\sum_{i=1}^{p} m_i = n$, $V_i = \{v_{i1}, v_{i2}, \cdots, v_{im_i}\}$, $\forall i$, $1 \leq i \leq p$ and $m_1 \leq m_2 \leq \cdots \leq m_p$.

**Output:**

**begin**

*Step 1.* If $m_p \geq m_1 + m_2 + \cdots + m_{p-1}$, then

color the vertices of $K_{m_1,m_2,\cdots,m_p}$ with distinct $\sum\limits_{i=1}^{p-1} m_i$ colors as follows:

$c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, \sum\limits_{i=1}^{p-1} m_i\}$ defined by,

for $j = 1$ to $m_1$ do

[

$c(v_{1j}) = j$

]

for $i = 2$ to $p-1$ do

[

for $j = 1$ to $m_i$ do

[

$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$

]

]

$c : V_p \to \{1, 2, \cdots, \sum\limits_{i=1}^{p-1} m_i\}$ a surjection.

But then $c$ is a *npc* coloring with maximum colors, resulting

$\psi_{s_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \sum\limits_{i=1}^{p-1} m_i$.

Stop.

else goto Step 2.

*Step 2.* If $\sum\limits_{i=1}^{p-1} m_i = m_p + 1$, then

color the vertices of $K_{m_1,m_2,\cdots,m_p}$ as follows:

$c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ defined by,

for $j = 1$ to $m_1$ do

[

$c(v_{1j}) = j$

]
for $i = 2$ to $p - 1$ do
[
for $j = 1$ to $m_i$ do
[
$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$
]
]
for $j = 2$ to $m_p$ do
[
$c(v_{pj}) = j$, $\forall j$
]
But then, $c$ is a *npc* coloring with maximum colors, resulting
$\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \sum_{i=1}^{p-1} m_i = \lceil \frac{m_1+m_2+\cdots+m_p}{2} \rceil = \lceil \frac{n}{2} \rceil$.
Stop.
else goto Step 3.

*Step 3.* If $m_p < \sum_{i=1}^{p-1} m_i - 1$, then

obtain a smallest positive integer $k$ such that $\lceil \frac{n}{2} \rceil \leq \sum_{i=1}^{k} m_i$.
If $k < p - 1$, then
color the vertices of $K_{m_1,m_2,\cdots,m_p}$ as follows:
$c : V(K_{m_1,m_2,\cdots,m_p}) \to \{1, 2, \cdots, (\lceil \frac{n}{2} \rceil + (p - k) + \lfloor \frac{(2k-p)}{2} \rfloor)\}$
defined by,
for $j = 1$ to $m_1$ do
[
$c(v_{1j}) = j$
]
for $i = 2$ to $k - 1$ do
[
for $j = 1$ to $m_i$ do
[
$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$
]
]
Also, when $i = k$ and for some $k_1$ satisfying $c(v_{kk_1}) = \lceil \frac{n}{2} \rceil$
for $j = 1$ to $k_1$ do
[
$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$
]

for $i = k+1$ to $p$ do
[
$c(v_{i1}) = \lceil \frac{n}{2} \rceil + i - k$
]
for $j = 1$ to $\lfloor \frac{2k-p}{2} \rfloor$ do
[
$c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lceil \frac{n}{2} \rceil + p - k + j$
]
and the unassigned vertices of $\bigcup_{i=k}^{p} V_i$ are colored using the colors from the set $\{\{1, 2, \cdots, \lceil \frac{n}{2} \rceil\} - \{c(v_{11}), c(v_{21}), \cdots, c(v_{k1})\}\}$ choosing each color at least once.
But then $c$ is a *npc* coloring with maximum colors, resulting $\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \frac{n}{2} \rceil + (p-k) + \lfloor \frac{(2k-p)}{2} \rfloor$.
Stop.
else goto Step 4.
else goto Step 4.

*Step 4.* In this case, $m_p < \sum_{i=1}^{p-1} m_i - 1$ and $k = p - 1$.
Color the vertices of $K_{m_1,m_2,\cdots,m_p}$ as follows:
for $j = 1$ to $m_1$ do
[
$c(v_{1j}) = j$
]
for $i = 2$ to $p - 2$ do
[
for $j = 1$ to $m_i$ do
[
$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$
]
]
Also, when $i = p - 1$ and for some $k_1$ satisfying $c(v_{(p-1)k_1}) = \lceil \frac{n}{2} \rceil$
for $j = 1$ to $k_1$ do
[
$c(v_{ij}) = c(v_{(i-1)(m_{i-1})}) + j$
]
Further,
If $(\lfloor \frac{n}{2} \rfloor - m_p) \leq \lfloor \frac{k}{2} \rfloor$, then
for $j = 1$ to $\lfloor \frac{n}{2} \rfloor - m_p$ do
[
$c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lceil \frac{n}{2} \rceil + j$

]

resulting,

$\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \sum_{i=1}^{p-1} m_i.$

Stop.

else

for $j = 1$ to $\lfloor \frac{n}{2} \rfloor - m_p$ do

[

$c(v_{(p-1)(m_{p-1}-(j-1))}) = c(v_{(p)(m_p-(j-1))}) = \lceil \frac{n}{2} \rceil + j$

]

resulting,

$\psi_{S_{nhd}}(K_{m_1,m_2,\cdots,m_p}) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor.$

Stop.

**end**

## 5. Conclusion

(i) We have obtained the neighborhood pseudo achromatic number of complete $p$-partite graph analyzing various cases and explained the unique procedure of coloring in various cases. Also, developed the simplified algorithm to find neighborhood pseudo achromatic number of complete $p$-partite graph.

(ii) Let $KI = K_{m_1,m_2,\ldots,m_p}$. Comparing with pseudo complete coloring, neighborhood pseudo complete coloring satisfies the stronger additional condition of neighborhood pseudo coloring, still it is possible to color for some classes of $KI$ with maximum number of colors such that $\psi_{S_{nhd}}(KI) = \psi_S(KI)$. But $\psi_{S_{nhd}}(KI) \neq \psi_S(KI)$ for some complete $p$-partite graphs from Observation 3.15.

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